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MODULAR ATOMIC EFFECT ALGEBRAS AND THE EXISTENCE OF SUBADDITIVE STATES[1]

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Lattice effect algebras generalize orthomodular lattices and MV-algebras. We describe all complete modular atomic effect algebras. This allows us to prove the existence of order-continuous subadditive states (probabilities) on them. For the separable noncomplete ones we show that the existence of a faithful probability is equivalent to the condition that their MacNeille completion is a complete modular effect algebra.

Keywords: Effect algebra, modular atomic effect algebra, subadditive state, MacNeille completion of an effect algebra

AMS Subject Classification: 03G12, 06F99, 81P10

1. INTRODUCTION

In recent years quantum effects and fuzzy events have been studied within a general algebraic framework called an effect algebra or, equivalently in some sense, a D-poset. Thus, the elements of these structures represent events that may be unsharp or imprecise ([4, 10, 11]). Moreover, lattice ordered effect algebras generalize orthomodular lattices [9] and MV-algebras ([1, 2]) – the algebraic structures which have proved their importance in the investigation of the phenomenon of uncertainty (see [3]).

It is known that there are (finite) effect algebras admitting no states and, hence, no probabilities ([16]). Greechie's example of an orthomodular lattice L admitting no states [6] is simultaneously an example of a lattice effect algebra admitting no states. This is because every orthomodular lattice L can be organized into a lattice effect algebra by putting (for $a, b \in L$) $a \oplus b = a \vee b$ iff $a \leq b'$. Clearly, in this case the notions of a state on orthomodular lattice L and a state on the derived lattice effect algebra coincide. If there is a state on a lattice effect algebra it need not be subadditive. We have shown in [17] that if a faithful subadditive state on a lattice effect algebra E exists *then E is separable and modular.* Nevertheless, it remained unanswered whether subadditive states on all such effect algebras exist. In the present paper we give a positive answer to this question. The proof is based on the fact that,

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as we show, every irreducible complete atomic modular effect algebra is of finite length and every complete atomic modular effect algebra is isomorphic to a direct product of irreducible ones. Moreover, we show that the only irreducible complete atomic effect algebras are either irreducible complete atomic modular ortholattices, or finite chains, or effect algebras of length 2. These facts give a full description of all complete atomic modular effect algebras and, as a consequence, a description of all atomic lattice effect algebras admitting order-continuous subadditive states (probabilities).

2. EFFECT ALGEBRAS, BASIC NOTIONS AND FACTS

Definition 2.1. (See [4].) A partial algebra $(E; \oplus, 0, 1)$ is called an *effect-algebra* if 0, 1 are two distinct elements and $\oplus$ is a partially defined binary operation on E which satisfies the following conditions for any $a, b, c \in E$:

(Ei) $b \oplus a = a \oplus b$ if $a \oplus b$ is defined,

(Eii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ if one side is defined,

(Eiii) for every $a \in P$ there exists a unique $b \in P$ such that $a \oplus b = 1$ (we put $a' = b$),

(Eiv) if $1 \oplus a$ is defined then $a = 0$.

We often denote the effect algebra $(E; \oplus, 0, 1)$ briefly by E. Moreover, if we write $a \oplus b = c$ for $a, b, c \in E$, then we mean both that $a \oplus b$ is defined and $a \oplus b = c$. In every effect algebra E we can define the partial operation $\ominus$ and the partial order $\leq$ by putting

$$a \leq b \text{ and } b \ominus a = c \text{ iff } a \oplus c \text{ is defined and } a \oplus c = b.$$

Since $a \oplus c = a \oplus d$ implies $c = d$, the operation $\ominus$ and the relation $\leq$ are well defined. If E with the defined partial order is a lattice (a complete lattice) then $(E; \oplus, 0, 1)$ is called a *lattice effect algebra* (a *complete effect algebra*). If, moreover, E is a modular or distributive lattice then E is called *modular* or *distributive* effect algebra.

Recall that a set $Q \subseteq E$ is called a *sub-effect algebra* of the effect algebra E if

(i) $1 \in Q$,

(ii) if out of elements $a, b, c \in E$ with $a \oplus b = c$ two are in Q, then $a, b, c \in Q$.

Assume that $(E_1; \oplus_1, 0_1, 1_1)$ and $(E_2; \oplus_2, 0_2, 1_2)$ are effect algebras. An injection $\varphi : E_1 \to E_2$ is called an *embedding* if $\varphi(1_1) = 1_2$ and for $a, b \in E_1$ we have $a \leq b'$ iff $\varphi(a) \leq (\varphi(b))'$ in which case $\varphi(a \oplus_1 b) = \varphi(a) \oplus_2 \varphi(b)$. We can easily see that then $\varphi(E_1)$ is a sub-effect algebra of E_2 and we say that E_1 and $\varphi(E_1)$ are *isomorphic*, or that E_1 *is up to isomorphism a sub-effect algebra of* E_2. We usually identify E_1 with $\varphi(E_1)$.

We say that a finite system $F = (a_k)_{k=1}^n$ of not necessarily different elements of an effect algebra $(E; \oplus, 0, 1)$ is $\oplus$-*orthogonal* if $a_1 \oplus a_2 \oplus \cdots \oplus a_n$ (written $\bigoplus_{k=1}^n a_k$

or $\bigoplus F$) exists in E. Here we define $a_1 \oplus a_2 \oplus \cdots \oplus a_n = (a_1 \oplus a_2 \oplus \cdots \oplus a_{n-1}) \oplus a_n$ supposing that $\bigoplus_{k=1}^{n-1} a_k$ exists and $\bigoplus_{k=1}^{n-1} a_k \leq a_n'$. An arbitrary system $G = (a_\kappa)_{\kappa \in H}$ of not necessarily distinct elements of E is called $\oplus$-orthogonal if $\bigoplus K$ exists for every finite $K \subseteq G$. We say that for a $\oplus$-orthogonal system $G = (a_\kappa)_{\kappa \in H}$ the element $\bigoplus G$ exists iff $\bigvee\{\bigoplus K | K \subseteq G,\ K \text{ is finite}\}$ exists in E and then we put $\bigoplus G = \bigvee\{\bigoplus K | K \subseteq G,\ K \text{ is finite}\}$ (we write $G_1 \subseteq G$ iff there is $H_1 \subseteq H$ such that $G_1 = (a_\kappa)_{\kappa \in H_1}$). We refer the reader to [18].

An effect algebra $(E; \oplus, 0, 1)$ is called *Archimedean* if for no nonzero element $e \in E$ the elements $ne = \underbrace{e \oplus e \cdots \oplus e}_{n \text{ times}}$ exist for all $n \in N$. An Archimedean effect algebra is called *separable* if every $\oplus$-orthogonal system of elements of E is at most countable. We can show that *every complete effect algebra is Archimedean* [19].

For an element x of an effect algebra E we write $\operatorname{ord}(x) = \infty$ if nx exists for every $n \in N$. We write $\operatorname{ord}(x) = n_x \in N$ if n_x is the greatest positive integer such that $n_x x$ exists in E. Clearly, in an Archimedean effect algebra $n_x < \infty$ for every $x \in E$.

Recall that elements x and y of a lattice effect algebra are called *compatible* (written $a \leftrightarrow b$) if $x \vee y = x \oplus (y \ominus (x \wedge y))$. For $x \in E$ and $Y \subseteq E$ we write $x \leftrightarrow Y$ iff $x \leftrightarrow y$ for all $y \in Y$. If every two elements of E are compatible then E is called an *MV-effect algebra.*

Every *finite chain* $0 < a < 2a < \cdots < 1 = n_a a$ is a distributive effect algebra in which every pair of elements is compatible, hence it is an MV-effect algebra.

An element a of an effect algebra E is called an *atom* if $0 \leq b < a$ implies $b = 0$ and E is called *atomic* if for every $x \in E$, $x \neq 0$ there is an atom $a \in E$ with $a \leq x$. Clearly every finite effect algebra is atomic.

For more details we refer the reader to (Dvurečenskij and Pulmannová [3]) and the references given therein. We review only a few properties.

Lemma 2.1. The elements of an effect algebra $(E; \oplus, 0, 1)$ satisfy the following properties:

(i) $a \oplus b$ is defined iff $a \leq b'$,

(ii) if $a \oplus b$ and $a \vee b$ exist then $a \wedge b$ exists and $a \oplus b = (a \wedge b) \oplus (a \vee b)$,

(iii) if $u \leq a$, $v \leq b$ and $a \oplus b$ is defined then $u \oplus v$ is defined,

(iv) [8] If E is a lattice and $Y \subseteq E$ with $\bigvee Y$ existing in E then $x \leftrightarrow Y \Rightarrow x \wedge (\bigvee Y) = \bigvee\{x \wedge y | y \in Y\}$ and $x \leftrightarrow \bigvee Y$.

3. IRREDUCIBLE COMPLETE ATOMIC MODULAR EFFECT ALGEBRAS

Recall that a direct product $\prod\{E_\kappa | \kappa \in H\}$ of effect algebras E_κ is a Cartesian product with $\oplus$, 0 and 1 defined "coordinatewise". An element $z \in E$ is called *central* if the intervals $[0, z]$ and $[0, z']$ with the inherited $\oplus$-operation are effect algebras in their own right and $E \cong [0, z] \times [0, z']$, (see [7]). The set $C(E) = \{z \in E | z \text{ is central}\}$ is called a *center* of E. If $C(E) = \{0, 1\}$ then E is called *irreducible.*

In every lattice effect algebra E the set $S(E) = \{x \in E | x \wedge x' = 0\}$ is an orthomodular lattice [8] and $B(E) = \{x \in E | x \leftrightarrow E\}$ is an MV-effect algebra such that $S(E)$ and $B(E)$ are sub-lattices and sub-effect algebras of E [14]. Moreover, we have shown in [12] that $z \in C(E)$ iff $x = (x \wedge z) \vee (x \wedge z')$ for all $x \in E$ which gives $C(E) = B(E) \cap S(E)$ for every lattice effect algebra E. Further, $S(E)$, $B(E)$ and $C(E)$ are closed with respect to all existing infima and suprema [18]. In general, $C(E) = \{0, 1\}$ does not imply $C(S(E)) = \{0, 1\}$.

Recall that the *length* of a finite chain is the number of its elements minus 1. The *length* (*height*) of a lattice L is finite if the supremum over the number of elements of chains in L equals to some natural number n and then $n - 1$ is called *length of the lattice L*.

Theorem 3.1. For every irreducible complete atomic modular effect algebra E at least one of the following conditions is satisfied.

(i) E is an irreducible complete atomic modular ortholattice.

(ii) E is a finite chain.

(iii) E is a horizontal sum of a family of Boolean algebras and chains, all of length 2.

Proof. (i) This is the case when $a \wedge a' = 0$ for every atom $a \in E$, because then $E = S(E) = \{x \in E | x \wedge x' = 0\}$. Indeed, if there is $x \in E$ with $x \wedge x' \neq 0$ then there exists an atom $a \in E$ with $a \leq x \wedge x'$ which gives $a \leq x \leq (x')' \leq a'$; a contradiction. Since $S(E)$ is an orthomodular lattice in every lattice effect algebra and E is modular, the equality $E = S(E)$ implies that E is a modular ortholattice. Moreover, $C(S(E)) = B(S(E)) = B(E) \cap S(E) = C(E) = \{0, 1\}$, because the compatibility in $S(E)$ coincides with the compatibility in derived effect algebra.

(ii) Assume now that there is an atom $a \in E$ such that $a \leq a'$ and let $a \in B(E)$. Then also $n_a a \in B(E)$ (by [14]) and, by [19, Theorem 2.4], $n_a a \in S(E)$. Thus $n_a a \in B(E) \cap S(E) = C(E) = \{0, 1\}$, which gives $n_a a = 1$. It follows that for every atom $b \in E$ we have $n_b b \leq 1 = n_a a$. Assume that there is an atom $b \in E$, $b \neq a$. Then either $n_b b = n_a a$, and hence by [19, Theorem 3.1] we have $2a = 2b = 1$ which gives $a \not\leftrightarrow b$. Or $n_b = 1$ and $n_a = 2$ which gives $1 = 2a = b \oplus b' = b \vee b'$, hence again $b \not\leftrightarrow a$, a contradiction. We conclude that E has a unique atom, hence $E = \{0, a, 2a, \ldots, 1 = n_a a\}$.

(iii) Finally, assume that there is an atom $a \in E$ such that $a \leq a'$ and $a \notin B(E)$. Then there exists an atom $b \in E$ with $b \not\leftrightarrow a$. As E is modular we have $[a \wedge b, b] \cong [a, a \vee b]$ which yields that $a \vee b$ covers a and hence there exists an atom $c \in E$ such that $a \oplus c = a \vee b$, which gives $c \leq a'$. Evidently, $c \neq b$ as $b \not\leq a'$. If $c \neq a$ then $a \oplus c = a \vee b$, which implies $b \leq a \vee c \leq a'$, a contradiction. Thus $c = a$ and $a \vee b = 2a$.

Let $p \in E$ be an atom. Then either $p \not\leftrightarrow a$ which, as we have just shown, implies that $p \leq p \vee a = 2a \leq n_a a$, or $p \leftrightarrow a$ and hence $p \leftrightarrow n_a a$ for every atom $p \in E$. By [18], for every $x \in E$ we have $x = \bigvee\{u \in E | u \leq x,\ u$ is a sum of finite sequence of atoms$\}$. Since $n_a a \leftrightarrow p$ for every atom p and hence $n_a a \leftrightarrow u$ for every finite sum

u of atoms, we conclude by Lemma 2.1, (iv), that $n_a a \leftrightarrow x$ for every $x \in E$. Thus $n_a a \in B(E) \cap S(E) = C(E) = \{0,1\}$, which implies that $n_a a = 1$. It follows that for every atom $p \in E$, the inequality $n_p p \leq n_a a = 1$ implies by [19, Theorem 3.1] that $n_p = 1$ or $n_p = 2$. If $n_p = 2$ then $[0, n_p p]$ is a chain $\{0, p, 1 = 2p\}$ and if $n_p = 1$ then $[0, n_p p]$ is a Boolean algebra $\{0, p, p', 1 = p \oplus p'\}$. □

Corollary 3.1. The unique example of a complete atomic modular effect algebra E with $C(E) = \{0,1\}$ and $C(S(E)) \neq \{0,1\}$ is a horizontal sum of the Boolean algebra $\{0, a, a', 1 = a \oplus a'\}$ and chain $\{0, b, 1 = 2b\}$.

In a lattice of finite length with 0 we define a *height function* as follows:

For $a \in L$ let $h(a)$ denotes the length of a longest maximal chain in $[0,a]$. Here a chain $P \subseteq [0,a]$ is called *maximal* if for any chain $Q \subseteq [0,a]$ we have $P \subseteq Q \Rightarrow P = Q$.

For a lattice of finite length the following conditions are equivalent:

(i) *L is modular,*

(ii) *$h(a) + h(b) = h(a \vee b) + h(a \wedge b)$ for all $a, b \in L$* (see [5, pp. 227–228]).

In [14] it has been shown that every maximal subset M of pairwise compatible elements of a lattice effect algebra E is a sub-effect algebra and a sublattice of E called a *block* of E. Moreover, E is a union of its blocks. Clearly, the blocks of E are MV-effect algebras.

Corollary 3.2. Every irreducible complete atomic modular effect algebra E is of finite length n which equals to the length of any block of E

Proof. For the case (i) of Theorem 3.1, see [9, p. 209]. For cases (ii) and (iii) the statement is evident. □

4. COMPLETE ATOMIC MODULAR EFFECT ALGEBRAS AND THE EXISTENCE OF SUBADDITIVE STATES

Recall that a map $\omega : E \to [0,1]$ is called a (finitely additive) state on an effect algebra E if $\omega(1) = 1$ and $x \leq y' \Rightarrow \omega(x \oplus y) = \omega(x) + \omega(y)$; ω is called *(o)-continuous* if $x_\alpha \xrightarrow{(o)} x \Rightarrow \omega(x_\alpha) \to \omega(x)$. Here for a net $(x_\alpha)_{\alpha \in \mathcal{E}}$ of elements of E we write $x_\alpha \xrightarrow{(o)} x$ if there exist a nondecreasing net $(u_\alpha)_{\alpha \in \mathcal{E}}$ and a nonincreasing net $(v_\alpha)_{\alpha \in \mathcal{E}}$ such that $u_\alpha \leq x_\alpha \leq v_\alpha$ for all $\alpha \in \mathcal{E}$ and $u_\alpha \uparrow x$ and $v_\alpha \downarrow x$. A state ω is called *σ-additive* if $\omega\left(\bigoplus_{n=1}^{\infty} x_n\right) = \sum_{n=1}^{\infty} \omega(x_n)$ for every $\oplus$-orthogonal sequence $(x_n)_{n=1}^{\infty}$ for which $\bigoplus_{n=1}^{\infty} x_n$ exists in E. A state ω on E is called *faithful* if $\omega(x) = 0$ implies $x = 0$.

If E is a lattice effect algebra then a state ω is called *subadditive* iff $\omega(x \vee y) \leq \omega(x) + \omega(y)$ for all $x, y \in E$. We have shown in [17] that a state ω is subadditive iff $\omega(a) + \omega(b) = \omega(a \vee b) + \omega(a \wedge b)$ for all $a, b \in E$ (ω is called a *valuation*) and this occurs iff $a \wedge b = 0 \Rightarrow \omega(a \vee b) = 0$. Moreover, if a faithful subadditive state on a lattice effect algebra exists then E is separable and modular, see [17]. It is a matter of a routine verification that for every complete separable effect algebra E and a faithful state ω on E the following conditions are equivalent:

(i) ω is σ-additive,

(ii) $x_n \downarrow 0 \Rightarrow \omega(x_n) \downarrow 0$,

(iii) $x_n \uparrow x \Rightarrow \omega(x_n) \uparrow \omega(x)$,

(iv) ω is (o)-continuous,

(v) $\omega(\bigoplus G) = \bigvee\{\sum_{x\in F} \omega(x) | F \subseteq G \text{ is finite}\}$ for every $\oplus$-orthogonal system G for which $\bigoplus G$ exists in E.

A σ-additive subadditive state on a lattice effect algebra will be called a *probability*.

Finally, recall that a lattice effect algebra E is called (o)*-continuous* if $x_\alpha \uparrow x \Rightarrow x_\alpha \wedge y \uparrow x \wedge y$ for all $x_\alpha, x, y \in E$, and it is iff the lattice operations $\vee$ and $\wedge$ are (o)-continuous.

Theorem 4.1. Every irreducible complete atomic modular effect algebra E possesses an (o)-continuous subadditive state. In fact, $\omega = \frac{h}{n}$, where n is the length of E and h is the height function on E.

Proof. Let $x \in E$. Then $[0, x]$ is a modular lattice of finite length and hence any two maximal chains of $[0, x]$ are of the same length $h(x)$ [5, p. 223, Theorem 1]. Moreover, every maximal chain of $[0, x]$ is of the form $0 < a_1 < a_1 \oplus a_2 < \cdots < a_1 \oplus a_2 \oplus \cdots \oplus a_k = x$, where $a_1, a_2, \ldots, a_k$ is a finite sequence of not necessarily different atoms of E and $k = h(x)$. Thus, if $y \leq x'$ and $y = b_1 \oplus b_2 \oplus \cdots \oplus b_\ell$ for some sequence of atoms $b_1, b_2, \ldots b_\ell$ of E, then $x \oplus y = a_1 \oplus \cdots \oplus a_k \oplus b_1 \oplus \cdots \oplus b_\ell$ and hence $h(x \oplus y) = h(x) + h(y)$. This proves that $\omega = \frac{h}{n}$ is a state on E. Since for all $x; y \in E$ we have $h(x \vee y) = h(x) + h(y) - h(x \wedge y)$, we conclude that $\omega = \frac{h}{n}$ is subadditive. The (o)-continuity of ω is trivial as E is of finite length. $\square$

Theorem 4.2. Let E be a complete atomic modular effect algebra. Then

(i) $E \cong L \times M \times E_0$, where L is a modular ortholattice, M is an MV-effect algebra and E_0 is a direct product of lattice effect algebras of length 2. The factors M, L and E_0 are complete and atomic, or trivial factors $\{0\}$.

(ii) E is (o)-continuous.

(iii) There is an (o)-continuous subadditive state on E.

(iv) There is a faithful (o)-continuous subadditive state on E iff $C(E)$ is separable.

Proof. (i) By [19, Theorem 3.2], $C(E)$ is a complete atomic Boolean algebra and by [19, Lemma 4.3] we obtain that $E \cong \prod\{[0, p] | p \text{ is an atom of } C(E)\}$. Since for every atom p of $C(E)$ the effect algebra $[0,p]$ is irreducible and any direct product of finite chains is an MV-effect algebra, the statement follows by Theorem 3.1.

(ii) For every atom p of $C(E)$ the effect algebra $[0,p]$ is of finite length and hence it is (o)-continuous. Thus, the direct product $\prod\{[0,p] | p$ is an atom of $C(E)\}$ is (o)-continuous as well. By part (i) we conclude that E is (o)-continuous.

(iii) If $C(E) = \{0,1\}$ the statement follows by Theorem 4.1. Let $C(E) \neq \{0,1\}$ and p is an atom of $C(E)$. Then by Theorem 4.1 there is a faithful subadditive state ω_p on $[0,p]$. Since $p \in C(E)$ for $x, y \in E$ we have $(x \vee y) \wedge p = (x \wedge p) \vee (y \wedge p)$ by Lemma 2.1, (iv). If $x \leq y'$ then $(x \oplus y) \wedge p = (x \wedge p) \oplus (y \wedge p)$ by [19, Lemma 4.1]. Thus ω defined by $\omega(x) = \omega_p(x \wedge p)$ for all $x \in E$ is a faithful subadditive state on E. Since by (ii) E is (o)-continuous, $x_\alpha \uparrow x \Rightarrow x_\alpha \wedge p \uparrow x \wedge p$ and hence $\omega_p(x_\alpha \wedge p) \uparrow \omega_p(x \wedge p)$, as $[0,p]$ has a finite length. It follows that $\omega(x_\alpha) \uparrow \omega(x)$, which proves that ω is (o)-continuous.

(iv) By [17, Theorem 2.8] the existence of a faithful state on E implies that E is separable and hence $C(E)$ is separable. Conversely, assume that $C(E)$ is separable. Let $K \subseteq N = \{1, 2, \dots\}$ and $A_{C(E)} = \{p_k | k \in K\}$ be the set of all atoms of $C(E)$. Let ω_k be faithful subadditive states on $[0, p_k]$, $k \in K$. Further, take $c_k \in (0,1) \subseteq R$ with $\sum_{k \in K} c_k = 1$. For every $x \in E$, let us put $\omega(x) = \sum_{k \in K} c_k \omega_k(x \wedge p_k)$. By similar reasonings as in part (iii), ω is a faithful subadditive state on E. Let us show that ω is (o)-continuous. Let $x_\alpha, x \in E$, $\alpha \in \mathcal{E}$, and $x_\alpha \uparrow x$. By Lemma 2.1, (iv), we have $x = x \wedge 1 = x \wedge \bigvee\{p_k | k \in K\} = \bigvee\{x \wedge p_k | k \in K\} = \bigvee\{(x \wedge p_1) \oplus (x \wedge p_2) \oplus \cdots \oplus (x \wedge p_n) | n \in K\}$ because $(x \wedge p_1) \oplus (x \wedge p_2) \oplus \cdots \oplus (x \wedge p_n) = (x \wedge p_1) \vee (x \wedge p_2) \vee \cdots \vee (x \wedge p_n)$, since for $k \neq \ell$ we have $p_k \leq p'_\ell$ and $p_k \wedge p_\ell = 0$ which gives $p_k \vee p_\ell = p_k \oplus p_\ell$ by Lemma 2.1, (ii). Since E is (o)-continuous, it is compactly generated by finite elements (i.e, by finite sums of atoms, see [19, Theorem 4.5]). It follows that for every $n \in K$ there is $\alpha_n \in \mathcal{E}$ such that $(x \wedge p_1) \oplus (x \wedge p_2) \oplus \cdots \oplus (x \wedge p_n) \leq x_{\alpha_n}$ and therefore we may assume $\alpha_1 \leq \alpha_2 \leq \dots$. Obviously $x_{\alpha_n} \uparrow x$. Further, $\omega(x) \geq \omega(x_{\alpha_n}) \geq \omega\big((x \wedge p_1) \oplus \cdots \oplus (x \wedge p_n)\big) = c_1 \omega_1(x \wedge p_1) + c_2 \omega_2(x \wedge p_2) + \cdots + c_n \omega_n(x \wedge p_n) \uparrow \omega(x)$ and hence $\omega(x_{\alpha_n}) \uparrow \omega(x)$ which gives $\omega(x_\alpha) \uparrow \omega(x)$. □

Corollary 4.1. For a complete atomic effect algebra E the following conditions are equivalent:

(i) There exists a faithful probability on E.

(ii) E is separable and modular.

Note that (i) $\Rightarrow$ (ii) has been proved in [17].

Finally, note that a lattice effect algebra admitting a subadditive state (not necessarily faithful) need not be modular. Nevertheless, every complete effect algebra admitting an (o)-continuous subadditive state can be decomposed into a direct product of two effect algebras at least one of which is modular, or it is modular.

Theorem 4.3. If for a complete effect algebra E the set $\mathcal{M} = \{\omega : E \to [0,1] \subseteq R | \omega$ is a subadditive (o)-continuous state$\}$ is nonempty then either E is modular, or there is $d_0 \in C(E)$ such that $d_0 \neq 0$, $[0, d_0]$ admits no (o)-continuous subadditive state and $[0, d_0']$ is modular.

Proof. For every $\omega \in \mathcal{M}$, we put $d_\omega = \{\bigvee\{x \in E | \omega(x) = 0\}$. Then, as we have shown in [16, Theorem 5.1], $\omega(d_\omega) = 0$, $E \cong [0, d_\omega] \times [0, d_\omega']$ and the restriction $\omega|_{[0,d_\omega']}$ is a faithful probability on $[0, d_\omega']$. This gives that $[0, d_\omega']$ is a separable modular effect algebra.

Put $d_0 = \bigwedge\{d_\omega | \omega \in \mathcal{M}\}$. Then $d_0 \in C(E)$, as $C(E)$ is a complete sublattice of E. Since $d_0' = \bigvee\{d_\omega' | \omega \in \mathcal{M}\}$ we see that $[0, d_\omega'] \subseteq [0, d_0']$, for all $\omega \in \mathcal{M}$. Let us show that $[0, d_0']$ is a modular lattice.

By [8], for every $x \in [0, d_0']$ we have $x = x \wedge d_0' = x \wedge \left(\bigvee\{d_\omega' | \omega \in \mathcal{M}\}\right) = \bigvee\{x \wedge d_\omega' | \omega \in \mathcal{M}\}$. Further, for every $\omega \in \mathcal{M}$ and $x, y, z \in [0, d_0']$ with $x \leq z$ we have $x \wedge d_\omega' \leq z \wedge d_\omega'$ and because $[0, d_\omega']$ is modular and $d_\omega' \leftrightarrow E$ we obtain $\left(x \vee (y \wedge z)\right) \wedge d_\omega' = (x \wedge d_\omega') \vee \left((y \wedge z) \wedge d_\omega'\right) = \left((x \wedge d_\omega') \vee (y \wedge d_\omega')\right) \wedge (z \wedge d_\omega') = \left((x \vee y) \wedge z\right) \wedge d_\omega'$. This yields $x \vee (y \wedge z) = \bigvee\left\{\left(x \vee (y \wedge z)\right) \wedge d_\omega' | \omega \in \mathcal{M}\right\} = \bigvee\left\{\left((x \vee y) \wedge z\right) \wedge d_\omega' | \omega \in \mathcal{M}\right\} = (x \vee y) \wedge z$. Thus $[0, d_0']$ is modular and evidently it is a complete sub-lattice of E. It follows that if E is not modular then $d_0' \neq 1$ and hence $d_0 \neq 0$.

Finally, let us show that $[0, d_0]$ admits no (o)-continuous subadditive state. Assume on the contrary that $d_0 \neq 0$ and there is an (o)-continuous subadditive state m on $[0, d_0]$. Then $\omega : E \to [0, 1]$ defined by $\omega(x) = m(x \wedge d_0)$, $x \in E$ is an (o)-continuous subadditive state on E. This follows from the facts that for all $x, y \in E$ we have $(x \vee y) \wedge d_0 = (x \wedge d_0) \vee (y \wedge d_0)$ and if $x \leq y'$ then $(x \oplus y) \wedge d_0 = (x \wedge d_0) \oplus (y \wedge d_0)$ by [19, Lemma 4.1]. Further, for $x_\alpha \downarrow x$, $x_\alpha, x \in E$ we have $x_\alpha \wedge d_0 \downarrow x \wedge d_0$ and hence $\omega(x_\alpha) = m(x_\alpha \wedge d_0) \downarrow m(x \wedge d_0) = \omega(x)$, which implies that ω is (o)-continuous by [19, Theorem 6.2]. Let $d_\omega = \bigvee\{x \in E | \omega(x) = 0\}$. Then $d_\omega \in C(E)$, $\omega(d_\omega) = 0$ and $d_0 \leq d_\omega$ which gives $\omega(d_0) = m(d_0) = 0$, a contradiction. This yields that $[0, d_0]$ admits no (o)-continuous probability.

It is well known that every poset $(P; \leq)$ has the *MacNeille completion* (completion by cuts). By J. Schmidt [20] the MacNeille completion of a poset P is any complete lattice $\widehat{P}$ into which the poset P can be supremum and infimum densely embedded, i.e., for each $x \in \widehat{P}$ there are $Q, M \subseteq P$ such that $x = \bigvee \varphi(M) = \bigwedge \varphi(Q)$, where $\varphi : P \to \widehat{P}$ is the embedding. We usually identify P with $\varphi(P)$.

A complete effect algebra $(\widehat{E}, \widehat{\oplus}, \widehat{0}, \widehat{1})$ is called a *MacNeille completion of an effect algebra* $(E; \oplus, 0, 1)$ if, up to isomorphism, E is a sub-effect algebra of $\widehat{E}$ and, as posets, $\widehat{E}$ is a MacNeille completion of E. It is known that there are (finite) effect algebras the MacNeille completion of which are not again effect algebras [13]. □

Corollary 4.2. If E is a complete effect algebra with $C(E) = \{0, 1\}$ then every (o)-continuous subadditive state on E is faithful.

Theorem 4.4. For an atomic lattice effect algebra E the following conditions are equivalent:

(i) There is a faithful probability ω on E.

(ii) The MacNeille completion of E is a separable complete atomic modular effect algebra

Proof. (i) $\Rightarrow$ (ii) If ω is a faithful probability on E then E is separable and hence ω is (o)-continuous. It follows by [17, Theorem 5.4] that the MacNeille completion $\widehat{E}$ of E is a separable complete atomic and modular effect algebra.

(ii) $\Rightarrow$ (i) In this case, by Theorem 4.1, there is a faithful probability $\widehat{\omega}$ on the MacNeille completion $\widehat{E}$ of E. Hence the restriction $\widehat{\omega}|_E$ is a faithful probability on E. $\square$

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