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MINIMAL POSITIVE REALIZATIONS: A SURVEY OF RECENT RESULTS AND OPEN PROBLEMS

LUCA BENVENUTI AND LORENZO FARINA

In this survey paper some recent results on the minimality problem for positive realizations are discussed. In particular, it is firstly shown, by means of some examples, that the minimal dimension of a positive realization of a given transfer function, may be much “larger” than its McMillan degree. Then, necessary and sufficient conditions for the minimality of a given positive realization in terms of positive factorization of the Hankel matrix are given. Finally, necessary and sufficient conditions for a third order transfer function with distinct real positive poles to have a third order positive realization are provided and some open problems related to minimality are discussed.

Keywords: positive systems, positive realizations

AMS Subject Classification: 93B27, 06F05

1. INTRODUCTION

The area of research presented in this paper was motivated by an interest of the authors in the field of *positive linear systems* (see, for general overviews, [8, 11, 19]). Positive systems are, by definition, systems whose variables can take only nonnegative values. From a general point of view, they should be considered as very particular. From a practical point of view, however, such systems are anything but particular since positive systems are often encountered in applications.

In fact, positive systems are, for instance, networks of reservoirs, industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems (memories, warehouses, ...), promotional systems, compartmental systems (frequently used when modelling transport and accumulation phenomena of substances in human body), water and atmospheric pollution models, stochastic models where state variables must be nonnegative since they represent probabilities and many other models commonly used in economy and sociology. Just to cite the most popular, consider the Leontieff model used by economists for predicting productions and prices [17], the Leslie model used to study age-structured population dynamics [18], the hidden Markov models [26] mainly adopted for speech recognition, the compartmental models [15] commonly encountered in pharmacokinetics and radio-nuclide tracer dynamics, the birth and death processes, relevant to the

analysis of queueing systems [27] and phase-type distributions [9, 22, 23]. More recently, it has been developed a MOS-based technology for discrete-time filtering, the so called “charge routing networks” (CRN), in which the state variables are positive since they represent quantity of electrical charge [3, 13]. Last, but not least, it’s worth noting the applications of the realization theory of positive systems to the design of fiber optic filters [6].

In recent times a number of issues regarding positive systems has been addressed (such as, for example, positive orthant reachability [28]) but, for a number of reasons, the so-called positive realization problem has been the most studied (see, for example, the references cited in [2, 10, 12, 21]). In the following we will restrict ourselves to consider only the SISO discrete-time case. The formulation of this problem is as follows.

The positive realization problem [1, 24] Given a strictly proper rational transfer function $G(z)$, the triple $\{A, b, c^T\}$ is said to be a *positive realization* if

$$G(z) = \sum_{k \geq 1} c^T A^{k-1} b z^{-k}$$

with A, b, c^T nonnegative (i. e. with nonnegative entries). The positive realization problem consists of providing answers to the questions:

- (*The existence problem*) Is there a positive realization $\{A, b, c^T\}$ of some finite dimension N and how it may be found?
- (*The minimality problem*) What is a minimal value for N ?
- (*The generation problem*) How can we generate all possible positive realizations?

In references [2, 10], the existence problem has been completely solved and a means of constructing such realizations is there also given. In this paper we shall not consider the interesting question of characterizing the relationship between equivalent realizations, and we shall concentrate on what we have termed the *minimality problem* (see, for example [4, 25, 29]).

In this paper firstly, we will show that the minimal dimension of a positive realization of a given transfer function, may be much “larger” than its McMillan degree. Secondly, we will present necessary and sufficient conditions for the minimality of a given positive realization in terms of positive factorization of the Hankel matrix. Finally, we will provide necessary and sufficient conditions for a third order transfer function with distinct real positive poles to have a third order positive realization and some open problems related to minimality.

2. NO RESTRICTIONS ON THE SIZE

It is known that, for transfer functions of degree 1 or 2, nonnegativity of the impulse response is a necessary and sufficient condition for the existence of a positive realization. Moreover, in those two cases, the minimal dimension of a positive realization

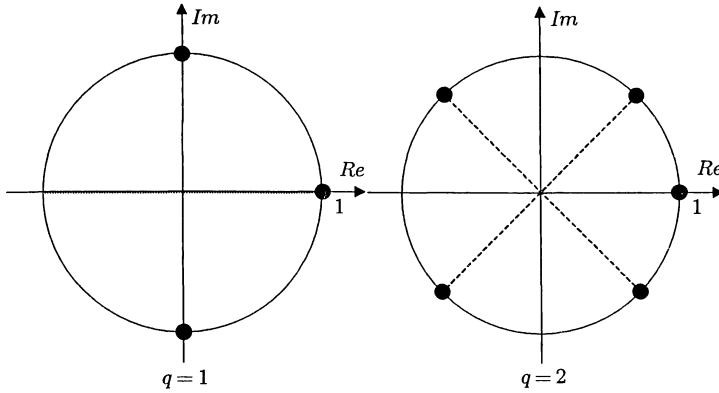


Fig. 1. Poles pattern of $G(z, q)$ for $q = 1$ and $q = 2$.

coincides with the degree of the transfer function [24]. On the other hand, the situation for the case of transfer functions of degree $n > 2$ is totally different. To show that the minimality problem for positive linear systems is inherently different from that of ordinary linear systems we shall make use of the following three examples.

Example 1. Consider the following positive realization

$$A = \begin{pmatrix} \overbrace{0 & 0 & \dots & 0 & 0 & 1}^{2^{q+1}} \\ 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & & & 0 & 0 \\ 0 & 0 & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \tag{1}$$

$$c^T = \left(\underbrace{0 \quad \dots \quad 0}_{2^q} \quad \underbrace{1 \quad \dots \quad 1}_{2^q} \right)$$

where the parameter q is an integer greater than or equal to 1. The dimension of the realization is $N(q) = 2^{q+1}$ while the corresponding transfer function

$$G(z, q) = \frac{1}{(z - 1)(z^{2^q} + 1)}, \quad q \geq 1$$

is of McMillan degree $n(q) = 2^q + 1$. By exploiting the rotational symmetry of the dominant poles of $G(z, q)$, we can prove that for any integer $q \geq 1$, the realization (1) is minimal as a positive linear system.

To see this, note that since $e^{\pi/2^q}$ and 1 are dominant poles of $G(z, q)$ (see Figure 1 for the cases $q = 1$ and $q = 2$), then from the Frobenius theorem, the spectrum of the dynamic matrix of any positive realization must remain unchanged under a rotation of $\pi/2^q$ radians. This implies that all the 2^{q+1} th roots of unity must belong to the spectrum of the dynamic matrix of any positive realization. Consequently, the minimal dimension for a positive realization of $G(z, q)$ is not smaller than 2^{q+1} .

Note that the difference between the dimension $N(q)$ of the minimal positive realization of the system and the corresponding transfer function McMillan degree $n(q)$

$$N(q) - n(q) = 2^q - 1$$

goes exponentially to ∞ as q increases.

For the sake of illustration consider the case $q = 1$. Then

$$G(z, 1) = \frac{1}{(z - 1)(z^2 + 1)}$$

and

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad c = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

Since the poles of $G(z, 1)$ are 1, i and $-i$, then the dynamic matrix of any minimal positive realization must include also the eigenvalue -1 ; consequently the above realization is minimal as a positive system.

As will appear in the sequel, this rotational symmetry of the spectrum of a nonnegative matrix, due to the specific dominant poles pattern, is not the only reason for non minimality (i. e. not jointly reachable and observable) in the positive realization problem.

Roughly speaking, we show next that the dimension of a positive realization may be “large” although the dominant eigenvalue is unique, so that no symmetry of the spectrum is required by the Frobenius theorem.

In fact, since a nonnegative matrix cannot have arbitrary eigenvalues, then the non dominant poles also have limitations. For this consider the sets Θ_n^ρ denoting the set of points in the complex plane that are eigenvalues of nonnegative $n \times n$ matrices with spectral radius ρ (see [20]). A full characterization of these sets has been given by Karpelevich [16]. For example, the set Θ_2^ρ consists of points on the segment $[-\rho, \rho]$ and the set Θ_3^ρ consists of points in the interior and on the boundary of the triangle with vertices $\rho, \rho e^{2\pi i/3}, \rho e^{4\pi i/3}$ and on the segment $[-\rho, \rho]$.

The sets Θ_3^ρ and Θ_4^ρ are depicted in Figure 2.

Example 2. Consider the following positive realization

$$A = \begin{pmatrix} 0 & 0.95 & 0 & 0.05 \\ 0.05 & 0 & 0.95 & 0 \\ 0 & 0.05 & 0 & 0.95 \\ 0.95 & 0 & 0.05 & 0 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad c = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \quad (2)$$

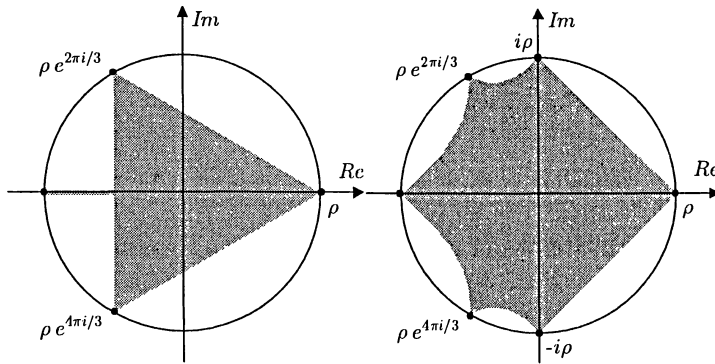


Fig. 2. The sets Θ_3^ρ and Θ_4^ρ .

of dimension 4. The corresponding transfer function

$$G(z) = \frac{*}{(z-1)(z^2+0.81)}$$

is of McMillan degree 3. Since the poles of $G(z)$ are $1, \pm 0.9i$ and they lie inside Θ_4^1 and not in Θ_3^1 , then the dynamic matrix of any minimal positive realization must be of dimension greater than 3. Therefore the fourth order positive realization (2) is minimal as a positive system.

This last mechanism, related to a specific poles pattern, is – again – not the only reason for non minimality in the positive realization problem even when the dominant eigenvalue is unique. In fact, the dimension of a positive realization may be “large” although the dominant eigenvalue is unique and no complex eigenvalues are present. This should be not surprising since positivity of the system implies restrictions not only on the dynamic matrix but on the input and output vectors also. The next theorem formalizes these restrictions from a geometric point of view:

Theorem 1. (Ohta et al [24]) Let $G(z)$ be a strictly proper rational transfer function of McMillan degree n and let $\{F, g, h^T\}$, with $F \in \mathbb{R}^{n \times n}$ and $g, h \in \mathbb{R}^n$ be a minimal (i. e. jointly reachable and observable) realization of $G(z)$. Then, $G(z)$ has a positive realization, if and only if there exists a polyhedral proper cone \mathcal{K} such that

1. $F\mathcal{K} \in \mathcal{K}$, i. e. \mathcal{K} is F -invariant;
2. $\mathcal{K} \in \mathcal{O}$;
3. $g \in \mathcal{K}$

where $\mathcal{O} = \{x \in \mathbb{R}^n | h^T F^k x \geq 0, k = 0, 1, \dots\}$ is called the observability cone.

As stated above, this theorem provides a geometrical interpretation of the positive realization problem: given any realization of a transfer function, then to any positive realization corresponds an invariant cone \mathcal{K} satisfying conditions 1–3 and vice versa. Moreover, the number of edges of the cone \mathcal{K} equals the dimension of the positive realization, so that the minimality problem can be equivalently stated as the problem of finding the invariant polyhedral proper cone \mathcal{K} with the minimal number of edges contained in the observability cone \mathcal{O} and containing the vector g .

We are now able to present the promised example, i. e. a “new” mechanism which leads to a positive realization of dimension larger than it’s McMillan degree.

Example 3. Consider the following positive realization:

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & \frac{63+4\sqrt{26}}{85} & 0 & 0 \\ 0 & \frac{22-4\sqrt{26}}{85} & \frac{63-4\sqrt{26}}{85} & 0 \\ 0 & 0 & \frac{22+4\sqrt{26}}{85} & 0 \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad c = \begin{pmatrix} 6 \\ 0 \\ 0 \\ 51 \end{pmatrix} \quad (3)$$

of dimension 4. The corresponding transfer function

$$G(z) = \frac{1}{z - 1} - \frac{25}{z - 0.4} + \frac{75}{z - 0.2} \quad (4)$$

is of McMillan degree 3 and has three real positive poles. Moreover, the impulse response of the system has two elements of the sequence equal to zero, that is

$$c^T A^2 b = c^T A^3 b = 0.$$

Suppose then that there exists a third order positive realization of $G(z)$. Hence, by taking any minimal (i. e. jointly reachable and observable) realization $\{F, g, h^T\}$ of $G(z)$, there exists a cone \mathcal{K} with three edges satisfying the conditions of Theorem 1. From conditions 1 and 3, it follows that $F^k g \in \mathcal{K}$ for $k = 0, 1, \dots$. Moreover, since in this case $h^T F^2 g = h^T F^3 g = 0$, then the following hold:

$$\begin{array}{lll} h^T(g) > 0 & h^T(Fg) > 0 & h^T(F^2g) = 0 \\ h^T F(g) > 0 & h^T F(Fg) = 0 & h^T F(F^2g) = 0 \\ h^T F^2(g) = 0 & h^T F^2(Fg) = 0 & h^T F^2(F^2g) > 0 \\ h^T F^3(g) = 0 & h^T F^3(Fg) > 0 & h^T F^3(F^2g) > 0 \\ h^T F^k(g) > 0 & h^T F^k(Fg) > 0 & h^T F^k(F^2g) > 0 \end{array}$$

for $k = 4, 5, \dots$. Consequently, the three vectors g, Fg and F^2g lie on different edges of the observability cone. Then, from condition 2 of Theorem 1, i. e. $\mathcal{K} \in \mathcal{O}$, the vectors g, Fg and F^2g are necessarily edges of \mathcal{K} so that \mathcal{K} is the polyhedral closed convex cone consisting of all finite nonnegative linear combinations of vectors g, Fg and F^2g , i. e.

$$\mathcal{K} = \text{cone}(g, Fg, F^2g).$$

Since $F^3g \notin \mathcal{K}$, then \mathcal{K} is not F -invariant, thus arriving at a contradiction. Therefore the fourth order positive realization (3) is minimal as a positive system.

Using similar arguments, it has been shown in reference [5] that the transfer function

$$G(z, N) = \frac{1}{z - 1} - 25 \cdot \frac{0.4^{4-N}}{z - 0.4} + 75 \cdot \frac{0.2^{4-N}}{z - 0.2}$$

admits a minimal positive realization of state space dimension not smaller than N , where the parameter N is an integer greater than or equal to 4. This is quite surprising since, in spite of the fact that we are dealing with the seemingly simple case of a third order transfer function with distinct positive real poles, the minimal positive realization may possibly have a “large” state space dimension.

3. DOES POSITIVE FACTORIZATION SUFFICE?

A well known result from system theory states that the minimal inner size of a factorization of the Hankel matrix H gives the minimal order of a realization. Since, obviously, the impulse response of a positive system is nonnegative, i. e. $g_k \geq 0$, then H has nonnegative entries. Moreover, given a minimal positive realization (A, b, c) of order N , the following hold

$$H = \begin{pmatrix} c^T b & c^T A b & c^T A^2 b & \dots \\ c^T A b & c^T A^2 b & c^T A^3 b & \dots \\ c^T A^2 b & c^T A^3 b & \ddots & \\ \vdots & \vdots & & \ddots \end{pmatrix} = \begin{pmatrix} c^T \\ c^T A \\ c^T A^2 \\ \vdots \end{pmatrix} (b \quad A b \quad A^2 b \quad \dots) = RS$$

where $R \in \mathbb{R}_+^{\infty \times N}$ and $S \in \mathbb{R}_+^{N \times \infty}$. As a consequence of the previous considerations, it is interesting to study whether the factorization of H into two nonnegative matrices (called *positive factorization*) of minimal inner size N implies the minimal order of a positive realization to be N . We show next, by means of an example, that this is not true.

Example 4. Consider the system with nonnegative impulse response $g_{1+4i} = 10$, $g_{2+4i} = 8$, $g_{3+4i} = 6$, $g_{4+4i} = 8$ for $i = 0, 1, \dots$, described by

$$\begin{cases} x(k+1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} x(k) + \begin{pmatrix} 8 \\ 1 \\ 1 \end{pmatrix} u(k) \\ y(k) = (1 \quad 1 \quad 1) x(k) \end{cases}$$

The dynamic matrix of the system has eigenvalues at 1, i and $-i$ so that from the Frobenius theorem, a third order positive realization does not exist. Nevertheless, a positive factorization of the Hankel matrix having inner size equal to 3 does exist.

To see this consider the Hankel matrix of the system

$$H = \begin{pmatrix} H_{11} & H_{12} & \dots \\ H_{21} & \ddots & \\ \vdots & & \ddots \end{pmatrix} \quad H_{ij} \in \mathbb{R}^{4 \times 4}$$

where, being the impulse response cyclic, $H_{ij} = H_{hk}$ for every integer i, j, h, k and

$$H_{ij} = \begin{pmatrix} 10 & 8 & 6 & 8 \\ 8 & 6 & 8 & 10 \\ 6 & 8 & 10 & 8 \\ 8 & 10 & 8 & 6 \end{pmatrix}.$$

Matrix H_{ij} can be factorized as

$$H_{ij} = \begin{pmatrix} 5 & 3 & 3 \\ 4 & 2 & 6 \\ 3 & 5 & 5 \\ 4 & 6 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} = PQ$$

so that

$$H = \begin{pmatrix} P \\ P \\ \vdots \end{pmatrix} (Q \quad Q \quad \dots).$$

We state now a necessary and sufficient condition [4] for the solution of the positive realization problem in terms of positive factorization of the Hankel matrix.

Theorem 2. The minimal state space dimension of a positive realization of a given transfer function $G(z)$ is the least integer N for which there exist nonnegative matrices $P \in \mathbb{R}_+^{n \times N}$, $S \in \mathbb{R}_+^{N \times \infty}$ and $U \in \mathbb{R}_+^{N \times N}$ such that

1. $H_t = PS$
2. $A_M P = PU$

where $H_t \in \mathbb{R}_+^{n \times \infty}$ is truncated Hankel matrix defined as

$$H = \begin{pmatrix} H_t \\ \vdots \end{pmatrix}$$

and A_M is the (Markov) canonical companion matrix of

$$G(z) = \frac{b_{n-1}z^{n-1} + b_{n-2}z^{n-2} + \dots + b_0}{z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \dots + a_0}$$

$$A_M = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{pmatrix}.$$

4. THIRD ORDER: THE MIST IS LIFTING

In this Section we gain partial insight into the positive minimality problem in the case of third order transfer functions. We shall also restrict attention to transfer functions with three distinct positive real poles. In this case, in reference [7], the following result is proved:

Theorem 3. Let

$$G(z) = \frac{r_1}{z - \lambda_1} + \frac{r_2}{z - \lambda_2} + \frac{r_3}{z - \lambda_3}$$

be a third order transfer function (i. e. $r_1, r_2, r_3 \neq 0$) with distinct positive real poles $\lambda_1 > \lambda_2 > \lambda_3 > 0$. Then, $G(z)$ has a third order positive realization if and only if the following conditions hold:

1. $r_1 > 0$
2. $r_1 + r_2 + r_3 \geq 0$
3. $(\lambda_1 - \bar{\eta}) r_1 + (\lambda_2 - \bar{\eta}) r_2 + (\lambda_3 - \bar{\eta}) r_3 \geq 0$
4. $(\lambda_1 - \eta)^2 r_1 + (\lambda_2 - \eta)^2 r_2 + (\lambda_3 - \eta)^2 r_3 \geq 0$ for all η such that $\bar{\eta} \leq \eta \leq \lambda_3$

where $\bar{\eta} = \max \left\{ 0, \frac{\lambda_1 + \lambda_2 + \lambda_3 - 2\sqrt{(\lambda_2 - \lambda_3)^2 + (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}}{3} \right\}$.

It is worth noting that the proof of the previous result, as presented in [7], is mainly geometric and heavily relies on the third-order assumption. For this reason, it appears very difficult to us extending that kind of proof to the higher order case. Nevertheless, this geometric approach may be fruitfully applied to the case in which either the assumption on the poles location is removed or the order of the minimal positive realization is not limited to equal the McMillan degree.

Example 5. Consider the transfer function (4) corresponding to the minimal positive fourth order realization (3). In this case $r_1 = 1, r_2 = -25, r_3 = 75, \lambda_1 = 1, \lambda_2 = 0.4, \lambda_3 = 0.2$ so that $\bar{\eta} = 0.0526$. It is easy to check that condition 4 of Theorem 3 does not hold. More precisely,

$$(\lambda_1 - \eta)^2 r_1 + (\lambda_2 - \eta)^2 r_2 + (\lambda_3 - \eta)^2 r_3 < 0$$

for all η such that $\bar{\eta} \leq \eta \leq \lambda_3$. Then, as shown in Example 3, a third order positive realization does not exist. It is worth noting that when considering – for example – the same transfer function with $r_2 \geq -16$, all conditions of Theorem 3 hold so that a third order positive realization exists. Then, for $r_2 < -16$ and $r_1 = 1$, $r_3 = 75$, we know that a third order positive realization does not exist and that for $r_2 = 25$ there is a fourth order positive realization. It is clear that further investigations are needed in order to clarify this intriguing situation.

5. OPEN PROBLEMS

As it is clear from the issues so far discussed, there are a considerable number of open problems related to minimality for positive systems. We just name a few of them. First of all, it is not clear what kind of mathematical “instruments” should be used to effectively tackle this problem. In fact, the geometric approach (i. e. that of working with invariant cones) has proved to be the right choice for determining the existence of a positive realization. By contrast, such approach, has lead to the determination of necessary and sufficient conditions for the third order case only. A different formulation, such as the factorization approach proposed by Picci and van Schuppen in reference [30], can be a viable and promising possibility.

Another important issue related to minimality of positive systems is the study of “hidden modes”, i. e. of the eigenvalues which possibly one has to add in order to obtain a minimal positive realization. A full characterization of this property may lead to a deeper and valuable insight into the problem. An obvious – but important – question is that of minimality of positive realizations for continuous–time systems. Lastly, we mention the MIMO case, which is not a straightforward extension of the SISO case, as for the existence problem.

Some other open problems related to minimality of positive systems are listed below:

- how are all positive minimal realizations connected?
- how can one simply figure, directly from the system’s parameters (say, residues and eigenvalues), the minimum number of samples of the impulse response to be checked in order to infer nonnegativity of the whole impulse response and how is this number related to minimality?
- how can one approximate a positive realization by a lower dimension one?
- find “tight” lower and upper bounds to minimal order of a positive realization¹.

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