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REGIME–SWITCHING MODELS OF TIME SERIES WITH CUBIC SPLINE TRANSITION FUNCTION IN GEODETIC APPLICATION[1]

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A new class of Smooth Transition Autoregressive models, based on cubic spline type transition functions, has been introduced and subjected to comparison with models based on the traditional logistic transition functions. A very high degree of similarity between the two model classes has been demonstrated.

The new class of models can be slightly preferable because of its more simple formal and geometrical structure that may enable users more convenient manipulation in statistical inference procedures.

Keywords: time series, regime-switching autoregressive models, logistic and cubic-spline transition functions

AMS Subject Classification: 62M10

1. INTRODUCTION

Smooth Transition Autoregressive models (STAR) have been extensively analyzed and applied by many authors during the last two decades. They have been introduced as a smooth alternative to Treshold Autoregressive models (representing nonlinear generalizations of autoregressive models) that assume different autoregressive models describing behaviour of an investigated time series y_t in different regimes.

We introduce and investigate cubic spline alternatives to the popular logistic transition functions. The new class of transition functions preserves most important properties of the logistic class while it may enable users more convenient manipulation in statistical inference procedures.

2. LOGISTIC MODEL

A formal representation of a 2-regimes STAR can be expressed by

$$y_t = \Phi_1(B)y_t[1 - G(y_{t-d};\gamma,c)] + \Phi_2(B)y_tG(y_{t-d};\gamma,c) + \epsilon_t \qquad (1)$$

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(see Teräsvirta [3]), where

ϵ_t is a white noise sequence with variance σ^2,

the autoregressive polynomials

$$\Phi_i(B) = \phi_{i,0} + \phi_{i,1}B + \cdots + \phi_{i,p_i}B^{p_i}, \qquad i = 1,2$$

in the shift operator B (defined by $By_t = y_{t-1}$) are related to regimes that are determined by values of a treshold variable y_{t-d} and its treshold level value c. A logistic transition function has the form

$$G(y_{t-d};\gamma,c) = \frac{1}{1+\exp(-\gamma[y_{t-d}-c])} \tag{2}$$

where γ is the smoothness parameter.
It is obvious that $G(c;\gamma,c) = \frac{1}{2}$ and $G(y;0,c) = \frac{1}{2}$ for any $y,c \in R$ and $\gamma \geq 0$.
If we put $q_t = y_{t-d} - c$ and

$$G^*(q;\gamma) = G(c+q;\gamma,c) - \frac{1}{2} = \frac{1}{1+\exp(-q\gamma)} - \frac{1}{2} \tag{3}$$

we can rewritte (1) in the form (see Franses and van Dijk [1])

$$y_t = \frac{1}{2}[\Phi_1(B) + \Phi_2(B)]y_t + [\Phi_2(B) - \Phi_1(B)]y_t G^*(q_t;\gamma) + \epsilon_t \tag{4}$$

which can be applied for testing linearity of the model (the hypothesis $\Phi_1(B) = \Phi_2(B)$ which is equivalent to the hypothesis $H_0 : \gamma = 0$ in (3)). In this test a third-order Taylor polynomial approximation to $G^*(q;\gamma)$ in the right neighborhood of $\gamma = 0$ was utilized.

Let us note that the function $G^*(q;\gamma)$ given by (3) is increasing in both variables q (for any fixed $\gamma > 0$) and γ (for any fixed $q > 0$) whereas $G^*(q;\gamma)$ is decreasing in γ for any fixed $q < 0$. Moreover, the following relations obviously hold:

$$G^*(0;\gamma) = 0 \qquad \text{for any } \gamma \geq 0 \tag{5}$$

$$G^*(q;0) = 0 = \lim_{\gamma\to 0} G^*(q;\gamma) \qquad \text{for any } q \in R \tag{6}$$

$$\lim_{q\to\infty} G^*(q;\gamma) = \frac{1}{2}; \qquad \lim_{q\to-\infty} G^*(q;\gamma) = -\frac{1}{2}; \qquad \text{for any } \gamma > 0 \tag{7}$$

$$\lim_{\gamma\to\infty} G^*(q;\gamma) = \begin{cases} \frac{1}{2} & \text{for } q > 0 \\ -\frac{1}{2} & \text{for } q < 0 \end{cases} \tag{8}$$

$$\frac{\partial G^*(q;\gamma)}{\partial\gamma} = \frac{q\,\exp(-q\gamma)}{[1+\exp(-q\gamma)]^2} \tag{9}$$

$$\frac{\partial G^*(q;\gamma)}{\partial q} = \frac{\gamma\ \exp(-q\gamma)}{[1+\exp(-q\gamma)]^2} \tag{10}$$

hence G^* is increasing in q for any fixed $\gamma > 0$ and $\frac{\partial G^*(q;\gamma)}{\partial q}$ is exponentially approaching 0 for any fixed $\gamma > 0$ and $|q| \to \infty$.
Furthermore

$$\frac{\partial^2 G^*(q;\gamma)}{\partial \gamma^2} = \frac{-q^2\ \exp(-q\gamma)[1-\exp(-q\gamma)]}{[1+\exp(-q\gamma)]^3} \tag{11}$$

$$\frac{\partial^2 G^*(q;\gamma)}{\partial q^2} = \frac{-\gamma^2\ \exp(-q\gamma)[1-\exp(-q\gamma)]}{[1+\exp(-q\gamma)]^3}. \tag{12}$$

Obviously, for any $\gamma > 0, G^*$ as a function of q is convex for $q < 0$ and concave for $q > 0$, thus the partial derivative $\frac{\partial G^*(q;\gamma)}{\partial q}$ attains its maxima $\frac{\gamma}{4}$ for q = 0.
Moreover

$$\frac{\partial^3 G^*(q;\gamma)}{\partial \gamma^3} = \frac{q^3\ \exp(-q\gamma)[1-4\exp(-q\gamma)+\exp(-2q\gamma)]}{[1+\exp(-q\gamma)]^4}. \tag{13}$$

From the equalities (9), (11) and (13) we get for $\gamma = 0$

$$\frac{\partial G^*(q;0)}{\partial \gamma} = \frac{q}{4}, \qquad \frac{\partial^2 G^*(q;0)}{\partial \gamma^2} = 0, \qquad \frac{\partial^3 G^*(q;0)}{\partial \gamma^3} = -\frac{q^3}{8}. \tag{14}$$

This yields a third-order Taylor approximation

$$T_3(q,\gamma) = \gamma \left[\frac{\partial G^*(q;\gamma)}{\partial \gamma}\right]_{\gamma=0} + \frac{1}{6}\gamma^3 \left[\frac{\partial^3 G^*(q;\gamma)}{\partial \gamma^3}\right]_{\gamma=0} = \frac{1}{4}\gamma q - \frac{1}{48}\gamma^3 q^3. \tag{15}$$

This has been used for testing of linearity in Franses and van Dijk [1] (where an error in the sign of the third-order term occurred). Note that $T_3(q,\gamma)$ as well as G^* and G are symmetric in the pair of variables q and γ since they depend only on the product $x = q\gamma$.

3. CUBIC SPLINE MODEL

Our idea is to find a class of third-order spline functions depending on a smoothing parameter γ that would resemble properties of $G^*(q,\gamma)$.

As we can see in the following Figures 1–3, the polynomials $T_3(q,\gamma)$ do not provide acceptable global approximations to $G^*(q;\gamma)$ as functions of q. To fix this shortcoming we introduce a class of polynomials

$$P(q,\gamma) = \begin{cases} -\frac{1}{2} & q < -\frac{3}{\gamma} \\ \frac{1}{4}q\gamma - \frac{1}{108}q^3\gamma^3 & -\frac{3}{\gamma} \le q \le \frac{3}{\gamma} \\ \frac{1}{2} & q > \frac{3}{\gamma} \end{cases}$$

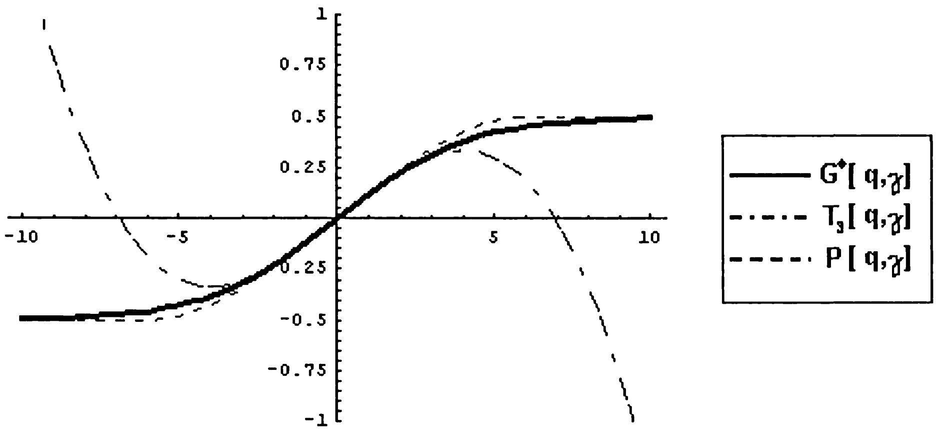

Fig. 1. $\gamma = 0.5$.

As we can observe in these figures, polynomials $P(q,\gamma)$ provide much better global approximations to $G^*(q,\gamma)$ than $T_3(q,\gamma)$. Moreover, the following relations obviously hold:

1.

$$P(0,\gamma) = 0 \qquad \text{for any} \quad \gamma \geq 0. \tag{16}$$

2. For any $q \neq 0, \gamma < \left|\frac{3}{q}\right|$ we have

$$P(q,\gamma) = \frac{1}{4}q\gamma - \frac{1}{108}q^3\gamma^3, \quad \text{thus} \lim_{\gamma \to 0} P(q,\gamma) = 0. \tag{17}$$

3. For any $\gamma > 0$ and $|q| \geq \left|\frac{3}{\gamma}\right|$ we have

$$P(q,\gamma) = \frac{1}{2}, \quad P(-q,\gamma) = -\frac{1}{2} \tag{18}$$

hence for any $\gamma > 0$

$$\lim_{q \to \infty} P(q,\gamma) = \frac{1}{2}, \quad \lim_{q \to -\infty} P(q,\gamma) = -\frac{1}{2} \tag{19}$$

and

$$\lim_{\gamma \to \infty} P(q,\gamma) = \begin{cases} \frac{1}{2} & \text{for} \quad q > 0 \\ -\frac{1}{2} & \text{for} \quad q < 0. \end{cases} \tag{20}$$

Moreover, the functions $P(q,\gamma)$ are odd in variable q

$$P(-q,\gamma) = -P(q,\gamma). \tag{21}$$

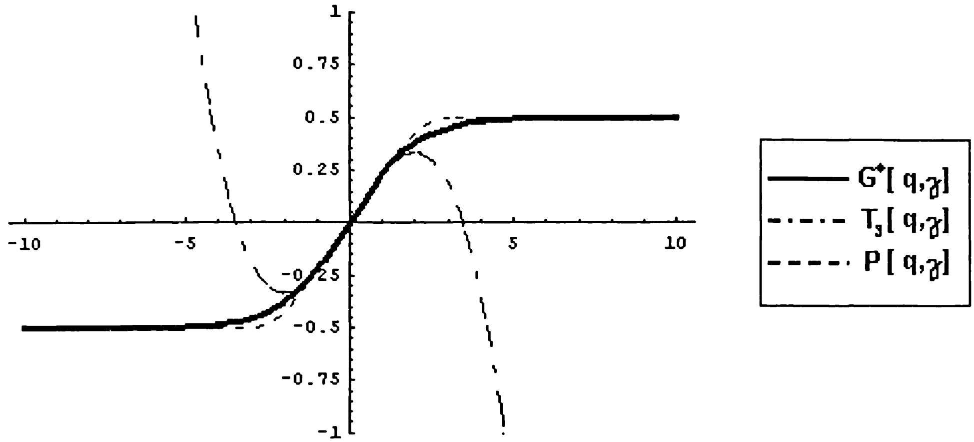

Fig. 2. $\gamma = 1$.

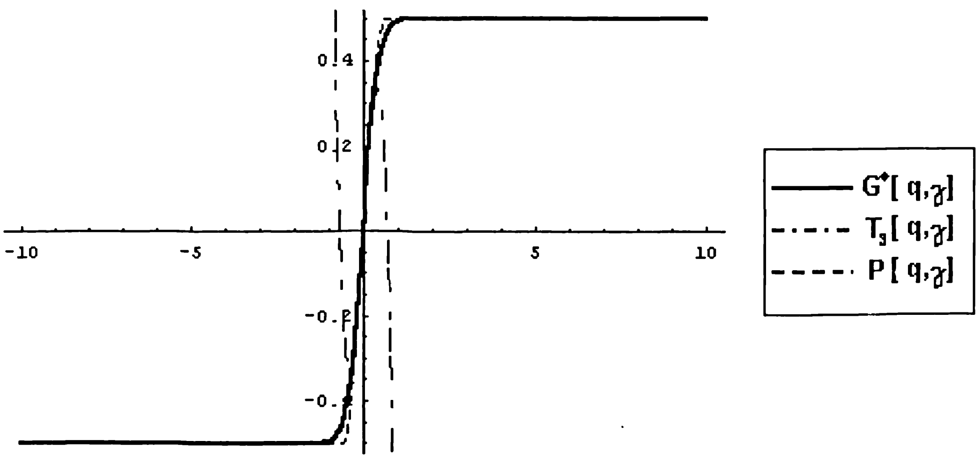

Fig. 3. $\gamma = 5$.

4.

$$\frac{\partial P(q;\gamma)}{\partial q} = \begin{cases} 0 & \text{for } q < -\frac{3}{\gamma} \\ \frac{1}{4}\gamma - \frac{1}{36}\gamma^3 q^2 = \frac{1}{4}\gamma\left(1 - \frac{1}{9}\gamma^2 q^2\right) & \text{for } -\frac{3}{\gamma} \le q \le \frac{3}{\gamma} \\ 0 & \text{for } q > \frac{3}{\gamma}. \end{cases} \tag{22}$$

For $|q| < 3\gamma^{-1}$ we have $q^2 < 9\gamma^{-2}, q^2\gamma^2 < 9$ and $\frac{\partial P(q;\gamma)}{\partial q} > 0$, hence $P(q,\gamma)$ is increasing in q for any $\gamma > 0$ and $|q| < 3\gamma^{-1}$. $\frac{\partial P(q;\gamma)}{\partial q}$ attains its maximum $\frac{\gamma}{4}$ for $q = 0$. $P(q,\gamma)$ is nondecreasing in γ for any fixed positive q with $P(q;\gamma) = \frac{1}{2}$ for $\gamma \ge \frac{3}{q}$. For any q negative $P(q;\gamma)$ is decreasing in γ for $\gamma \le -\frac{3}{q}$ with $P(q;\gamma) = -\frac{1}{2}$ for $\gamma \ge -\frac{3}{q}$.

Moreover,

$$\left[\frac{\partial P(q;\gamma)}{\partial q}\right]_{q=\frac{3}{\gamma}} = 0 = \left[\frac{\partial P(q;\gamma)}{\partial q}\right]_{q=-\frac{3}{\gamma}},$$

hence for any fixed γ, $P(q;\gamma)$ is a nondecreasing spline function in q on R.

5.

$$\frac{\partial^2 P(q;\gamma)}{\partial q^2} = \begin{cases} 0 & \text{for} \quad q < -\frac{3}{\gamma} \\ -\frac{1}{18}\gamma^3 q & \text{for} \quad -\frac{3}{\gamma} \le q \le \frac{3}{\gamma} \\ 0 & \text{for} \quad q > \frac{3}{\gamma}. \end{cases}$$

Hence for any $\gamma > 0$, $P(q;\gamma)$ as a function of q is convex for $q < 0$ and concave for $q > 0$.

Let us recall similarities in the properties of $G^*(q,\gamma)$ and $P(q,\gamma)$:

a)

$$G^*(0,\gamma) = P(0,\gamma) = 0 \qquad \text{for any } \gamma > 0.$$

b)

$$G^*(q,\gamma) + G^*(-q,\gamma) = P(q;\gamma) + P(-q;\gamma) = 0 \qquad \text{for any } q \in R \text{ and } \gamma > 0.$$

c)

$$\frac{\partial G^*(q;\gamma)}{\partial q} > 0, \quad \frac{\partial P(q;\gamma)}{\partial q} \ge 0 \qquad \text{for all } q \in R \text{ and } \gamma > 0.$$

Thus for any fixed $\gamma > 0, G^*(q;\gamma)$ is increasing and $P(q,\gamma)$ is nondecreasing in q. Moreover

$$\frac{\partial G^*(0;\gamma)}{\partial \gamma} = \frac{\partial P(0;\gamma)}{\partial \gamma} = \frac{\gamma}{4}.$$

d)

$$\frac{\partial^2 G^*(q;\gamma)}{\partial q^2} < 0 \quad \text{and} \quad \frac{\partial^2 P(q;\gamma)}{\partial q^2} \le 0 \quad \text{for any } q > 0 \text{ and } \gamma > 0.$$

Hence for any fixed $\gamma > 0$ the functions $G^*(q;\gamma)$ and $P(q;\gamma)$ are concave for $q > 0$ and convex for $q < 0$.

e) For any $\gamma > 0$

$$\lim_{q\to\infty} G^*(q;\gamma) = \lim_{q\to\infty} P(q;\gamma) = \frac{1}{2}$$

and

$$\lim_{q\to-\infty} G^*(q;\gamma) = \lim_{q\to-\infty} P(q;\gamma) = -\frac{1}{2}.$$

f) For any $q > 0$, $G^*(q;\gamma)$ is increasing in γ and $P(q;\gamma)$ is nondecreasing in γ,

$$\lim_{\gamma\to\infty} G^*(q;\gamma) = \lim_{\gamma\to\infty} P(q;\gamma) = \frac{1}{2}$$

and

$$\lim_{\gamma\to-\infty} G^*(q;\gamma) = \lim_{\gamma\to-\infty} P(q;\gamma) = -\frac{1}{2}.$$

g) For any $q \in R$

$$\lim_{\gamma \to 0} G^*(q;\gamma) = \lim_{\gamma \to 0} P(q;\gamma) = 0.$$

Inspecting behaviour of the difference $\frac{\partial P(q;\gamma)}{\partial \gamma} - \frac{\partial G^*(q;\gamma)}{\partial \gamma}$ we conclude that for any fixed $\gamma > 0$ the difference $P(q,\gamma) - G^*(q,\gamma)$ is positive for $q > 0$, increasing on the interval $(0, q_0)$, where $q_0 \approx \frac{2.58}{\gamma}$, and decreasing on (q_0, ∞). The maximal difference $P(q_0,\gamma) - G^*(q_0,\gamma) \approx 0.056$ independently of γ.

Similarly we can obtain that the difference $G^*(q,\gamma) - T_3(q,\gamma)$ is positive for any $\gamma > 0, q > 0$. $T_3(q,\gamma)$ is better approximation to $G^*(q,\gamma)$ than $P(q,\gamma)$ on the interval $(0, q_1)$, where $q_1 \approx \frac{1.966}{\gamma}$ and $G^*(q_1,\gamma) - T_3(q_1,\gamma) = P(q_1,\gamma) - G^*(q_1,\gamma) \approx 0.044$ independently of γ. The maximum of differences $[(P(q,\gamma) - G^*(q,\gamma)) - (G^*(q,\gamma) - T_3(q,\gamma)]$ is obtained in $q = q_2 \approx \frac{1.482}{\gamma}$ and its value is 0.013 independently of γ.

4. APPLICATION

The time series of values of vertical coordinate v consists of results of permanent position at POTS (Potsdam Observatory) station by the GPS (Global Positioning System) technology. The series covers the interval from January 1, 2001 to December 31, 2002. Each element of the series represents the value obtained during the 24-hour interval of observation of GPS satellites ($n = 730$).

The GPS is the satellite navigation system applied in geodesy for precise determination of position expressed in Cartesian geocentric coordinate system) on the Earth's surface. The elements of time series of POTS coordinates are derived from solutions of European network of permanent GPS monitoring stations (about 40 stations distributed all over the continent).

We calculated optimal 2-regime models with both logistic and polynomial transition functions and threshold variables y_{t-d} for $d = 1,2,3,4,5$. The best fit were received in both logistic and polynomial transition function classes for $d = 5$ and resulting model parameters estimates $(c, \gamma, \phi_{1,0}, \phi_{1,1}, \phi_{2,0}, \phi_{2,1}, \sigma^2)$ equal to:

$$(2.7; 15; -0.025; 0.967; 0.239; 0.930; 0.511)$$

for the logistic case and

$$(2.7; 0.5; -0.024; 0.968; 0.228; 0.932; 0.511)$$

for the polynomial case.

The values of the vector parameters estimates for both models are extremaly similar except for the slope parameter γ. Moreover, there are minimal differences between autoregressive coefficients $\phi_{1,1}$ and $\phi_{2,1}$ for both transition function classes. This inspires testing of linearity. Fitting AR(1) model for the same data we receive estimates $(\phi_0, \phi_1, \sigma^2) = (-0.001; 0.975; 0.517)$.

For both logistic and polynomial model we can apply the same testing procedure that was indicated in Franses and van Dijk [1]. We calculate an auxiliary regression model with explaining variables $y_{t-1}; y_{t-5} \cdot y_{t-5}^2; y_{t-5}^3; y_{t-1} \cdot y_{t-5}; y_{t-1} \cdot y_{t-5}^2; y_{t-1} \cdot y_{t-5}^3$. The residual variance estimate for this model is $\sigma^2 = 0.523$.

The ANOVA test for AR(1) submodel yields F-ratio 0.452 which is much lower than the mean of the F-distribution and thus highly nonsignificant. Hence neither logistic nor polynomial STAR alternative to AR(1) model can be accepted as an alternative to AR(1) model.

Next we repeated the same kind of analyses for the corresponding time series of North–South coordinates of the same station. After removing the trend and cyclical component we received the best fit for both logistic and polynomial transition function classes for delay $d = 2$ with parameter estimates $(c, \gamma, \phi_{1,0}, \phi_{1,1}, \phi_{2,0}, \phi_{2,1}, \sigma^2)$ equal to

$$(-0.32; 0.5; -1.37; 0.729; 1.17; 0.272; 0.06579)$$

for logistic case and

$$(-0.32; 0.5; -0.098; 0.498; 2.425; 0.04; 0.06577)$$

for polynomial case.

We see that the estimates of c, γ and σ^2 have practically the same value while coefficients of AR(1) model differ considerably between transition function classes and between individual regimes in each of these classes. Repeating the F-test for regression model with explanatory variables $y_{t-1}; y_{t-2}; y_{t-2}^2; y_{t-2}^3; y_{t-1} \cdot y_{t-2}; y_{t-1} \cdot y_{t-2}^2; y_{t-1} \cdot y_{t-2}^3$ and its AR(1) submodel we receive F-ratio value 14.912 which is very highly significant (beyond the ranges of any standardly used statistical tables). Therefore, AR(1) model is rejected against both logistic and polynomial two regimes alternatives. Since both these models provide practically the same quality of fit, decision between them is not an easy problem. We can prefer the polynomial alternative because of its slightly better fit and more simple formal structure.

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