Pedro Miranda; Michel Grabisch
p-symmetric bi-capacities

Kybernetika, Vol. 40 (2004), No. 4, [421]--440

Persistent URL: http://dml.cz/dmlcz/135605

KYBERNETIKA — VOLUME 40 (2004), NUMBER 4, PAGES 421–440

p-SYMMETRIC BI-CAPACITIES[1]

PEDRO MIRANDA AND MICHEL GRABISCH

Bi-capacities have been recently introduced as a natural generalization of capacities (or fuzzy measures) when the underlying scale is bipolar. They allow to build more flexible models in decision making, although their complexity is of order 3^n, instead of 2^n for fuzzy measures. In order to reduce the complexity, the paper proposes the notion of p-symmetric bi-capacities, in the same spirit as for p-symmetric fuzzy measures. The main idea is to partition the set of criteria (or states of nature, individuals, ...) into subsets whose elements are all indifferent for the decision maker.

Keywords: bi-capacity, bipolar scales, p-symmetry

AMS Subject Classification: 28E05, 03H05, 28C05

1. INTRODUCTION

Recently, Grabisch and Labreuche [9, 10] have defined the concept of bi-capacity. Bi-capacities have their origin in situations in decision making where a bipolar scale is used.

In many problems dealing with bipolar scales, models based on classical fuzzy measures (as for example those based on Choquet integral, Šipoš integral or even on CPT (Cumulative Prospect Theory)) may fail to represent the preferences of the decision maker [10]. However, some of these situations could be solved through bi-capacities [10]. In this sense, Choquet integral with respect to a bi-capacity generalizes these three models.

On the other hand, fuzzy measures need $2^n - 2$ coefficients to be completely determined; in order to reduce this complexity, we have defined the concept of p-symmetric fuzzy measures. These measures reflect the fact that some criteria may be indistinguishable for the decision maker and they constitute a middle term between symmetric fuzzy measures and general fuzzy measures. In a previous paper [15], we have shown some of their properties, paying special attention to the representation of p-symmetric fuzzy measures and the decomposition of Choquet integral.

[1]This is a revised and extended version of the paper [17], presented at AGOP conference. The research has been supported in part by FEDER-MCYT grant number BFM2001-3515.

The aim of the paper is to define p-symmetric bi-capacities in a similar way as p-symmetric fuzzy measures have been defined for fuzzy measures. The definition is based on the extension of the concepts of indifferent elements and subsets of indifference, thus keeping as far as possible the construction of p-symmetric fuzzy measures.

The rest of the paper is organized as follows: We introduce the basic concepts on fuzzy measures that are needed throughout the paper in the next section. In Sections 3 and 4 we give a brief survey about p-symmetric fuzzy measures and bi-capacities, respectively. In Section 5 we define p-symmetric bi-capacities and study some of their properties, in particular looking for a simple representation. In Section 6 we deal with Choquet integral with respect to a p-symmetric bi-capacity. Finally, in Section 7 we briefly study the definition of p-symmetry for k-ary capacities. We finish with some conclusions.

2. BASIC BACKGROUND ON FUZZY MEASURES

We consider a finite set of n elements (states of nature, criteria, etc.) $X = \{x_1, \dots, x_n\}$, and denote $\mathcal{P}(X)$ the set of subsets of X. Subsets of X will be denoted by $A, B, \dots$ and also by $A_1, A_2, \dots$.

Definition 1. (See [2, 3, 19].) A *(discrete) fuzzy measure* or *capacity* over $(X, \mathcal{P}(X))$ is a mapping $\mu : \mathcal{P}(X) \to [0, 1]$ such that

— $\mu(\emptyset) = 0$, $\mu(X) = 1$ (boundary conditions).

— $\forall A, B \in \mathcal{P}(X)$ such that $A \subseteq B$, then $\mu(A) \leq \mu(B)$ (monotonicity).

From a mathematical point of view, fuzzy measures are a generalization of probabilities, exchanging additivity by monotonicity.

Fuzzy measures have proved to be a very valuable tool in multicriteria decision making (see e. g. [6]).

Symmetric fuzzy measures are a special case of fuzzy measures that are the basis for p-symmetric fuzzy measures.

Definition 2. A fuzzy measure μ is said to be *symmetric* if it satisfies for any $A, B \in \mathcal{P}(X)$,

$$|A| = |B| \Rightarrow \mu(A) = \mu(B).$$

An equivalent representation of fuzzy measures is given by the Möbius inverse. This transformation is an invertible linear transform of set functions, and is a fundamental notion in fuzzy measure theory [1].

Definition 3. (See [18].) Let μ be a set function (not necessarily a fuzzy measure) on X. The *Möbius transform (or inverse)* of μ is another set function m_μ on X defined by

$$m_\mu(A) := \sum_{B\subseteq A}(-1)^{|A\setminus B|}\mu(B), \quad \forall\, A \subseteq X. \tag{1}$$

Identifying $\mathcal{P}(X)$ with $\{0,1\}^{|X|}$, a subset $A \subseteq X$ can be denoted by a vector $(y_1, \dots, y_n)$ with $y_i = 1$ whenever $x_i \in A$ and $y_i = 0$ otherwise. Then, $\mu(A)$ can be rewritten as $\mu(y_1, \dots, y_n)$ and μ becomes a pseudo-boolean function [12]. We will call this notation of μ as the *vectorial notation.*

Consider now a mapping $f : X \to \mathbb{R}^+$; this mapping may represent the scores of an object on each criterium. In order to aggregate the different values, the so-called fuzzy integrals are used. One of the most important is the Choquet integral:

Definition 4. (See [2].) The *Choquet integral* of a function

$$f : X \to \mathbb{R}^+$$

with respect to a fuzzy measure μ on X is defined by

$$\mathcal{C}_\mu(f) := \sum_{i=1}^{n}(f(x_{(i)}) - f(x_{(i-1)}))\mu(B_i), \tag{2}$$

where $\{x_{(1)}, \dots, x_{(n)}\}$ is a permutation of the set $\{x_1, \dots, x_n\}$ satisfying

$$0 =: f(x_{(0)}) \leq f(x_{(1)}) \leq \dots \leq f(x_{(n)}),$$

and $B_i := \{x_{(i)}, \dots, x_{(n)}\}$.

The Choquet integral is a generalization of the concept of expected value.

It must be noted that the Choquet integral is properly defined for any set function vanishing on the empty set: monotonicity is not mandatory in (2).

Remark also that, as we need to rank the values of f and look for the different B_i, the Choquet integral has a high computational cost, except for some particular situations, e. g. probabilities or symmetric fuzzy measures.

Assume now that f is a real-valued function, not restricted to $\mathbb{R}^+$. Then, the Choquet integral can be defined through two different ways.

Definition 5. (See [23].) Let μ be a fuzzy measure over X and $f : X \to \mathbb{R}$ a real mapping. The *symmetric Choquet integral* or *Šipoš integral* is defined by

$$\check{\mathcal{C}}_\mu(f) := \mathcal{C}_\mu(f^+) - \mathcal{C}_\mu(f^-), \tag{3}$$

with $f^+ := f \vee 0$, $f^- := (-f)^+$.

Definition 6. Let μ be a fuzzy measure over X and $f : X \to \mathbb{R}$ a real mapping. The *asymmetric Choquet integral* is defined by

$$\mathcal{C}_\mu(f) := \mathcal{C}_\mu(f^+) - \mathcal{C}_{\bar{\mu}}(f^-), \tag{4}$$

with $\bar{\mu}(A) := 1 - \mu(A^c)$.

The Cumulative Prospect Theory (CPT) model is a generalization of these two definitions:

Definition 7. (See [22].) Let μ_1, μ_2 be two fuzzy measures over X and $f : X \to \mathbb{R}$ a real mapping. The *Cumulative Prospect Theory model* with respect to μ_1, μ_2 is defined by

$$CPT(f) := \mathcal{C}_{\mu_1}(f^+) - \mathcal{C}_{\mu_2}(f^-). \tag{5}$$

3. p-SYMMETRIC FUZZY MEASURES

The main drawback of fuzzy measures is their complexity. We need $2^n - 2$ coefficients to properly define a fuzzy measure. Then, some subfamilies of fuzzy measures have been defined in an attempt to reduce complexity, e. g. k-additive measures [7], λ-measures [21], or more generally decomposable measures [4, 24]. Specially appealing are k-additive measures, as they fill the gap between probability measures and general fuzzy measures (see [8, 14] for details).

In the same spirit of k-additive measures, p-symmetric measures appear as a middle term between symmetric fuzzy measures and general fuzzy measures. They reduce the complexity of fuzzy measures and provide a generalization of the idea of symmetry.

Let us now turn to the complexity of the Choquet integral. It can be proved [5, 6] that the Choquet integral w.r.t. a symmetric fuzzy measure is an OWA operator [25]; then, the Choquet integral w.r.t. a symmetric fuzzy measure has a low complexity. Therefore, we define p-symmetric fuzzy measures in a way such that the corresponding Choquet integral has a reduced complexity.

The definition of p-symmetric fuzzy measure is based on the concept of indifferent elements and subsets of indifference.

Definition 8. (See [15].) Given two elements x_i, x_j of the universal set X and a fuzzy measure μ over X, we say that x_i and x_j are *indifferent elements* for μ if and only if

$$\forall A \subseteq X \backslash \{x_i, x_j\}, \ \mu(A \cup x_i) = \mu(A \cup x_j). \tag{6}$$

If we consider the vectorial notation, two elements $x_i, x_j \in X$ are *indifferent* if

$$\mu(y_1, \dots, 1, \dots, 0, \dots, y_n) = \mu(y_1, \dots, 0, \dots, 1, \dots, y_n),$$

$$\forall y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_{j-1}, y_{j+1}, \dots, y_n. \tag{7}$$

Indeed, in Eq. (7) we are permuting values 0 and 1 (the possible values for any coordinate) in positions i and j.

In decision making, the definition of indifferent elements reflects the fact that criteria x_i and x_j are equivalent, so that we do not care about which one is fulfilled.

This can be extended to more than two elements through subsets of indifference.

Definition 9. (See [15].) Given a subset A of X, we say that A is a *subset of indifference* for a fuzzy measure μ over X if and only if $\forall B_1, B_2 \subseteq A$, $|B_1| = |B_2|$ and $\forall C \subseteq X \backslash A$, we have

$$\mu(B_1 \cup C) = \mu(B_2 \cup C). \tag{8}$$

From this definition, any two elements of the same subset of indifference are indifferent elements in the sense of Definition 8.

Remark that, for a given subset $A \subseteq X$, any arbitrary vector $(y_1, \dots, y_n)$ representing some subset $B \subseteq X$, can be written as a pair of vectors (y_A, y_{-A}) where y_A denotes the coordinates y_i such that $x_i \in A$ and y_{-A} the coordinates of all $x_i \in A^c$. Therefore, we can also identify any permutation π on A with the corresponding permutation on y_A; then, with some abuse of notation, the effect of permutation π on y_A will be denoted by $y_{\pi(A)}$, and if $\pi(x_i) = x_j$, then $y_{\pi(i)}$ will denote the value y_j. Then, if we use the vectorial notation, Definition 9 translates into

Definition 10. Let μ be a fuzzy measure over X and consider $A \subseteq X$. We say that A is a *subset of indifference* if $\forall y_A$

$$\mu(y_A, y_{-A}) = \mu(y_{\pi(A)}, y_{-A}), \quad \forall y_{-A}, \tag{9}$$

for any permutation π on A.

Definition 11. Let Π_1, Π_2 be two partitions of X. We say that Π_2 is *coarser* than Π_1, denoted $\Pi_1 \sqsubseteq \Pi_2$ if

$$\forall A_1 \in \Pi_1, \ \exists A_2 \in \Pi_2, A_1 \subseteq A_2.$$

Remark that given two partitions of X, $\Pi_1 = \{A_1, \dots, A_p\}$ and $\Pi_2 = \{B_1, \dots, B_q\}$ such that $\Pi_1 \neq \Pi_2$, it is always possible to find i, j, k satisfying

$$A_i \cap B_k \neq \emptyset, A_j \cap B_k \neq \emptyset \text{ or } B_i \cap A_k \neq \emptyset, B_j \cap A_k \neq \emptyset. \tag{10}$$

Now, we define p-symmetric fuzzy measures as follows:

Definition 12. (See [15].) Given a fuzzy measure μ over X, we say that μ is a *p-symmetric fuzzy measure* if and only if the coarsest partition of the universal set in subsets of indifference is formed of p non-empty subsets $A_1, \ldots, A_p$. The partition $\{A_1, \ldots, A_p\}$ is called the *indifference partition* or *basis* of the p-symmetric fuzzy measure.

We will prove below (Corollary 1) that we have a unique coarsest partition, so that the concepts of p-symmetry and indifference partition are well-defined.

A symmetric fuzzy measure is just a 1-symmetric fuzzy measure. Remark that we are dealing with the coarsest partition. Otherwise, a symmetric measure could be also considered as a k-symmetric fuzzy measure $\forall\, k \in \{1, \ldots, n\}$.

For a p-symmetric fuzzy measure w.r.t. the indifference partition $\{A_1, \ldots, A_p\}$, we can identify $B \subseteq X$ with a p-dimensional vector $(b_1, \ldots, b_p)$ whose coordinates are given by $b_i = |A_i \cap B|$, $\forall\, i = 1, \ldots, p$.

This property allows a reduction in the complexity of the measure:

Lemma 1. (See [16].) Let μ be a p-symmetric fuzzy measure on X w.r.t. the indifference partition $\{A_1, \ldots, A_p\}$. Then, it can be represented in a $(|A_1| + 1) \times \cdots \times (|A_p| + 1)$ matrix whose coefficients are defined by

$$M(i_1, \ldots, i_p) := \mu(i_1, \ldots, i_p), i_j \in \{0, \ldots, |A_j|\}.$$

This representation also allows a simple way to compute the coefficients $\mu(B_j)$ of the Choquet integral:

Theorem 1. (See [16].) Let μ be a p-symmetric fuzzy measure w.r.t. the indifference partition $\{A_1, \ldots, A_p\}$ and consider the matrix representation of Lemma 1. Then, the corresponding values of $\mu(B_i)$ in Definition 4 can be computed by finding a path between the "points" $(0, \ldots, 0)$ and $(|A_1|, \ldots, |A_p|)$ applying the following rule: Assume B_{j-1} corresponds to point $(i_1, \ldots, i_p)$ and $x_{(j)} \in A_k$, then $\mu(B_j)$ is in the point $(i_1, \ldots, i_k + 1, \ldots, i_p)$ (see Figure 1 for an example with a 2-symmetric fuzzy measure).

The number of possible paths is given in next lemma.

Lemma 2. (See [16].) Let μ be a p-symmetric fuzzy measure on X with respect to the indifferent partition $\{A_1, \ldots, A_p\}$. Then, the number of paths from $(0, \ldots, 0)$ to $(|A_1|, \ldots, |A_p|)$ is the multinomial number

$$\binom{n}{|A_1|, \ldots, |A_p|} := \frac{n!}{|A_1|! \cdots |A_p|!}. \tag{11}$$

The concept of p-symmetry permits also a decomposition of Choquet integral, as the next proposition shows. This relates to the IEC (inclusion-exclusion coverings) [20] and belongs to the field of Choquet decomposition.

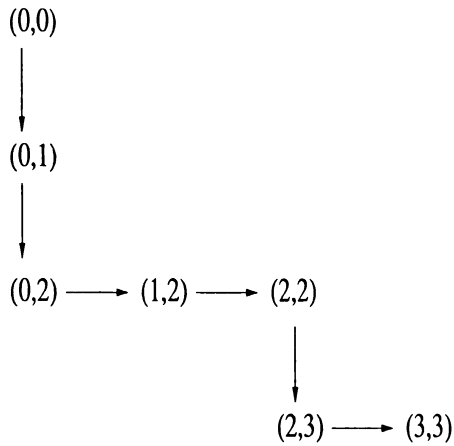

Fig. 1. Possible path from (0,0) to (3,3) when $|A_1| = 3$ and $|A_2| = 3$.

Proposition 1. (See [16].) Let μ be a p-symmetric fuzzy measure on X with respect to the partition $\{A_1, \ldots, A_p\}$, and suppose $\mu(A_i) > 0, \forall i \in \{1, \ldots, p\}$. Then, the Choquet integral is given by

$$\mathcal{C}_\mu(f) = \sum_{i=1}^{p} \mu(A_i)\mathcal{C}_{\mu_{A_i}}(f) + \sum_{B \not\subseteq A_j, \forall j} m(B) \bigwedge_{x_i \in B} f(x_i), \tag{12}$$

where μ_{A_i} is defined by

$$\mu_{A_i}(C) := \frac{\mu(A_i \cap C)}{\mu(A_i)}, \quad \forall C \subseteq X, \tag{13}$$

and m denotes the Möbius transform of μ.

4. BI-CAPACITIES

Despite the good properties of fuzzy measures, in some practical situations dealing with bipolar scales, Choquet and Šipoš integral, or more generally a CPT model, are not sufficiently appropriate to model and solve the practical problem [9]. Therefore, a generalization is needed. It is in this context where bi-capacities appear.

Definition 13. (See [9, 10].) Let us denote

$$\mathcal{Q}(X) := \{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X) \text{ s.t. } A \cap B = \emptyset\}.$$

A *bi-capacity* is a mapping $\nu : \mathcal{Q}(X) \to \mathbb{R}$ satisfying

— $\nu(\emptyset, \emptyset) = 0$, $\nu(X, \emptyset) = 1$, $\nu(\emptyset, X) = -1$.

— $\nu(\cdot,\cdot)$ is increasing w.r.t. set inclusion in the first coordinate and decreasing in the second.

From this definition, we have $\nu(\emptyset, C) \leq 0$, $\nu(C, \emptyset) \geq 0$, $\forall\, C \subseteq X$.

Remark that $\mathcal{Q}(X)$ can be identified with $\{-1,0,1\}^{|X|}$ and $(A,B) \in \mathcal{Q}$ can be denoted by $(y_1,\dots,y_n)$ with $y_i = 1$ if $x_i \in A, y_i = -1$ if $x_i \in B$, and $y_i = 0$ otherwise. Consequently, $\nu(A,B)$ can be rewritten as $\nu(y_1,\dots,y_n)$ in vectorial notation, and thus, the number of different coefficients that we need to completely determine a bi-capacity is 3^n. Three of these coefficients are fixed by the conditions of Definition 13.

Choquet integral w.r.t. a bi-capacity is given in next definition.

Definition 14. (See [10].) Given a mapping $f : X \to \mathbb{R}$ and a bi-capacity ν, the *Choquet integral* of f w.r.t. ν is defined by

$$F_\nu(f) := \mathcal{C}_{\mu_{X^+}}(|f|), \tag{14}$$

where $|f|$ is the absolute value of f, $X^+ = \{x \in X \mid f(x) \geq 0\}$, $X^- = X \backslash X^+$, and

$$\mu_{X^+}(A) := \nu(A \cap X^+, A \cap X^-), \quad \forall\, A \subseteq X. \tag{15}$$

This definition generalizes the definitions of Choquet integral and Šipoš integral w.r.t a fuzzy measure, and also the CPT model, as desired.

It must be remarked that μ_{X^+} is a set function, but it is not a fuzzy measure in general.

5. p-SYMMETRIC BI-CAPACITIES

Let us now turn to the definition of p-symmetric bi-capacities. This definition should maintain the essence of p-symmetric measures; therefore, it should represent the fact that some elements may be indistinguishable for the decision maker.

The first step is to translate the concepts of indifferent elements and subset of indifference for bi-capacities.

Definition 15. Let ν be a bi-capacity and consider $x_i, x_j \in X$. We say that x_i and x_j are *indifferent elements* if for any $(A,B) \in \mathcal{Q}(X)$, $A, B \subseteq X\backslash\{x_i,x_j\}$ the following holds:

$$\nu(x_i \cup A, B) = \nu(x_j \cup A, B), \tag{16}$$

$$\nu(A, x_i \cup B) = \nu(A, x_j \cup B), \tag{17}$$

$$\nu(A \cup x_j, x_i \cup B) = \nu(A \cup x_i, x_j \cup B). \tag{18}$$

With this definition, if the decision maker considers x_i and x_j indifferent, he does not care about which one of them is in the coalition (in both arguments of the bi-capacity). Moreover, if both elements appear, one in the first argument and the other in the second one, he does not care about which one of them is in each argument. This means that the decision maker is indifferent about these criteria, in the sense that any coalition has exactly the same importance with any of them. At this point, (16) and (17) are natural extensions of (6). Let us turn to Eq. (18); it must be remarked that this condition cannot be derived from the two first, as next example shows:

Example 1. Let us consider $X = \{x_1, x_2, x_3\}$ and the bi-capacity ν defined by

$A\backslash B$	$\emptyset$	x_1	x_2	x_1, x_2	x_3	x_1, x_3	x_2, x_3	X
$\emptyset$	0	−0.1	−0.3	−0.5	−0.3	−0.5	−0.6	−1
x_1	0.1		0		0		−0.2	
x_2	0.3	0			0	−0.1		
x_1, x_2	0.5				0.2			
x_3	0.3	0	−0.4					
x_1, x_3	0.5		0.1					
x_2, x_3	0.6	0.4						
X	1							

Then, this bi-capacity satisfies the two first conditions for x_2, x_3. However, it fails for the third condition, as $\nu(\{x_1, x_2\}, \{x_3\}) = 0.2, \nu(\{x_1, x_3\}, \{x_2\}) = 0.1$.

Of course, the bi-capacity in this example does not fulfill our intuition about indifferent elements; therefore, in order to translate the concept of indifferent elements, the three conditions are needed.

If we consider the vectorial notation, Definition 15 can be written in the following form:

Definition 16. Let ν be a bi-capacity and consider $x_i, x_j \in X$. We say that x_i and x_j are *indifferent elements* if $\forall\, y_i, y_j$

$$\nu(y_1, \dots, y_i, \dots, y_j, \dots, y_n) = \nu(y_1, \dots, y_{\pi(i)}, \dots, y_{\pi(j)}, \dots, y_n),$$

$$\forall\, y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_{j-1}, y_{j+1}, \dots, y_n, \tag{19}$$

with $\pi : \{i, j\} \to \{i, j\}$ a permutation.

It is easy to see that (19) recovers (16), (17) and (18) of Definition 15 for the two possible permutations π and different values of y_i, y_j, so that both definitions are equivalent.

Let us now turn to the extension of subset of indifference for bi-capacities. This concept must reflect the fact that all elements in it are indifferent. Therefore, this definition should extend Definition 16 in the same way Definition 9 extends Definition 8. Considering the vectorial notation, we can extend the concept of subset of indifference (Definitions 9 and 10) as follows:

Definition 17. Let ν be a bi-capacity and consider $A \subseteq X$. We say that A is a *subset of indifference* if $\forall\, y_A$

$$\nu(y_A, y_{-A}) = \nu(y_{\pi(A)}, y_{-A}), \quad \forall\, y_{-A}, \tag{20}$$

for any permutation π on A.

For a subset of indifference A, the following can be proved:

Lemma 3. A subset $A \subseteq X$ is a subset of indifference if and only if for any $B_1, B_2, B_3, B_4 \subseteq A$ such that $|B_1| = |B_3|, |B_2| = |B_4|, B_1 \cap B_2 = \emptyset, B_3 \cap B_4 = \emptyset$ the following condition holds:

$$\nu(C \cup B_1, D \cup B_2) = \nu(C \cup B_3, D \cup B_4), \quad \forall\, (C, D) \in \mathcal{Q},\ C,\ D \subseteq X \setminus A. \tag{21}$$

Proof. $\Rightarrow$) It suffices to show that there exists a permutation π on A leading to (21). But for this it suffices to consider a permutation π mapping B_1 on B_3 and B_2 on B_4. As $|B_1| = |B_3|$, $|B_2| = |B_4|$, $B_1 \cap B_2 = \emptyset, B_3 \cap B_4 = \emptyset$, it is clear that we can find such a permutation, whence the result.

$\Leftarrow$) Consider y_A. Then, we can define

$$B_1 = \{x_i \in A \,|\, y_i = 1\}, \quad B_2 = \{x_i \in A \,|\, y_i = -1\}.$$

Given a permutation π on A, we define

$$B_3 = \{x_i \in A \,|\, y_{\pi(i)} = 1\}, \quad B_4 = \{x_i \in A \,|\, y_{\pi(i)} = -1\}.$$

We have $B_1 \cap B_2 = \emptyset$, $B_3 \cap B_4 = \emptyset$; on the other hand $|B_1| = |B_3|$, $|B_2| = |B_4|$, as π is a permutation. Then, by (21)

$$\nu(C \cup B_1, D \cup B_2) = \nu(C \cup B_3, D \cup B_4), \quad \forall\, (C, D) \in \mathcal{Q},\ C, D \subseteq X \setminus A,$$

or, in other words, $\forall\, y_A$

$$\nu(y_A, y_{-A}) = \nu(y_{\pi(A)}, y_{-A}), \quad \forall\, y_{-A}.$$

This completes the proof. □

This means that all elements in A are indistinguishable for the decision maker. Indeed, we do not care about which subset of the subset of indifference is in the coalition; we only care about the number of elements of the subset of indifference that are contained.

We can then define p-symmetric bi-capacities from subsets of indifference, as it has been done for the case of fuzzy measures.

Definition 18. Given a bi-capacity ν, we say that ν is a *p-symmetric bi-capacity* if and only if the coarsest partition of the universal set X in subsets of indifference is $\{A_1, \dots, A_p\}, A_i \neq \emptyset, \forall\, i \in \{1, \dots, p\}$.

Let us first show that Definition 18 makes sense, i. e. there is always a unique coarsest partition of subsets of indifference. For this, we need a preliminary lemma.

Definition 19. Given $S \subseteq \mathbb{R}$ and $\mu : S^n \to \mathbb{R}$, we say that a subset $A \subseteq \{1, \dots, n\}$ satisfies the *exchange property* ($\mathcal{EP}$) if $\forall\, y_A$

$$\mu(y_A, y_{-A}) = \mu(y_{\pi(A)}, y_{-A}), \ \forall\, \pi \text{ permutation on } A, \forall\, y_{-A}.$$

Lemma 4. Let $S \subseteq \mathbb{R}$ and $\mu : S^n \to \mathbb{R}$ be a mapping. We consider the family

$$\mathcal{P} := \{\{A_1, \dots, A_p\} \text{ partition of } \{1, \dots, n\} \mid A_i \text{ satisfies } \mathcal{EP}, \forall\, i\}.$$

Then, there exists a top element in $\mathcal{P}$.

Proof. Assume that there exist two different partitions, denoted $\{A_1, \dots, A_p\}$ and $\{B_1, \dots, B_q\}$, in $\mathcal{P}$ which are maximal.

As they are different, we know from (10) that it is always possible to find i, j, k satisfying

$$A_i \cap B_k \neq \emptyset, A_j \cap B_k \neq \emptyset \quad \text{or} \quad B_i \cap A_k \neq \emptyset, B_j \cap A_k \neq \emptyset.$$

Assume that there exist i, j, k such that $A_i \cap B_k \neq \emptyset$, $A_j \cap B_k \neq \emptyset$, and let us show that in this case $A_i \cup B_k$ satisfies $\mathcal{EP}$.

Let us consider π a permutation on $A_i \cup B_k$ and show that $\forall\, y_{A_i \cup B_k}$

$$\mu(y_{A_i \cup B_k}, y_{-(A_i \cup B_k)}) = \mu(y_{\pi(A_i \cup B_k)}, y_{-(A_i \cup B_k)}), \quad \forall\, y_{-(A_i \cup B_k)}$$

holds. Any permutation can be decomposed in a composition of transpositions, i. e. permutations interchanging only two elements [13]. Consequently, it suffices to show the result for any transposition σ_{rs} interchanging r and s. We have three different cases:

- If $r, s \in A_i$, then σ_{rs} is indeed a transposition on A_i and consequently, since A_i satisfies the exchange property, the result holds.
- Similarly, if $r, s \in B_k$, the result holds.
- The final case arises when $r \in A_i \setminus B_k, s \in B_k \setminus A_i$. As $A_i \cap B_k \neq \emptyset$, consider $t \in A_i \cap B_k$. Then, $\sigma_{rs} = \sigma_{tr}\sigma_{ts}\sigma_{rt}$, and we have

$$\mu(y_{A_i \cup B_k}, y_{-(A_i \cup B_k)}) = \mu(y_{\sigma_{rt}(A_i \cup B_k)}, y_{-(A_i \cup B_k)})$$
$$= \mu(y_{\sigma_{ts}\sigma_{rt}(A_i \cup B_k)}, y_{-(A_i \cup B_k)}) = \mu(y_{\sigma_{tr}\sigma_{ts}\sigma_{rt}(A_i \cup B_k)}, y_{-(A_i \cup B_k)}),$$

  whence the result.

Thus, $A_i \cup B_k$ satisfies $\mathcal{EP}$, but then so does $A_i \cup B_k \cup A_j$, whence $A_i \cup A_j$ satisfies $\mathcal{EP}$; this is a contradiction with the fact that $\{A_1, \dots, A_p\}$ is maximal. This concludes the proof. □

Taking successively $S = \{0, 1\}$ and $S = \{-1, 0, 1\}$ we get:

Corollary 1. Given a fuzzy measure μ, there exists a unique coarsest partition of X in subsets of indifference.

Corollary 2. Given a bi-capacity ν, there exists a unique coarsest partition of X in subsets of indifference.

Remark 1. In the following, we will call the coarsest partition of X in subsets of indifference the *indifference partition* or *basis* of the bi-capacity ν.

In the case of a p-symmetric bi-capacity w.r.t the partition $\{A_1, \dots, A_p\}$, in order to define $\nu(B, C)$ it suffices to consider two p-dimensional vectors $(b_1, \dots, b_p)$ and $(c_1, \dots, c_p)$ with $b_i = |A_i \cap B|$, $c_i = |A_i \cap C|$, $\forall i = 1, \dots, p$ by Lemma 3. In the following, when dealing with a p-symmetric bi-capacity, we will identify $(B, C) \in \mathcal{Q}(X)$ with $((b_1, \dots, b_p), (c_1, \dots, c_p)) \in \mathbb{N}^{2p}$.

Let us now turn to the representation of p-symmetric bi-capacities. Let us first consider a 1-symmetric bi-capacity on X, with $|X| = n$; then, in order to know $\nu(A, B)$, it suffices to consider $|A|, |B|$. For fixed $|A|$, the possible values for $|B|$ are $0, \dots, n - |A|$. This allows us to represent this bi-capacity as in Table 1.

Table 1. Representation of symmetric bi-capacities.

	0	1	$\dots$	$n-1$	n
0	$\nu(0,0)$	$\nu(1,0)$	$\dots$	$\nu(n-1,0)$	$\nu(n,0)$
1	$\nu(0,1)$	$\nu(1,1)$	$\dots$	$\nu(n-1,1)$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$		
n	$\nu(0,n)$				

However, in order to keep the structure of p-symmetric fuzzy measures, it should be desirable to obtain a matrix representation of p dimensions. Hence, Table 1 should be rewritten as a vector containing

$$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

elements. In general, the following can be proved:

Proposition 2. A p-symmetric bi-capacity ν with respect to the indifference partition $\{A_1, \dots, A_p\}$ can be represented in a p-dimensional matrix, denoted by M, of dimensions

$$\frac{(|A_1|+1)(|A_1|+2)}{2} \times \cdots \times \frac{(|A_p|+1)(|A_p|+2)}{2}.$$

Consequently, the number of coefficients needed to define the bi-capacity is

$$\prod_{i=1}^{p} \frac{(|A_i|+1)(|A_i|+2)}{2}.$$

(Three of these coefficients are known by the conditions of Definition 13.)

Proof. In order to define $\nu(B,C)$ it suffices to consider $((b_1,\dots,b_p),(c_1,\dots,c_p))$, with $b_i = |A_i \cap B|, c_i = |A_i \cap C|$, as explained before. Let us start with $(b_1,\dots,b_p)$. Remark that b_i varies between 0 and $|A_i|$. Therefore, the different possibilities for $(b_1,\dots,b_p)$ can be written in a $(|A_1|+1)\times\cdots\times(|A_p|+1)$ matrix. Let us denote this matrix by N.

Let us fix b_i; then, c_i varies between 0 and $|A_i| - b_i$. Consequently, for fixed $(b_1,\dots,b_p)$, the possible values that $(c_1,\dots,c_p)$ can attain determine another matrix $M_{(b_1,\dots,b_p)}$ of dimensions $(|A_1|-b_1+1)\times\cdots\times(|A_p|-b_p+1)$. Next matrix is an example for $M_{(b_1,b_2)}$ for fixed b_1, b_2.

$$\begin{pmatrix} \nu((b_1,b_2),(0,0)) & \dots & \nu((b_1,b_2),(0,|A^c|-b_2)) \\ \vdots & \vdots & \vdots \\ \nu((b_1,b_2),(|A|-b_1,0)) & \dots & \nu((b_1,b_2),(|A|-b_1,|A^c|-b_2)) \end{pmatrix}.$$

Therefore, in position $(b_1,\dots,b_p)$ of matrix N, we have not a number but a p-dimensional matrix of dimensions $(|A_1|-b_1+1)\times\cdots\times(|A_p|-b_p+1)$. Now, it must be noted that for fixed b_i the number of rows of matrix $M_{(b_1,\dots,b_i,\dots,b_p)}$ on the ith dimension is always $|A_i|-b_i+1$ for any other values of $b_1,\dots,b_{i-1},b_{i+1},\dots,b_p$. This implies that a p-symmetric bi-capacity can be represented in a "big" p-dimensional matrix. The number of rows of such a matrix for the ith coordinate is given by

$$\sum_{b_i=0}^{|A_i|}(|A_i|-b_i+1) = \frac{(|A_i|+1)(|A_i|+2)}{2}.$$

This concludes the proof. □

This representation will allow us to compute Choquet integral in a simple way, as we will see in the following section.

6. CHOQUET INTEGRAL W.R.T. A p-SYMMETRIC BI-CAPACITY

Let us now turn to the problem of obtaining the Choquet integral w.r.t. a p-symmetric bi-capacity with basis $\{A_1,\dots,A_p\}$.

First, let us show for the p-symmetric case that we can use the matrix representation to compute Choquet integral in a similar way as in Theorem 1. For a mapping $f: X \to \mathbb{R}$, the Choquet integral is defined in terms of a set function μ_{X^+}, depending on $X^+ = \{x \in X | f(x) \geq 0\}$ (Eq. (14)), and defined from ν (Eq. (15)). Once μ_{X^+} is defined, it is necessary to find the coefficients of the corresponding

$\mu_{X^+}(B_i)$, $i = 1, \ldots, n$. It follows from Definition 4 that $B_i = \{x_{(i)}, \ldots, x_{(n)}\}$, where $\{x_{(1)}, \ldots, x_{(n)}\}$ is a permutation of the set $\{x_1, \ldots, x_n\}$ satisfying

$$0 =: |f|(x_{(0)}) \leq |f|(x_{(1)}) \leq \cdots \leq |f|(x_{(n)}).$$

Let us consider B_i. Then, $B_i = B_{i+1} \cup \{x_{(i)}\}$. Let us define

$$d_j := |A_j \cap B_{i+1} \cap X^+|, e_j := |A_j \cap B_{i+1} \cap X^-|, \quad j = 1, \ldots, p.$$

Remark that in this case,

$$\mu_{X^+}(B_{i+1}) = \nu(B_{i+1} \cap X^+, B_{i+1} \cap X^-) = \nu((d_1, \ldots, d_p), (e_1, \ldots, e_p)),$$

as ν is a p-symmetric bi-capacity. On the other hand, by Proposition 2, we know that ν can be represented in a p-dimensional matrix M and consequently, for suitable values of $i_1, \ldots, i_p$, it is $\nu((d_1, \ldots, d_p), (e_1, \ldots, e_p)) = M[i_1, \ldots, i_p]$. We conclude that

$$\mu_{X^+}(B_{i+1}) = M[i_1, \ldots, i_p].$$

Let us now turn to $\mu_{X^+}(B_i)$. Then, we have two different situations:

— Suppose $f(x_{(i)}) < 0$ and $x_{(i)} \in A_j$. Then, $x_{(i)} \in A_j \cap X^-$. Then,

$$\mu_{X^+}(B_i) = \nu((d_1, \ldots, d_p), (e_1, \ldots, e_j + 1, \ldots, e_p)).$$

As $(d_1, \ldots, d_p)$ remains the same, we have to consider again the p-dimensional matrix $M_{(d_1, \ldots, d_p)}$ and look for the position of vector $(e_1, \ldots, e_j + 1, \ldots, e_p)$; this position can be obtained from $(i_1, \ldots, i_p)$ just adding one to the jth coordinate. Therefore,

$$\mu_{X^+}(B_{(i)}) = M[i_1, \ldots, i_j + 1, \ldots, i_p].$$

— On the other hand, if $f(x_{(i)}) > 0$ and $x_{(i)} \in A_j$, then

$$\mu_{X^+}(B_{(i)}) = \nu((d_1, \ldots, d_j + 1, \ldots, d_p), (e_1, \ldots, e_p)).$$

In this case, we have to look for vector $(e_1, \ldots, e_p)$ in $M_{(d_1, \ldots, d_j+1, \ldots, d_p)}$. This can be obtained from $(i_1, \ldots, i_p)$ just taking into account that $M_{(d_1, \ldots, d_p)}$ has $|A_j| - d_j + 1$ rows on the jth direction. Therefore, i_j should be increased by $|A_j| - d_j + 1 - e_j + e_j = |A_j| - d_j + 1$.

Then, we have shown that it is possible to obtain the coefficients for computing the Choquet integral from the matrix representation of Proposition 2, starting from position $(0, \ldots, 0)$. We conclude that for a p-symmetric bi-capacity ν with respect to the indifference partition $\{A_1, \ldots, A_p\}$, the coefficients for computing the Choquet integral can be obtained by finding a path from $\nu((0, \ldots, 0), (0, \ldots, 0))$ to $\nu((X^+ \cap A_1, \ldots, X^+ \cap A_p), (X^- \cap A_1, \ldots, X^- \cap A_p))$.

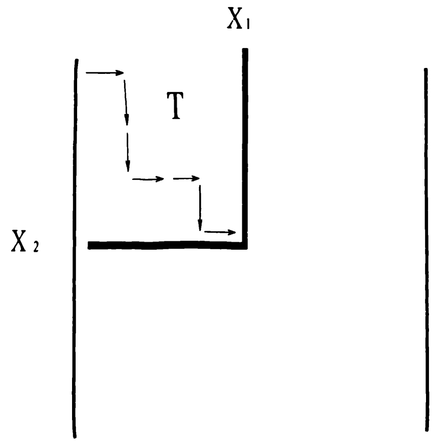

Fig. 2. $T \equiv$ Part of matrix representation used for fixed X^+ $(X_1 = X^+ \cap A_1, X_2 = X^+ \cap A_2)$.

Remark 2. It is important to remark that the end point of the paths varies with X^+. Indeed X^+ determines the part of M that is used for computing the Choquet integral, as it defines the values of $(X^+ \cap A_1, \dots, X^+ \cap A_p), (X^- \cap A_1, \dots, X^- \cap A_p)$. (see Figure 2 for an example for the 2-symmetric case).

It must be also noted that for a given $f : X \to \mathbb{R}$, even if ν is a p-symmetric bi-capacity, μ_{X^+} is not a p-symmetric set function in general. This is because, for $B \subseteq X$, the value $\mu_{X^+}(B)$ depends not only on $|B \cap A_i|$, $i = 1, \dots, p$ but also on X^+ (and X^-). However, the following can be proved:

Lemma 5. Let ν be a p-symmetric bi-capacity with respect to the indifference partition $\{A_1, \dots, A_p\}$. Then, for any $X^+ \subseteq X$, $X^- = X \backslash X^+$, the set function μ_{X^+} defined by

$$\mu_{X^+}(C) := \nu(C \cap X^+, C \cap X^-), \quad \forall C \subseteq X,$$

is a $2p$-symmetric set function w.r.t. the indifference partition $\{B_1, \dots, B_p, C_1, \dots, C_p\}$ (at most) with $B_i := A_i \cap X^+$, $C_i := A_i \cap X^-$, $i = 1, \dots, p$.

Proof. It is straightforward to check that $\{B_1, \dots, B_p, C_1, \dots, C_p\}$ is indeed a partition of X.

Let $D, E \subseteq X$ such that for any $i = 1, \dots, p$, it is

$$d_i := |B_i \cap D| = |B_i \cap E| =: e_i, \quad d'_i := |C_i \cap D| = |C_i \cap E| =: e'_i.$$

Then, we have

$$\begin{aligned} \mu_{X^+}(D) &= \nu(D \cap X^+, D \cap X^-) = \nu((d_1, \dots, d_p), (d'_1, \dots, d'_p)) \\ &= \nu((e_1, \dots, e_p), (e'_1, \dots, e'_p)) = \mu_{X^+}(E), \end{aligned}$$

as ν is a p-symmetric bi-capacity. □

Corollary 3. Let ν be a p-symmetric bi-capacity with respect to the indifference partition $\{A_1, \ldots, A_p\}$ and consider $X^+ \subseteq X$. Define $X^- = X \backslash X^+$. Then, the number of possible paths from $(0, \ldots, 0), (0, \ldots, 0))$ to $((|A_1 \cap X^+|, \ldots, |A_p \cap X^+|), (|A_1 \cap X^-|, \ldots, |A_p \cap X^-|))$ is the multinomial number

$$\binom{n}{|A_1 \cap X^+|, \ldots, |A_p \cap X^+|, |A_1 \cap X^-|, \ldots, |A_p \cap X^-|}.$$

Proof. The proof is straightforward from Lemmas 5 and 2. □

Now, let us turn to the Choquet decomposition. As a consequence of Lemma 5, we can decompose Choquet integral as it is done in Proposition 1.

Proposition 3. Let ν be a p-symmetric bi-capacity with respect to the indifference partition $\{A_1, \ldots, A_p\}$ and $f : X \to \mathbb{R}$. Define $B_i = A_i \cap X^+$, $C_i = A_i \cap X^-$, $i = 1, \ldots, p$. Then, the expression for Choquet integral is

$$F_\nu(f) = \sum_{\mu_{X^+}(B_i)>0} \mathcal{C}_{\mu'_{B_i}}(|f|) + \sum_{\mu_{X^+}(C_i)<0} \mathcal{C}_{\mu'_{C_i}}(|f|)$$

$$+ \sum_{D \not\subseteq B_i, C_i} m_{X^+}(D) \bigwedge_{x_i \in D} |f|(x_i), \tag{22}$$

where $\mu'_{B_i}, \mu'_{C_i} : \mathcal{P}(X) \to [0, 1]$ are defined by

$$\mu'_{B_i}(D) := \nu(B_i \cap D, \emptyset), \quad \mu'_{C_i}(D) := \nu(\emptyset, C_i \cap D), \quad \forall\, D \subseteq X. \tag{23}$$

Moreover, if $\mu_{X^+}(B_i), \mu_{X^+}(C_i) \neq 0, \forall\, i$, then,

$$F_\nu(f) = \sum_{i=1}^{p} \mu_{X^+}(B_i) \mathcal{C}_{\mu_{B_i}}(|f|) + \sum_{i=1}^{p} \mu_{X^+}(C_i) \mathcal{C}_{\mu_{C_i}}(|f|)$$

$$+ \sum_{D \not\subseteq B_i, C_i} m_{X^+}(D) \bigwedge_{x_i \in D} |f|(x_i), \tag{24}$$

where $\mu_{B_i}, \mu_{C_i} : \mathcal{P}(X) \to [0, 1]$ are defined by

$$\mu_{B_i}(D) := \frac{\nu(B_i \cap D, \emptyset)}{\nu(B_i, \emptyset)}, \quad \mu_{C_i}(D) := \frac{\nu(\emptyset, C_i \cap D)}{\nu(\emptyset, C_i)}, \quad \forall\, D \subseteq X. \tag{25}$$

Proof. As μ_{X^+} is a $2p$-symmetric fuzzy measure w.r.t. $(B_1, \ldots, B_p, C_1, \ldots, C_p)$ (Lemma 5), we can apply Proposition 1, thus obtaining as Choquet integral

$$\sum_{i=1}^{p} [\mu_{X^+}(B_i) \mathcal{C}_{\mu_{B_i}}(|f|) + \mu_{X^+}(C_i) \mathcal{C}_{\mu_{C_i}}(|f|)] + \sum_{D \not\subseteq B_i, C_i} m_{X^+}(D) \bigwedge_{x_i \in D} |f|(x_i),$$

with μ_{B_i}, μ_{C_i} defined as in (25).

On the other hand, $\mu'_{B_i} \geq 0, \mu'_{C_i} \leq 0$, whence (24) holds. The first part of the proposition is straightforward; it just suffices to introduce $\mu_{X^+}(B_i)$ in the definitions of μ_{B_i} and μ_{C_i} respectively. The second part of the proposition is also straightforward; it just suffices to normalize the set functions μ'_{B_i}, μ'_{C_i}. □

Remark 3. When dealing with symmetric bi-capacities, we have just one subset of indifference (X itself). Then, we have just one B_i and one C_i, namely X^+ and X^-, respectively, and (22) turns into

$$F_\nu(f) = \nu(X^+, \emptyset)\mathcal{C}_{\mu'_{X^+}}(|f|) + \nu(\emptyset, X^-)\mathcal{C}_{\mu'_{X^-}}(|f|) + \sum_{D \not\subseteq X^+, X^-} m_{X^+}(D) \bigwedge_{x_i \in D} |f|(x_i),$$

with $\mu'_{X^+}, \mu'_{X^-} : \mathcal{P}(X) \to [0,1]$ defined by

$$\mu'_{X^+}(D) := \nu(X^+ \cap D, \emptyset), \quad \mu'_{X^-}(D) := \nu(\emptyset, D \cap X^-), \quad \forall D \subseteq X.$$

Even more, for μ_{B_i}, μ_{C_i} the following can be proved:

Lemma 6. The set functions $\mu_{B_i}, \mu_{C_i} : \mathcal{P}(X) \to [0,1]$ defined by

$$\mu_{B_i}(D) := \frac{\nu(B_i \cap D, \emptyset)}{\nu(B_i, \emptyset)}, \quad \mu_{C_i}(D) := \frac{\nu(\emptyset, C_i \cap D)}{\nu(\emptyset, C_i)}, \quad \forall D \subseteq X,$$

are indeed fuzzy measures.

Proof. Let us prove the result for μ_{B_i}.

— $\mu_{B_i}(\emptyset) = \frac{\nu(\emptyset \cap B_i, \emptyset)}{\nu(B_i, \emptyset)} = \frac{\nu(\emptyset, \emptyset)}{\nu(B_i, \emptyset)} = 0.$

— $\mu_{B_i}(X) = \frac{\nu(X \cap B_i, \emptyset)}{\nu(B_i, \emptyset)} = \frac{\nu(B_i, \emptyset)}{\nu(B_i, \emptyset)} = 1.$

— If $D \subseteq E$, then

$$\mu_{B_i}(D) = \frac{\nu(D \cap B_i, \emptyset)}{\nu(B_i, \emptyset)} \leq \frac{\nu(E \cap B_i, \emptyset)}{\nu(B_i, \emptyset)} = \mu_{B_i}(E). \tag{26}$$

For μ_{C_i}, the proof is similar. In this case, it must be noted that $\nu(\emptyset, D \cap C_i) \geq \nu(\emptyset, E \cap C_i)$ whenever $D \subseteq E$ but this inequality reverses because $\nu(\emptyset, C_i) < 0$ as $\nu(\emptyset, C_i) \neq 0$ and the conditions of Definition 13. □

7. p-SYMMETRIC k-ARY CAPACITIES

Capacities can be extended to more than two dimensions through the so-called k-ary capacities, defined by Grabisch and Labreuche in [11]. Roughly speaking, while a bi-capacity is a function ν on the lattice $3^{|X|}$ (criteria with positive evaluation and criteria with negative evaluation), a k-ary capacity is a function on the lattice $k^{|X|}$ representing the overall score of some k-ary alternative $(\alpha_1, \dots, \alpha_k)$ representing reference levels of interest.

The corresponding concept of p-symmetry in this situation lays again on the concept of indifferent elements and subsets of indifference.

If we consider the vector representation, it comes out that in this case the possible values are not $S = \{-1, 0, 1\}$ but $S = \{\alpha_1, \dots, \alpha_k\}$. However, the concepts of indifferent elements and subset of indifference remain the same.

Definition 20. Let ν be a k-ary capacity and consider $x_i, x_j \in X$. We say that x_i and x_j are *indifferent elements* if $\forall\, y_i, y_j$

$$\mu(y_1, \dots, y_i, \dots, y_j, \dots, y_n) = \mu(y_1, \dots, y_{\pi(i)}, \dots, y_{\pi(j)}, \dots, y_n),$$

$$\forall\, y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_{j-1}, y_{j+1}, \dots, y_n, \tag{27}$$

with $\pi : \{i, j\} \to \{i, j\}$ any permutation.

Definition 21. Let ν be a k-ary capacity and consider $A \subseteq X$. We say that A is a *subset of indifference* if $\forall\, y_A$

$$\nu(y_A, y_{-A}) = \nu(y_{\pi(A)}, y_{-A}), \quad \forall\, y_{-A}, \tag{28}$$

for any π permutation on A.

From these definitions, p-symmetry can be defined the same way as before:

Definition 22. Given a k-ary capacity ν, we say that ν is a *p-symmetric k-ary capacity* if and only if the coarsest partition of the universal set X in subsets of indifference is $\{A_1, \dots, A_p\}, A_i \neq \emptyset,\ \forall\, i \in \{1, \dots, p\}$.

As a consequence of Lemma 4 with $S = \{\alpha_1, \dots, \alpha_k\}$, we have

Corollary 4. Given a k-ary capacity ν, there exists a unique coarsest partition of X in subsets of indifference.

Then, the concept of p-symmetric k-ary capacity is well defined. Finally, it can be easily checked that all properties for p-symmetric fuzzy measures and p-symmetric bi-capacities also hold for general k-ary capacities.

8. CONCLUSIONS

In this paper we have presented the concept of p-symmetric bi-capacities. The definition follows the line started with p-symmetric fuzzy measures; therefore, p-symmetric bi-capacities try to reflect the fact that some criteria may be equivalent for the decision maker.

We have also introduced a new notation related to k-ary capacities from which it is possible to define the concepts of indifferent elements and subsets of indifference in terms of permutations. This notation allows a simple way to prove some results on bi-capacities.

We have proved that p-symmetric bi-capacities provide a reduction in the number of necessary values to completely define the bi-capacity. Moreover, we have shown that they can be represented in a p-dimensional matrix, in a way similar to p-symmetric fuzzy measures.

We have proved that we can translate the properties regarding Choquet integral; first, we have proved that Choquet integral can be computed through a path in the matrix representation; then, we have obtained a decomposition of Choquet integral as a weighted sum of Choquet integrals w.r.t. fuzzy measures and an error term. This last result is especially interesting, as we have obtained a decomposition in terms of fuzzy measures while Choquet integral for a bi-capacity is usually a Choquet integral w.r.t. a set function, not necessarily monotone.

Finally, we have proposed a definition of p-symmetric k-ary capacities. It must be remarked here that using the definitions in terms of permutations, no change is needed.

(Received October 2, 2003.)

REFERENCES

[1] A. Chateauneuf and J.-Y. Jaffray: Some characterizations of lower probabilities and other monotone capacities through the use of Möbius inversion. Math. Social Sci. *17* (1989), 263–283.
[2] G. Choquet: Theory of capacities. Annales de l'Institut Fourier *5* (1953), 131–295.
[3] D. Denneberg: Non-additive Measures and Integral. Kluwer, Dordrecht 1994.
[4] D. Dubois and H. Prade: A class of fuzzy measures based on triangular norms. Internat. J. Gen. Systems *8* (1982), 43–61.
[5] M. Grabisch: Pattern classification and feature extraction by fuzzy integral. In: 3rd European Congress on Intelligent Techniques and Soft Computing (EUFIT, Aachen, Germany, August 1995), pp. 1465–1469.
[6] M. Grabisch: Fuzzy measures and integrals: A survey of applications and recent issues. In: Fuzzy Sets Methods in Information Engineering: A Guide Tour of Applications (D. Dubois, H. Prade, and R. Yager, eds.), Wiley, New York 1997, pp. 507–530.
[7] M. Grabisch: k-order additive discrete fuzzy measures. In: Proc. 6th Internat. Conference on Information Processing and Management of Uncertainty in Knowledge-based Systems (IPMU), Granada (Spain), 1996, pp. 1345–1350,
[8] M. Grabisch: k-order additive discrete fuzzy measures and their representation. Fuzzy Sets and Systems *92* (1997), 167–189.
[9] M. Grabisch and C. Labreuche: Bi-capacities. In: Proc. First Internat. Conference on Soft Computing and Intelligent Systems (SCIC), Tsukuba (Japan), 2002.

[10] M. Grabisch and C. Labreuche: Bi-capacities for decision making on bipolar scales. In: Proc. Seventh Meeting of the EURO Working Group on Fuzzy Sets (EUROFUSE) Varenna (Italy), September 2002, pp. 185–190.
[11] M. Grabisch and Ch. Labreuche: Capacities on lattices and k-ary capacities. In: 3rd Internat. Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT 2003), Zittau, Germany, September 2003, pp. 304–307.
[12] P. L. Hammer and R. Holzman: On approximations of pseudo-boolean functions. Z. Oper. Res. – Math. Methods Oper. Res. *36* (1992), 3–21.
[13] T. W. Hungerford: Algebra. Springer–Verlag, Berlin 1980.
[14] R. Mesiar: k-order additive measures. Internat. J. Uncertainty, Fuzziness and Knowledge-based Systems *7* (1999), 423–428.
[15] P. Miranda and M. Grabisch: p-symmetric fuzzy measures. In: Proc. Ninth Internat. Conference of Information Processing and Management of Uncertainty in Knowledge-based Systems (IPMU), Annecy (France), July 2002, pp. 545–552.
[16] P. Miranda, M. Grabisch, and P. Gil: p-symmetric fuzzy measures. Internat. J. Uncertainty, Fuzziness and Knowledge-based Systems *10* (2002), 105–123. Supplement.
[17] P. Miranda and E. F. Combarro, and P. Gil: p-symmetric bi-capacities. In: Proc. Second Internat. Summer School on Aggregation Operators and Their Applications (AGOP), Alcalà de Henares (Spain), 2003, pp. 123–128.
[18] G. C. Rota: On the foundations of combinatorial theory I. Theory of Möbius functions. Z. Wahrschein. und Verwandte Gebiete *2* (1964), 340–368.
[19] M. Sugeno: Theory of Fuzzy Integrals and Its Applications. PhD Thesis, Tokyo Institute of Technology, 1974.
[20] M. Sugeno, K. Fujimoto, and T. Murofushi: A hierarchical decomposition of Choquet integral model. Internat. J. Uncertainty, Fuzziness and Knowledge-based Systems *1* (1995), 1–15.
[21] M. Sugeno and T. Terano: A model of learning based on fuzzy information. Kybernetes *6* (1977), 157–166.
[22] A. Tversky and D. Kahneman: Advances in prospect theory: cumulative representation of uncertainty. J. Risk and Uncertainty *5* (1992), 297–323.
[23] J. Šipoš: Integral with respect to a pre-measure. Math. Slovaca *29* (1979), 141–155.
[24] S. Weber: $\perp$-decomposable measures and integrals for archimedean t-conorms $\perp$. J. Math. Anal. Appl. *101* (1984), 114–138.
[25] R. R. Yager: On ordered weighted averaging aggregation operators in multicriteria decision making. IEEE Trans. Systems Man Cybernet. *18* (1988), 183–190.

Pedro Miranda, Department of Statistics and O.R., Faculty of Mathematical Sciences – Complutense University of Madrid, Plaza de Ciencias 3, 28040 Madrid. Spain.
e-mail: pmiranda@mat.ucm.es

Michel Grabisch, University Paris I – Panthéon-Sorbonne – LIP6, 8, rue du Capitaine Scott, 75015 Paris. France.
e-mail: Michel.Grabisch@lip6.fr