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Kybernetika, Vol. 39 (2003), No. 4, [493]--506

Persistent URL: http://dml.cz/dmlcz/135548

KYBERNETIKA — VOLUME 39 (2003), NUMBER 4, PAGES 493–506

ON THE OBSERVABILITY OF FUZZY SECOND ORDER CONTROL SYSTEMS[1]

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In this paper, the observability of fuzzy logic second order control system is studied from the aspect of fuzzy differential equations. The fuzzy observability in the weak sense is created using the concept of "likelihood" to indicate on which level and along which solution the state is most likely observable. One of the initial state range has been derived with the given input and output. The result generalizes the previous results.

Keywords: fuzzy differential equation, second order control system, fuzzy solution, likelihood

AMS Subject Classification: 93B07, 93C42

1. INTRODUCTION

In recent years there has been considerable effort in the investigation of abstract second order differential equations directly rather than to convert them into first order systems. Much of this effort was inspired by partial second order equations which serve as models for various problems in continuum mechanics. A useful machinery for the study of abstract second order equations is the theory of strongly continuous cosine families. For these reasons, there has been an increasing interest in studying equations that can be described in the form of abstract fuzzy second order equations.

Generally, several systems are mostly related to uncertainty and unexactness. The problem of unexactness is considered in general exact science and that of uncertainty is considered as vagueness or fuzzy and accident. Ding et al [4] combine differential equations with fuzzy sets to form a fuzzy logic system and analysed the observability. For fuzzy concepts recently the author [1] established the theory of metric space of fuzzy sets. In particular, Kaleva [7] researched the fuzzy differential equations and Cauchy problem. Seikkala [9] proved the existence and uniqueness of the fuzzy solution for the following systems:

$$x'(t) = f(t, x(t)), \qquad x(0) = x_0$$

[1]This work was supported by grant No. 2000-1-10300-003-2 from the Basic Search Program of the Korea Science and Engineering Foundation.

where f is a continuous mapping from $\mathbb{R}^+ \times \mathbb{R}$ into $\mathbb{R}$ and x_0 is a fuzzy number. The fuzzy differential equations are kept in mind to describe the fuzzy logic system, which has several attractive features discussed in the last section. First the observability is analysed.

This paper is to investigate the observability of the following nonlinear fuzzy second order control system (in short FSOCS)

$$x''(t) = A(t)x(t) + B(t)U(t) + f(t), \quad t \in \mathbb{R}, \quad x(0) = x_0, \quad x'(0) = x_1 \quad (1.1)$$

$$y(t) = E(t)x(t) + D(t)U(t) \quad (1.2)$$

where A, B, C, and D are matrices whose elements are continuous functions and $x(0) = x_0$, $x'(0) = x_1$ are given initial conditions. If the inputs $U(t)$ are crisp, then it is the classical control system, when $U(t)$ are the fuzzy inputs we have the FSOCS. In this paper, instead of the controllability problem, the observability problem is to be concerned about the initial state x_0 of the system be always identified by observing the output y and the input $U(t)$ over a finite time.

2. MATHEMATICAL PRELIMINARIES

Let E^n be the fuzzy space based on $\mathbb{R}^n$, let A, B, D and E be crisp continuous matrices, let $f : J \to E^n$ be a continuous fuzzy mapping and X, Y and U be fuzzy sets. We call the system

$$\begin{aligned} X''(t) &= A(t)X(t) + B(t)U(t) + f(t), \qquad t > 0, \\ &\qquad X(0) = \{x(0)\}, \quad X'(0) = \{x'(0)\} \\ Y(t) &= D(t)X(t) + E(t)U(t) \end{aligned} \quad (2.1)$$

where $A(t)$ is a generator of continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ in fuzzy sets. Then above equation (2.1) can be represented by the fuzzy mild form:

$$\begin{aligned} X(t) &= C(t)X_0 + S(t)X_1 + \int_0^t S(t-s)B(s)U(s)\mathrm{d}s + f(s), \\ Y(t) &= D(t)X(t) + E(t)U(t), \end{aligned} \quad (2.2)$$

where $S(t)x = \int_0^t C(s)x\mathrm{d}s$, $x \in E^n$, $t \in \mathbb{R}$. In order to define the likelihood for the solution of equation (2.1), we construct a function h such that

$$h : \mathbb{R}^n \times P_{KC}(\mathbb{R}^n) \to \mathbb{R}^1 \cup \{\infty\}$$

and by setting

$$h(\omega, \Omega) = \sup_f \left\{ \left(\int_0^1 |f(s) - \omega|^2 \mathrm{d}s \right)^{1/2} \middle| f : [0,1] \to \Omega, \int_0^1 f(s)\mathrm{d}s = \omega \right\} \quad (2.3)$$

with the understanding that $h(\omega, \Omega) = -\infty$ if $\omega \notin \Omega$. Here $h(\omega, \Omega)$ has been interpreted as the maximum variance among all random variables supported inside Ω, whose mean is ω. Let $A(t)$ be denote the fundamental matrix of the equation

$$x''(t) = A(t)x(t) + f(t).$$

We present the following definition of the weak form:

Definition 2.1. Let X be a weak solution (2.1) and X_α be the α-level of X. Then likelihood of the solution on level α of (2.1) is

$$L_\alpha(X) = \sup_{x_\alpha \in X_\alpha} \Big[\int_0^T h^2 \Big(x_\alpha(t), C(t)x_0 \quad + \quad S(t)x_1 + \int_0^t S(t-s)B(s)U_\alpha(s)\mathrm{d}s \\ + \quad \int_0^t S(t-s)f(s)\mathrm{d}s \Big)\mathrm{d}t \Big]^{1/2}.$$

Thus, the maximum of $L_\alpha(X)$ is the "most likely" solution also well defined by Liapunov's theorem. Further if $\omega \in \Omega$, the existence of the function $f : [0,1] \to \Omega$ for which the $\sup_f$ in (2.3) is exactly attained (see [3]). Thus, the function h actually denotes the maximum.

3. FORMATION OF FUZZY SECOND ORDER CONTROL SYSTEM

If $U(t)$ is a crisp, the deterministic SOCS is given by

$$\begin{aligned} x''(t) &= A(t)x(t) + B(t)U(t) + f(t), \quad t > 0, \quad x(0) = x_0, \quad x'(0) = x_1, &(3.1)\\ y(t) &= D(t)x(t) + E(t)U(t). &(3.2) \end{aligned}$$

The above equation (3.1)–(3.2) are FSOCS if the inputs $U(t)$ are fuzzy. Let $0 < \alpha \leq 1$ and consider the differential inclusions

$$\begin{aligned} x_\alpha''(t) &= A(t)x_\alpha(t) + B(t)U(t) + f_\alpha(t), \qquad t \in J &(3.3)\\ x(0) &= x_0, \qquad x'(0) = x_1. &(3.4) \end{aligned}$$

First we proof that the solution set X^α of (3.3)–(3.4) is nonempty, compact and convex in $C(J, \mathbb{R}^n)$. Let

$$\begin{aligned} M_1 &= \max_{t\in J} ||C(t)||, & M_2 &= \max_{t\in J} ||S(t)|| \\ N_1 &= \max_{t\in J} ||f(t)||, & N_2 &= \max_{t\in J} ||B(t)|| \quad N_3 = \max_{t\in J} ||u(t)||, \ u(t) \in U^\alpha(t), \end{aligned}$$

then we see that [6] there is a selection $u(t) \in U^\alpha(t)$ such that

$$x(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)B(s)U(s)\mathrm{d}s + \int_0^t S(t-s)f(s)\mathrm{d}s.$$

Thus, we have

$$\begin{aligned} ||x(t)|| &\leq ||C(t)|| ||x(0)|| + ||S(t)|| ||x_1|| \\ &\quad + \int_0^t ||S(t-s)B(s)U(s)||\mathrm{d}s + \int_0^t ||S(t-s)f(s)||\mathrm{d}s \\ &\leq M_1||x(0)|| + M_2||x_1|| + M_2N_2M_3T + M_2N_1T. \end{aligned}$$

From this we get X^α is bounded. Next, we prove that X^α is equicontinuous. In fact, for each $x \in X^\alpha$ and for any $t_1, t_2 \in J$, we have

$$\begin{aligned}
\|x(t_2) - x(t_1)\| &\leq \|C(t_2) - C(t_1)\|\|x_0\| + \|S(t_2) - S(t_1)\|\|x_1\| \\
&\quad + \left\| \int_0^{t_2} S(t_2 - s)B(s)u(s)\mathrm{d}s - \int_0^{t_1} S(t_1 - s)B(s)u(s)\mathrm{d}s \right\| \\
&\quad + \left\| \int_0^{t_2} S(t_2 - s)f(s)\mathrm{d}s - \int_0^{t_1} S(t_1 - s)f(s)\mathrm{d}s \right\| \\
&\leq \|C(t_2) - C(t_1)\|\|x_0\| + \|S(t_2) - S(t_1)\|\|x_1\| \\
&\quad + \int_0^{t_1} \|S(t_2 - s)B(s)u(s)\mathrm{d}s - S(t_1 - s)B(s)u(s)\mathrm{d}s\| \\
&\quad + \int_{t_1}^{t_2} \|S(t_2 - s)B(s)u(s)\mathrm{d}s\| \\
&\quad + \int_0^{t_1} \|S(t_2 - s)f(s)\mathrm{d}s - S(t_1 - s)f(s)\mathrm{d}s\| \\
&\leq \|C(t_2) - C(t_1)\|\|x_0\| + \|S(t_2) - S(t_1)\|\|x_1\| \\
&\quad + N_2 N_3 \int_0^{t_1} \|S(t_2 - s)\mathrm{d}s - S(t_1 - s)\mathrm{d}s\| \\
&\quad + M_2 N_2 N_3 (t_2 - t_1) + N_1 \int_0^{t_1} \|S(t_2 - s) - S(t_1 - s)\mathrm{d}s\|.
\end{aligned}$$

Since $C(t)$ and $S(t)$ are uniformly continuous and X^α is equicontinuous. From the Arzela–Ascoli theorem [3] we know that X^α is compact. Indeed, it is sufficient to prove that it is closed. Let $x_k \in X^\alpha$ and $x_k \to x$ for each integer $k > 0$ then there is a $u_k \in U^\alpha$ such that

$$x_k(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t - s)B(s)u_k(s)\mathrm{d}s + \int_0^t S(t - s)f(s)\mathrm{d}s.$$

Since $x_k \in L_{\mathbb{R}^n}(J)$, there exists a subsequence $\{u_{k_j}\}$ of $\{u_k\}$ such that $\{u_{k_j}\}$ converges weakly to $u \in L_{\mathbb{R}^n}(J)$ (see [5, p. 292]). From Mazur's theorem, there exists a convex combination of u_{k_j}, say $\sum_j \lambda_j u_{k_j}$ which converges strongly to u. Since $\sum_j \lambda_j = 1$, we have

$$\begin{aligned}
\sum_j \lambda_j x_{k_j}(t) = \sum_j \lambda_j C(t)x_0 + \sum_j \lambda_j S(t)x_1 \quad &+ \quad \int_0^t S(t - s)B(s) \sum_j \lambda_j u_{k_j}(s)\mathrm{d}s \\
&+ \quad \sum_j \lambda_j \int_0^t S(t - s)f(s)\mathrm{d}s. \qquad (3.5)
\end{aligned}$$

Taking the limit on (3.5) and using of Fatou's lemma, we obtain

$$x(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t - s)B(s)u(s)\mathrm{d}s + \int_0^t S(t - s)f(s)\mathrm{d}s.$$

Since $U^\alpha(t)$ is convex and closed, $\sum_j \lambda_j u_{k_j}(t) \in U^\alpha(t)$ and thus $\sum_j \lambda_j u_{k_j}(t) \to u(t) \in U^\alpha(t)$. From continuity of x we conclude that $x \in X^\alpha$. This proves the closeness of X^α.

We now show that X^α is convex. Let $x_1, x_2 \in X^\alpha$, then there are $u_1(t), u_2(t) \in U^\alpha(t)$ such that

$$\begin{aligned} x_1(t) &= C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)B(s)u_1(s)\mathrm{d}s + \int_0^t S(t-s)f(s)\mathrm{d}s \\ x_2(t) &= C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)B(s)u_2(s)\mathrm{d}s + \int_0^t S(t-s)f(s)\mathrm{d}s. \end{aligned}$$

Let $x(t) = \lambda x_1(t) + (1-\lambda)x_2(t)$, $0 \le \lambda \le 1$, then

$$\begin{aligned} x(t) &= \lambda \left\{ C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)B(s)u_1(s)\mathrm{d}s + \int_0^t S(t-s)f(s)\mathrm{d}s \right\} \\ &\quad + (1-\lambda)\left\{ C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)B(s)u_2(s)\mathrm{d}s + \int_0^t S(t-s)f(s)\mathrm{d}s \right\} \\ &= \lambda C(t)x_0 + (1-\lambda)C(t)x_0 + \lambda S(t)x_1 + (1-\lambda)S(t)x_1 \\ &\quad + \int_0^t S(t-s)B(s)[\lambda u_1(s) \; + \; (1-\lambda)u_2(s)]\mathrm{d}s \\ &\quad + \int_0^t S(t-s)[\lambda f(s) + (1-\lambda)f(s)]\mathrm{d}s. \end{aligned}$$

Since $U^\alpha(t)$ is convex, we see that

$$\lambda u_1(t) + (1-\lambda)u_2(t) \in U^\alpha(t).$$

Thus, we have

$$x(t) \in C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)B(s)u(s)\mathrm{d}s + \int_0^t S(t-s)f(s)\mathrm{d}s,$$

that is $x \in X^\alpha$. Therefore X^α is convex. Consequently, X^α is nonempty, compact and convex in $C(J, \mathbb{R}^n)$. Thus from Arzela–Ascoli theorem [3] we know that $X^\alpha(t)$ is compact in $\mathbb{R}^n$ for every $t \in J$. Obviously $X^\alpha(t)$ is convex in $\mathbb{R}^n$ and we have $X^\alpha(t) \in P_{KC}(\mathbb{R}^n)$, for every $t \in J$.

Next, we want to show that α_j varies in $[0,1]$, the family $X^\alpha(t)$ forms a fuzzy set in E^n. In order to obtain this result, we need to check all the three conditions of the following theorem:

Theorem 3.1. [8] If $u \in E^n$, then

(i) $[u]^\alpha \in P_{KC}(\mathbb{R}^n)$ for all $0 \le \alpha \le 1$,

(ii) $[u]^{\alpha_2} \subset [u]^{\alpha_1}$ for all $0 \le \alpha_1 \le \alpha_2 \le 1$,

(iii) if (α_k) is a nondecreasing sequence converging to $\alpha > 0$, then $[u]^\alpha = \bigcap_{0<\alpha\le 1} [u]^{\alpha_k}$.

Proof. We have already proved the first condition of Theorem 3.1. Now we need to prove conditions (ii) and (iii) of Theorem 3.1.

Let $0 \leq \alpha_1 \leq \alpha_2 \leq 1$. Since $[U(t)]^{\alpha_2} \subset [U(t)]^{\alpha_1}$, we have $S^1_{[U(t)]^{\alpha_2}} \subset S^1_{[U(t)]^{\alpha_1}}$ and the following inclusion

$$\begin{aligned} x''_\alpha(t) &\in A(t)x_\alpha(t) + B(t)[U(t)]^{\alpha_2} + f(t) \\ &\subset A(t)x_\alpha(t) + B(t)[U(t)]^{\alpha_1} + f(t). \end{aligned}$$

Thus we obtain

$$\begin{aligned} x_{\alpha_2}(t) &\in C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)B(s)S^1_{[U(t)]^{\alpha_2}}\mathrm{d}s + \int_0^t S(t-s)f(s)\mathrm{d}s \\ &\subset C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)B(s)S^1_{[U(t)]^{\alpha_1}}\mathrm{d}s + \int_0^t S(t-s)f(s)\mathrm{d}s. \end{aligned}$$

This implies that $X^{\alpha_2} \subset X^{\alpha_1}$, and thus $X^{\alpha_2}(t) \subset X^{\alpha_1}(t)$. This implies that the condition (ii) in Theorem 3.1 is fulfilled. In order to prove the condition (iii), let (α_k) be a nondecreasing sequence converging to $\alpha > 0$. We need first to prove that $X^\alpha(t) = \bigcap_{k\geq 1} X^{\alpha_k}(t)$. Since

$$[U(t)]^\alpha = \bigcap_{k\geq 1} [U(t)]^{\alpha_k},$$

we have

$$S^1_{[U(t)]^\alpha} = S^1_{\bigcap_{k\geq 1}[U(t)]^{\alpha_k}}.$$

Thus we obtain

$$\begin{aligned} x_\alpha(t) &\in C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)B(s)\bigcap_{k\geq 1}[U(s)]^{\alpha_k}\mathrm{d}s + \int_0^t S(t-s)f(s)\mathrm{d}s \\ &\subset C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)B(s)[U(s)]^{\alpha_k}\mathrm{d}s + \int_0^t S(t-s)f(s)\mathrm{d}s, \ k \geq 1. \end{aligned}$$

Hence we have

$$X^\alpha \subset X^{\alpha_k},$$

which yields $X^\alpha \subset \bigcap_{k\geq 1} X^{\alpha_k}$. This proves one direction of inclusion. To prove other direction, let x be the solution to following the inclusions:

$$x''_{\alpha_k}(t) \in A(t)x_{\alpha_k}(t) + B(t)[U(t)]^{\alpha_k} + f(t), \quad k \geq 1.$$

Then we have

$$x(t) \in C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)B(s)S^1_{[U(t)]^{\alpha_k}}\mathrm{d}s + \int_0^t S(t-s)f(s)\mathrm{d}s$$

and thus

$$\begin{aligned} x(t) &\in C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)B(s)\bigcap_{k\geq 1} S^1_{[U(t)]^{\alpha_k}}\mathrm{d}s + \int_0^t S(t-s)f(s)\mathrm{d}s \\ &\subset C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)B(s)\mathrm{d}s S^1_{[U(t)]^\alpha}\mathrm{d}s + \int_0^t S(t-s)f(s)\mathrm{d}s. \end{aligned}$$

This means that $x \in X^{\alpha}$. Therefore $\bigcap_{k\geq 1} X^{\alpha_k} \subset X^{\alpha}$. □

Now applying Theorem 3.1, there exists $X(t) \in E^n$, $t \in J$ such that $X^{\alpha}(t)$ is a solution set to the differential inclusion (3.3)–(3.4). Thus we proved that instead of equation (3.1)–(3.2), the equations are FSOCS which can be rewritten as

$$X''(t) = A(t)X(t) + B(t)U(t) + f(t), \quad X(0) = \{x_0\}, \quad X'(0) = \{x_1\}, \tag{3.6}$$
$$Y(t) = D(t)X(t) + E(t)U(t) \tag{3.7}$$

4. OBSERVABILITY OF FUZZY SECOND ORDER CONTROL SYSTEM

Consider the FSOCS (3.6)–(3.7), the concept of observability is described with the following problem: given system (3.6)–(3.7) and its inputs and outputs over a finite interval J, calculate the range of the initial state x_0. For this purpose we give the following definition:

Definintion 4.1. The state $x_0 \neq 0$ of system (3.6)–(3.7) is said to be likely observable at level α over the interval J if the knowledge of the α-level input $U(t)$ and the α-level output $Y(t)$ over J suffice to determine the range of x_0. If the likelihood of the solution on level α reaches the maximum, then the solution (3.6)–(3.7) is said to be most likely observable at α-level.

Let $||A||$ be the norm of the matrix A. Then we have the sufficient condition for the observability of the system (3.6)–(3.7) is given by the following theorem:

Theorem 4.2. System (3.6)–(3.7) is likely observable on level α over the interval J if $E(T)C(T)$ is nonsingular. Further, let $u_0(t)$ and $y_0(t)$ be the center points of $U(t)$ and $Y(t)$, respectively, and let x_α be the possible initial point on α-level then we have the range estimation for initial value on α-level given by

$$\begin{aligned} ||x_\alpha - x_0|| \leq \; & ||[E(T)C(T)]^{-1}|| \Bigg(||E(T)|| ||S(T)|| \max_{x_\alpha \in X_\alpha} ||x_\alpha(T) - x_1(T)|| \\ & + \max_{y_\alpha(T)\in Y_\alpha(T)} ||y_\alpha(T) - y_0(T)|| + ||D(T)|| \max_{u_\alpha(T)\in U_\alpha(T)} ||u_\alpha(T) - u_0(T)|| \\ & + ||E(T)|| \max_{0\leq t\leq T} ||S(t)|| \max_{0\leq t\leq T} ||B(t)|| \int_0^T \max_{u_\alpha \in U_\alpha(t)} ||u_\alpha(t) - u_0(t)|| \mathrm{d}t \\ & + ||E(T)|| \max_{0\leq t\leq T} ||S(t)|| \max_{0\leq t\leq T} ||B(t)|| \int_0^T \max_{f_\alpha \in F_\alpha(t)} ||f_\alpha(t) - f_0(t)|| \mathrm{d}t \Bigg). \end{aligned} \tag{4.1}$$

Proof. From Section 3, for the system (3.6)–(3.7), $X_\alpha(T)$ is given by

$$X_\alpha(T) = C(T)x_0 + S(T)x_1 + \int_0^T S(T-s)B(s)U_\alpha(s)\mathrm{d}s + \int_0^T S(T-s)f_\alpha(s)\mathrm{d}s,$$

then we have

$$\begin{aligned} Y_\alpha(T) &= E(T)\left(C(T)x_0 + S(T)x_1 + \int_0^T S(T-s)B(s)U_\alpha(s)\mathrm{d}s \right.\\ &\left. + \int_0^T S(T-s)f_\alpha(s)\mathrm{d}s\right) + D(T)U_\alpha(T). \end{aligned}$$

Thus we get

$$\begin{aligned} [E(T)C(T)]x_0 &\in Y_\alpha(T) - E(T)S(T)x_1 - DU_\alpha(T) \\ &- E(T)\int_0^T S(T-s)B(s)U_\alpha(s)\mathrm{d}s - E(T)\int_0^T S(T-s)f_\alpha(s)\mathrm{d}s. \end{aligned}$$

Let x_α be the possible initial value, then we can rewrite the above as

$$\begin{aligned} [E(T)C(T)]x_\alpha(0) &\in Y_\alpha(T) - E(T)S(T)x_\alpha - E(T) - DU_\alpha(T) \\ &-E(T)\int_0^T S(T-s)B(s)U_\alpha(s)\mathrm{d}s) \\ &-E(T)\int_0^T S(T-s)f_\alpha(s)\mathrm{d}s. \end{aligned} \tag{4.2}$$

We can also have

$$\begin{aligned} [E(T)C(T)]x_0 - g(0)] &\in Y_0(T) - E(T)S(T)x_1 - D(T)u_0(T)) \\ &-E(T)\int_0^T S(T-s)B(s)u_0(s)\mathrm{d}s \\ &-E(T)\int_0^T S(T-s)f_\alpha(s)\mathrm{d}s. \end{aligned} \tag{4.3}$$

Combining (4.2) and (4.3), we can estimate the distance between x_α, x_0 and x_1 as follows:

$$\begin{aligned} ||E(T)C(T)(x_\alpha - x_0)|| \leq \max \mathrm{d}\Bigg(&\bigg[Y_0(T) - E(T)S(T)x_1 - Du_0(T) \\ &-E(T)\int_0^T S(T-s)B(s)u_0(s)\mathrm{d}s - E(T)\int_0^T S(T-s)f_0(s)\mathrm{d}s\bigg], \\ &\bigg[Y_\alpha(T) - E(T)S(T)x_\alpha - DU_\alpha(T) \\ &-E(T)\int_0^T S(T-s)B(s)U_\alpha(s)\mathrm{d}s - E(T)\int_0^T S(T-s)f_\alpha(s)\mathrm{d}s\bigg]\Bigg) \end{aligned}$$

Finally, we obtain

$$\begin{aligned}
\|x_\alpha - x_0\| \;\leq\; & \|[E(T)C(T)]^{-1}\| \Bigg(\|E(T)\|\|S(T)\| \max_{x_\alpha \in X_\alpha} \|x_\alpha(T) - x_1(T)\| \\
+ & \max_{y_\alpha(T) \in Y_\alpha(T)} \|y_\alpha(T) - y_0(T)\| + \|D(T)\| \max_{u_\alpha(T) \in U_\alpha(T)} \|u_\alpha(T) - u_0(T)\| \\
+ & \|E(T)\| \max_{0 \leq t \leq T} \|S(t)\| \max_{0 \leq t \leq T} \|B(t)\| \int_0^T \max_{u_\alpha \in U_\alpha(t)} \|u_\alpha(t) - u_0(t)\| \mathrm{d}t \\
+ & \|E(T)\| \max_{0 \leq t \leq T} \|S(t)\| \max_{0 \leq t \leq T} \|B(t)\| \int_0^T \max_{f_\alpha \in F_\alpha(t)} \|f_\alpha(t) - f_0(t)\| \mathrm{d}t \Bigg). \quad \square
\end{aligned}$$

5. COMPUTATION OF $L_\alpha(X)$

Let $\Omega \in P_{KC}(\mathbb{R}^n)$. Its Chebyshev center $c(\Omega)$ is the unique point $\bar{\omega} \in \Omega$ where the function

$$\phi_\Omega(x) = \max_{\omega \in \Omega} \|w - x\|$$

attains its global minimum. The Chebyshev radius of Ω is then

$$r(\Omega) = \max_{\omega \in \Omega} \|\omega - c(\Omega)\|.$$

Let

$$\Omega^* = \overline{co}\{\omega \in \Omega : \|\omega - c(\Omega)\| = r(\Omega)\}.$$

Proposition 5.1. (Bressan [2]) Let $\omega \in \mathbb{R}^n$, $\Omega \in P_{KC}(\mathbb{R}^n)$. Then

$$h(\omega, \Omega)^2 \leq r^2(\Omega) - \|\omega - c(\Omega)\|^2.$$

Furthermore, if $\Omega = \Omega^*$, then

$$h(\omega, \Omega)^2 = r^2(\Omega) - \|\omega - c(\Omega)\|^2.$$

A fuzzy set W is called a fuzzy box if W_α is a box in $\mathbb{R}^n$, $0 < \alpha \leq 1$, i.e.,

$$W_\alpha = [a_1^\alpha, b_1^\alpha] \times \cdots \times [a_n^\alpha, b_n^\alpha] \subset \mathbb{R}^n.$$

Let

$$U_\alpha^i(t) = [a_1^\alpha(t), b_1^\alpha(t)] \times \cdots \times [a_n^\alpha(t), b_n^\alpha(t)] \subset U_\alpha(t)$$

be the biggest box contained in $U_\alpha(t)$ and let

$$U_\alpha^0 = [c_1^\alpha(t), \mathrm{d}_1^\alpha(t)] \times \cdots \times [c_n^\alpha(t), \mathrm{d}_n^\alpha(t)] \supset U_\alpha(t)$$

be the smallest box containing $U_\alpha(t)$. Similarly let

$$Y_\alpha^i(t) = [e_1^\alpha(t), f_1^\alpha(t)] \times \cdots \times [e_n^\alpha(t), f_n^\alpha(t)] \subset Y_\alpha(t)$$

be the biggest box contained in $Y_\alpha(t)$ and let

$$Y_\alpha^0 = [g_1^\alpha, h_1^\alpha] \times \cdots \times [g_n^\alpha, h_n^\alpha] \supset Y_\alpha(t)$$

be the smallest box containing $Y_\alpha(t)$.

Proposition 5.2. If $U(t)$ is a fuzzy box for each t, then the likelihood of the solution on level α is

$$\begin{aligned} L_\alpha(X) &= \sup_{x_\alpha \in X_\alpha} \Bigg[\int_0^T r^2 \left(C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)B(s)U_\alpha(s)\mathrm{d}s \right) \mathrm{d}t \\ &- \int_0^T \left\| x_\alpha(t) - c\left(C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)B(s)U_\alpha(s)\mathrm{d}s \right) \right\|^2 \mathrm{d}t \Bigg]^{1/2} \end{aligned} \quad (5.1)$$

Proof. Since $U(t)$ is a fuzzy box for each t, then $U_\alpha(t) = U_\alpha^*(t)$. It is easy to verify that

$$\int_0^t S(t-s)B(s)U_\alpha(s)\mathrm{d}s = \left(\int_0^t S(t-s)B(s)U_\alpha(s)\mathrm{d}s \right)^*$$

hence the second equality of Proposition 5.1 holds, that is, $\Omega = \Omega^*$. This implies (5.1).

Since usually the input is not necessarily a box, we have to use the biggest and smallest boxes to approximate it, so that we can find the likelihood of the solution on each level.

6. EXAMPLE

Consider a FSOCS represented by

$$\begin{aligned} x''(t) &= \lambda x(t) + U(t) + e^{-t}, \quad x(0) = x_0, \quad x'(0) = x_1 \quad (6.1) \\ y(t) &= x(t). \quad (6.2) \end{aligned}$$

Assume that

$$U_\alpha = 1 - \alpha, \qquad Y_\alpha = \frac{1}{2} + \sqrt{1-\alpha}.$$

Let

$$A = \lambda, \quad B = C = 1, \quad D = 0 \quad \text{and} \quad T = \pi/2.$$

The solution of (6.1) at the α-level is given by

$$x_\alpha(t) \in C(t)x_0 + S'(t)x_1 + \int_0^t S'(t-s)U_\alpha \mathrm{d}s + \int_0^t S(t-s)e^{-s}\mathrm{d}s, \quad t \in [-T, T],$$

where A generates a cosine family $C(t) = \cos t$, so that $S(t) = \sin t$. Hence, the equation (6.2) becomes

$$C(t)x_0 - S(t)x_1 + \int_0^t S(t-s)e^{-s}\mathrm{d}s \in y_\alpha - \int_0^t S(t-s)U_\alpha \mathrm{d}s.$$

Let x_α be the possible initial value on level α, we can rewrite the above inclusion as

$$C(t)x_\alpha + S(t)x_1 + \int_0^t S(t-s)e^{-s}\mathrm{d}s \in y_\alpha - \int_0^t S(t-s)U_\alpha \mathrm{d}s.$$

If $\alpha \neq 1$, let u_0 and y_0 be the center points of U_α and y_α respectively, then we have

$$C(t)x(0) + S(t)x_1 + \int_0^t S(t-s)e^{-s}\mathrm{d}s \in y_0 - \int_0^t S(t-s)U_0\mathrm{d}s.$$

Thus, we have

$$C(t)[x_\alpha - x_0] = \left\{y_\alpha - \int_0^t S(t-s)U_\alpha \mathrm{d}s - \left(y_0 - \int_0^t S(t-s)U_0\mathrm{d}s\right), \quad u_\alpha \in U_\alpha\right\}.$$

Hence, we get

$$\|x_\alpha - x_0\| \leq \left(\|y_\alpha - y_0\| + T \max_{u_\alpha \in U_\alpha} \|u_\alpha - u_0\|\right).$$

Since

$$\|y_\alpha - y_0\| = (1-\alpha)^{1/2}, \qquad \|u_\alpha - u_0\| = 1 - \alpha,$$

we have

$$\|x_\alpha - x_0\| \leq (1-\alpha)^{1/2} + T(1-\alpha).$$

Figure 6.1 shows that when T becomes larger, the range becomes larger. This means that time increases the difficulty to determine the initial value. As $\alpha \to 1$,

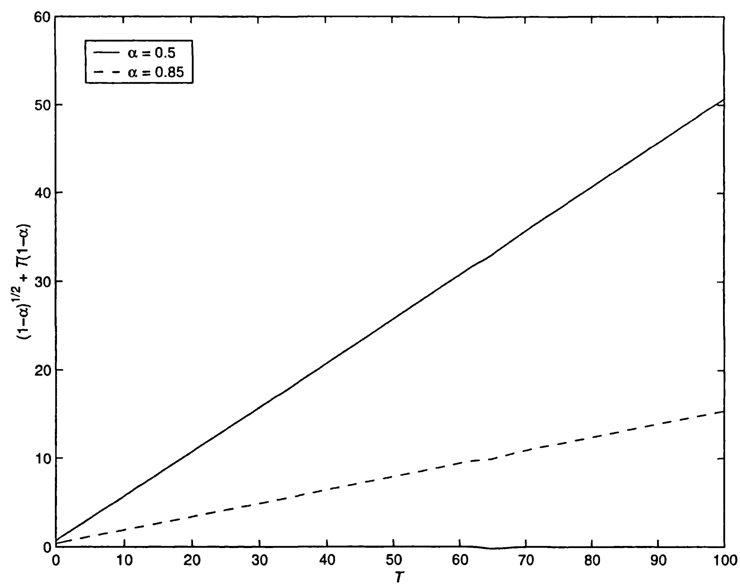

Fig. 6.1. The upper bounds of $\|x_\alpha - x_0\|$ with different T.

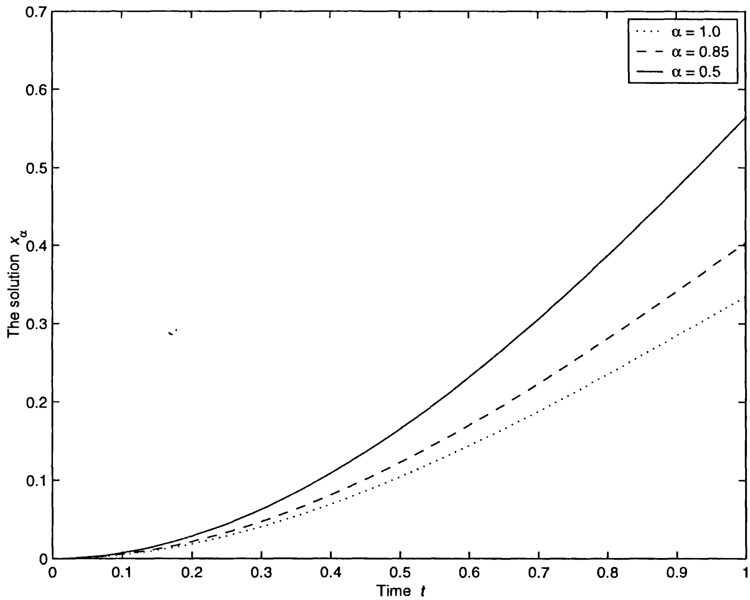

Fig. 6.2. The solution at different α-level in (6.3).

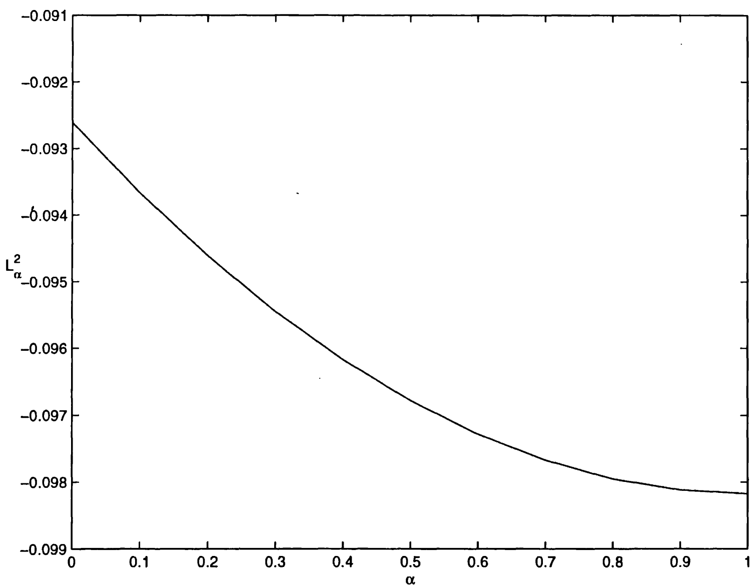

Fig. 6.3. The likelihood of the solution on different α-levels.

then $\|x_\alpha - x_0\| \to 0$, thus we will get closer and closer to our initial value x_0. Now, we determine which level is most likely to help us to determine the range of the initial value. Since

$$\begin{aligned}\cos tx_0 &+ \sin tx_1 + \int_0^t \sin(t-s)e^{-s}\mathrm{d}s + \int_0^t \sin(t-s)(1-\alpha)\mathrm{d}s \\ &= x_0\cos t + x_1 \sin t + \int_0^t \sin(t-s)e^{-s}\mathrm{d}s + (1-\alpha)\int_0^t \sin(t-s)\mathrm{d}s\end{aligned}$$

the Chebyshev radius of the L. H. S. of the above equation is

$$\begin{aligned}& r^2\left(\cos tx_0 + \sin tx_1 + \int_0^t \sin(t-s)e^{-s}\mathrm{d}s + \int_0^t \sin(t-s)(1-\alpha)\mathrm{d}s\right) \\ &= (1-\alpha)^2\left[\int_0^t \sin(t-s)\mathrm{d}s\right]^2\end{aligned}$$

and the Chebyshev center as

$$\begin{aligned}c\Big(\cos tx_0 + \sin tx_1 &+ \int_0^t \sin(t-s)e^{-s}\mathrm{d}s + \int_0^t \sin(t-s)(1-\alpha)\mathrm{d}s\Big) \\ &= x_0\cos t + x_1\sin t + \int_0^t \sin(t-s)e^{-s}\mathrm{d}s.\end{aligned}$$

Furthermore we have

$$\begin{aligned}\Big\| x_\alpha(t) - c\Big(\cos tx_0 + \sin tx_1 &+ \int_0^t \sin(t-s)e^{-s}\mathrm{d}s + \int_0^t \sin(t-s)(1-\alpha)\mathrm{d}s\Big)\Big\|^2 \\ &= \alpha_1^2 \max\left[\int_0^t \sin(t-s)\mathrm{d}s\right]^2.\end{aligned}$$

Clearly, the likelihood of the solution on α-level is given by $L_\alpha(X)$, where

$$\begin{aligned}L_\alpha(X) &= \sup_{\alpha_1}\left[\int_0^T (1-\alpha)^2\left\{\int_0^t \sin(t-s)\mathrm{d}s\right\}^2 \mathrm{d}t - \int_0^T \alpha_1^2 \max\{\sin(t-s)\mathrm{d}s\}\,\mathrm{d}t\right]^{1/2} \\ &= \sup_{\alpha_1}\left\{(1-\alpha)^2\int_0^T (1-\cos t)^2\mathrm{d}t - \alpha_1^2\int_0^T \max(1-\cos t)^2\mathrm{d}t\right\}^{1/2} \\ &= \sup_{\alpha_1}\left\{(1-\alpha)^2\int_0^T (1-2\cos t+\cos^2 t)\mathrm{d}t - 4\alpha_1^2 T\right\}^{1/2} \\ &= \sup_{\alpha_1}\left\{(1-\alpha)^2(3\pi/4-2)\mathrm{d}t - 2\alpha_1^2\pi\right\}^{1/2} \\ &= \frac{1}{16}\left\{(1-\alpha)^2(3\pi-8) - 8\pi\right\}^{1/2}.\end{aligned}$$

The solution with likelihood $\frac{1}{16}\left\{(1-\alpha)^2(3\pi-8)-8\pi\right\}^{1/2}$ is

$$\begin{aligned} x_\alpha(t) &= x_0\cos t + x_1 \sin t + \int_0^t \sin(t-s)e^{-s}\mathrm{d}s + (1-\alpha)(1-\cos t) \\ &= (x_0 - \frac{1}{2})\cos t + (x_1 + \frac{1}{2})\sin t + \frac{1}{2}e^{-t} + (1-\alpha)(1-\cos t). \qquad (6.3) \end{aligned}$$

Figure 6.2 is the solution when $\alpha = 0.5$, 0.85 and 1 in (6.3). Figure 6.3 shows the likelihood of the solution on different α-levels.

ACKNOWLEDGEMENT

This work was supported by grant No. 2000-1-10300-003-2 from the Basic Search Program of the Korea Science and Engineering Foundation. Prof. Jong Yeoul Park was supported by an BK21 project.

(Received October 15, 2001.)

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