Kybernetika

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Kybernetika, Vol. 39 (2003), No. 1, [75]--98

Persistent URL: `http://dml.cz/dmlcz/135509`

KYBERNETIKA — VOLUME 39 (2003), NUMBER 1, PAGES 75–98

ON CONTINUOUS CONVERGENCE AND EPI-CONVERGENCE OF RANDOM FUNCTIONS
Part I: Theory and Relations

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Continuous convergence and epi-convergence of sequences of random functions are crucial assumptions if mathematical programming problems are approximated on the basis of estimates or via sampling. The paper investigates "almost surely" and "in probability" versions of these convergence notions in more detail. Part I of the paper presents definitions and theoretical results and Part II is focused on sufficient conditions which apply to many models for statistical estimation and stochastic optimization.

Keywords: continuous convergence, epi-convergence, stochastic programming, stability
AMS Subject Classification: 90C15, 90C31, 60B10

1. INTRODUCTION

Often a decision maker has to deal with a programming problem which contains unknown parameters. Then, usually, he will estimate the parameters and solve the surrogate problem obtained in this way. And he hopes that the solution of the surrogate problem is a good approximation to the solution of the true problem. Thus there is a need for conditions ensuring that this hope is justified, conditions on the form of the true problem and on the behavior of the estimates.

There are many papers dealing with the approximation of mathematical programming problems. Especially stability theory of parametric programming and the theory of epi-convergence yield a lot of helpful results (cf. [1, 4, 16]).

When the surrogate problems are random – as in the case of estimated parameters – additional considerations are necessary to adopt the deterministic results to the random setting.

This problem was – for the almost surely case – mainly dealt with in the framework of stochastic programming and Markovian decision processes. Meanwhile a lot

[1]The research has been partially supported by Deutsche Forschungsgemeinschaft under grant No. 436TSE113/40.

[2]The research has been partially supported by the Ministry of Education, Youth and Sports of the Czech Republic under Project MSM 113200008 and by the Grant Agency of the Czech Republic under grant No. 201/02/0621.

of qualitative and quantitative results are available (cf. [2, 3, 7, 8, 10, 13, 14, 15, 17, 19, 20, 21, 22, 25, 26, 28, 29, 30, 31]).

The methods of investigation are usually adjusted to the special framework. For instance the approximation of the true probability measure by the empirical measure based on independent samples or the availability of consistent estimates are often employed.

But there are also many problems where weakly consistent estimates or dependent samples are only accessible. Then, mostly, one cannot deal with the a. s. setting, and one may ask for weaker convergence notions such as (semi)convergence in probability for optimal values and optimal solution sets.

In [11, 12, 26] special large deviations results are given, which offer the possibility to derive statements on convergence in probability. General stability statements in terms of convergence in probability are proved in [27].

The similarity of the results in [27] to the "almost surely" case gave reason to consider the relations between convergence almost surely and in probability in more detail. The investigations are done for one-sided forms of epi- and/or continuous convergence. The results can be combined in several forms to derive statements for epi-convergence or continuous convergence as well.

When approximating optimization problems with constraints, in general, it is not necessary to impose convergence of the objective functions on the whole domain. Therefore we consider convergence restricted to a convergence region X.

We prove equivalent characterizations for so-called lower (or upper) semicontinuous approximations and epi-upper approximations almost surely at X which pave the way for the examination of the connections between convergence almost surely and in probability. They show immediately that the different notions for convergence almost surely imply those for convergence in probability. Furthermore, it is clarified to what extent convergence in probability can be characterized by convergence almost surely of subsequences. Roughly spoken, if lower or upper semicontinuous approximations are considered on the whole domain and for epi-upper approximations, convergence in probability is equivalent to the fact that each subsequence contains a subsequence which converges almost surely in the sense under consideration. However, this is no longer true if semicontinuous convergence is restricted to a non-trivial subset X.

The connection between epi-convergence of a sequence of functions and the behavior of corresponding minimal values and sets of "argmins" is well investigated and utilized in many papers on stability in stochastic programming. Implications which may be drawn if half-sided approximations only are assumed are scattered in the literature (cf. [4, 16, 27]). In order to make the present paper self-contained, corresponding results are proved independently.

2. DESCRIPTION OF THE CONSIDERED PROBLEM

Let a complete probability space $[\Omega, \mathcal{A}, P]$ be given and suppose that a random optimization problem

$$(\mathbb{P}_0) \qquad \min_{x\in\Gamma_0(\omega)} f_0(x,\omega)$$

is approximated by a sequence of surrogate problems

$$(\mathbb{P}_n) \qquad \min_{x\in\Gamma_n(\omega)} f_n(x,\omega), \quad n\in\mathbb{N},\ \omega\in\Omega,$$

where $\Gamma_n|\ \Omega \to 2^{\mathbb{R}^p}$, $n \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}$, denotes a multifunction with measurable graph, i.e. Graph $\Gamma_n \in \mathcal{A}\otimes\Sigma^p$, and the function $f_n|\ \mathbb{R}^p\times\Omega \to \overline{\mathbb{R}}$, $n\in\mathbb{N}_0$, is supposed to be $(\Sigma^p\otimes\mathcal{A},\ \overline{\Sigma})$-measurable. Here Σ denotes the σ-field of Borel sets of $\mathbb{R}$ and $\overline{\Sigma}$ is the σ-field of Borel sets of $\overline{\mathbb{R}}$, i.e. generated by Σ and $\{+\infty\},\{-\infty\}$. Consequently, Σ^p denotes the σ-field of Borel sets of $\mathbb{R}^p$.

Although our main interest is in deterministic original problems, which are approximated relying on estimates, we here allow for random original problems in order to show that the relations between the convergence notions under consideration also hold for random original problems. Furthermore, random original problems occur if one deals with stochastic processes.

We usually write the "full" form $f_n(x,\cdot)$ instead of $f_n(x)$ for random functions (and $f_n(x,\omega)$ for the realizations), because we sometimes deal with random functions and deterministic functions simultaneously and hence have to distinguish clearly between them.

The constraint set Γ_n may be specified by inequality constraints:

$\Gamma_n(\omega) = \{x\in\mathbb{R}^p|\ g_n^j(x,\omega)\le 0, j\in J\}$, where the functions $g_n^j|\ \mathbb{R}^p\times\Omega\to\mathbb{R}$, $n\in\mathbb{N}_0$, $j\in J$ have to satisfy the same measurability conditions as f_n, J is a countable index set.

By $\tilde{f}_n$ we denote the modified objective functions

$$\tilde{f}_n(x,\ \omega) := \begin{cases} f_n(x,\ \omega) & \text{if } x\in\Gamma_n(\omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.1}$$

Having graph of Γ_n measurable, the function $\tilde{f}_n$ is $(\Sigma^p\otimes\mathcal{A},\overline{\Sigma})$-measurable. Obviously, f_n can be regarded as a modified objective function. Therefore, in the following, we shall introduce continuous convergence, epi-convergence and a concept of approximations (almost surely, in probability or in the deterministic sense) for the functions f_n, $n\in\mathbb{N}$.

3. ALMOST SURE CONVERGENCE OF RANDOM FUNCTIONS

Let us start with deterministic functions.

Definition 3.1. Let $\{h_n,\ n\in\mathbb{N}_0\}$ be a family of deterministic functions $h_n|\mathbb{R}^p\to\overline{\mathbb{R}}$. By $EL_*h_n(x_0)$ we denote the epi-limes inferior of $(h_n)_{n\in\mathbb{N}}$ at $x_0\in\mathbb{R}^p$ and by $EL^*h_n(x_0)$ the epi-limes superior:

$$EL_*h_n(x_0) := \sup_{V\in\mathcal{N}(x_0)} \liminf_{n\to+\infty} \inf_{x\in V} h_n(x), \tag{3.1}$$

$$EL^*h_n(x_0) := \sup_{V\in\mathcal{N}(x_0)} \limsup_{n\to+\infty} \inf_{x\in V} h_n(x). \tag{3.2}$$

(We denote by $\mathcal{N}(x_0)$ the neighborhood system of x_0.)

These two limits can be equivalently described using convergent sequences.

Lemma 3.1. Let $\{h_n,\ n \in \mathbb{N}_0\}$ be a family of deterministic functions $h_n|\ \mathbb{R}^p \to \overline{\mathbb{R}}$. Then

$$\liminf_{n\to+\infty} h_n(x_n) \geq EL_* h_n(x_0) \text{ for each sequence } x_n \to x_0 \text{ and} \tag{3.3}$$
$$\text{there is a sequence } \hat{x}_n \to x_0 \text{ such that } \liminf_{n\to+\infty} h_n(\hat{x}_n) = EL_* h_n(x_0),$$
$$\limsup_{n\to+\infty} h_n(x_n) \geq EL^* h_n(x_0) \text{ for each sequence } x_n \to x_0 \text{ and} \tag{3.4}$$
$$\text{there is a sequence } \tilde{x}_n \to x_0 \text{ such that } \limsup_{n\to+\infty} h_n(\tilde{x}_n) = EL^* h_n(x_0).$$

The epi-limes inferior and the epi-limes superior of a sequence of functions provide us with a powerful tool for the problem $(\mathbb{P}_n)$ investigation. Let us define notions convenient for our task. The notation was chosen because of the close relationship to the lower semicontinuity of a function of two variables, see [10], [16].

Definition 3.2. A sequence $(h_n)_{n\in\mathbb{N}}$ satisfying the inequality

$$EL_* h_n(x_0) \geq h_0(x_0) \tag{3.5}$$

at a point x_0 will be called a *lower semicontinuous approximation to h_0 at x_0*, we shall abbreviate this property by $h_n \xrightarrow[\{x_0\}]{\mathrm{l}} h_0$.

Definition 3.3. A sequence $(h_n)_{n\in\mathbb{N}}$ will be called an *upper semicontinuous approximation to h_0 at x_0* $\left(h_n \xrightarrow[\{x_0\}]{\mathrm{u}} h_0\right)$ if (3.5) is satisfied for $\{-h_n,\ n \in \mathbb{N}_0\}$.

Definition 3.4. A sequence $(h_n)_{n\in\mathbb{N}}$ fulfilling the relation

$$EL^* h_n(x_0) \leq h_0(x_0), \tag{3.6}$$

$(h_n)_{n\in\mathbb{N}}$ is called an *epi-upper approximation to h_0 at x_0* $\left(h_n \xrightarrow[\{x_0\}]{\text{epi-u}} h_0\right)$.

Definition 3.5. A sequence $(h_n)_{n\in\mathbb{N}}$ with $h_n \xrightarrow[\{x_0\}]{\mathrm{l}} h_0$ and $h_n \xrightarrow[\{x_0\}]{\mathrm{u}} h_0$ is continuously convergent to h_0 at x_0 $\left(h_n \xrightarrow[\{x_0\}]{} h_0\right)$, and a sequence satisfying (3.5) and (3.6) is epi-convergent to h_0 at x_0 $\left(h_n \xrightarrow[\{x_0\}]{\text{epi}} h_0\right)$.

The above definitions are formulated for single points, only. Using a natural idea we can extend them for subsets of $\mathbb{R}^p$.

Definition 3.6. For X subset of $\mathbb{R}^p$, we define

$$h_n \xrightarrow[X]{1} h_0 \iff \forall x_0 \in X : \; h_n \xrightarrow[\{x_0\}]{1} h_0, \tag{3.7}$$

$$h_n \xrightarrow[X]{u} h_0 \iff \forall x_0 \in X : \; h_n \xrightarrow[\{x_0\}]{u} h_0, \tag{3.8}$$

$$h_n \xrightarrow[X]{\text{epi-u}} h_0 \iff \forall x_0 \in X : \; h_n \xrightarrow[\{x_0\}]{\text{epi-u}} h_0, \tag{3.9}$$

$$h_n \xrightarrow[X]{\text{epi}} h_0 \iff \forall x_0 \in X : \; h_n \xrightarrow[\{x_0\}]{\text{epi}} h_0, \tag{3.10}$$

$$h_n \xrightarrow[X]{} h_0 \iff \forall x_0 \in X : \; h_n \xrightarrow[\{x_0\}]{} h_0. \tag{3.11}$$

The lower semicontinuous approximation is connected with continuity of a function.

Definition 3.7. Let $X \subset \mathbb{R}^p$. A function $h|\ \mathbb{R}^p \to \overline{\mathbb{R}}$ is called

lower semicontinuous (l.s.c.) at X $\iff$ $\forall x_0 \in X : EL_* h(x_0) \geq h(x_0)$ (or equivalently $h \xrightarrow[X]{1} h$);

upper semicontinuous (u.s.c.) at X $\iff$ $-h$ is l.s.c. at X;

continuous at X being both l.s.c. and u.s.c. at X (or equivalently $h \xrightarrow[X]{} h$);

lower semicontinuous (l.s.c.) on X $\iff$ $\forall x_0 \in X$ $\sup\limits_{V \in \mathcal{N}(x_0)} \liminf\limits_{n \to +\infty} \inf\limits_{x \in V \cap X} h(x) \geq h(x_0)$;

upper semicontinuous (u.s.c.) on X $\iff$ $-h$ is l.s.c. on X;

continuous on X being both l.s.c. and u.s.c. on X.

The above defined approximations are closely related to stability of optimal value and optimal solutions of an optimization problem. They provide a deeper analysis of the stability problem than epi-convergence itself.

Definition 3.8. For a deterministic function $h|\ \mathbb{R}^p \to \overline{\mathbb{R}}$ we denote

$$\varphi(h) = \inf_{x \in \mathbb{R}^p} h(x) \tag{3.12}$$

and for each $\alpha \in \overline{\mathbb{R}}$ we define

$$\text{level}_\alpha(h) = \{x \in \mathbb{R}^p : (x) \leq \alpha\} \quad \text{and} \quad \Psi(h;\alpha) = \bigcap_{\delta > \alpha} \text{clo}\,(\text{level}_\delta(h)). \tag{3.13}$$

Definition 3.9. For a couple of non-empty sets $A, B \subset \mathbb{R}^p$ we consider distance between a point $a \in A$ and the set B given by

$$d_p(a, B) = \inf_{b\in B} d_p(a, b) \tag{3.14}$$

and the excess from the set A to the set B defined by

$$\operatorname{excess}(A, B) = \sup_{a\in A} d_p(a, B), \tag{3.15}$$

where d_p denotes the Euclidean distance in $\mathbb{R}^p$.
For convenience, we set $\operatorname{excess}(\emptyset, B) = \operatorname{excess}(\emptyset, \emptyset) = 0$, $\operatorname{excess}(A, \emptyset) = +\infty$.

If the functions depend on random element the setup of approximations can be naturally combined with the almost sure validity.

Definition 3.10. Let $\{f_n,\ n \in \mathbb{N}_0\}$ be a family of functions $f_n |\ \mathbb{R}^p \times \Omega \to \overline{\mathbb{R}}$. Then the sequence $(f_n)_{n\in\mathbb{N}}$ is said to be

i) a lower semicontinuous approximation almost surely to f_0 at X (notation $f_n \xrightarrow[X]{\text{l-a.s.}} f_0$) if $P\left\{\omega :\ f_n(\cdot, \omega) \xrightarrow[X]{\text{l}} f_0(\cdot, \omega)\right\} = 1$,

ii) an upper semicontinuous approximation almost surely to f_0 at X (notation $f_n \xrightarrow[X]{\text{u-a.s.}} f_0$) if $-f_n \xrightarrow[X]{\text{l-a.s.}} -f_0$,

iii) continuously convergent almost surely to f_0 at X (notation $f_n \xrightarrow[X]{\text{a.s}} f_0$) if $(f_n \xrightarrow[X]{\text{l-a.s.}} f_0) \wedge (f_n \xrightarrow[X]{\text{u-a.s}} f_0)$,

iv) an epi-upper approximation almost surely to f_0 at X (notation $f_n \xrightarrow[X]{\text{epi-u-a.s.}} f_0$) if $P\left\{\omega :\ f_n(\cdot, \omega) \xrightarrow[X]{\text{epi-u}} f_0(\cdot, \omega)\right\} = 1$,

v) epi-convergent almost surely to f_0 at X (notation $f_n \xrightarrow[X]{\text{epi-a.s}} f_0$) if $(f_n \xrightarrow[X]{\text{l-a.s.}} f_0) \wedge (f_n \xrightarrow[X]{\text{epi-u-a.s.}} f_0)$.

The set X plays the role of the "convergence region", because, in general, we do not need (especially continuous) convergence on the whole $\mathbb{R}^p$ (cf.[27]); Theorem 3.6 is showing that. Epi-convergence almost surely as dealt with by [23, 24] corresponds to our definition with $X = \mathbb{R}^p$.

The following propositions gather up equivalent characterizations of the lower semicontinuous approximation a.s. and of the epi-upper approximation a.s. They will be latter employed to introduce the setup of approximations in probability. Partly Propositions are inspired by the results in [23]. The mentioned characterizations are valid for closed convergence regions, only. Therefore, we make the following agreement.

Agreement. In the sequel, the set X is always assumed to be a closed subset of $\mathbb{R}^p$.

Let $\operatorname{Epi} f_n(\cdot,\omega)$ denote the epi-graph of $f_n(\cdot,\omega)$. Under our measurability conditions the multifunction $\operatorname{Epi} f_n$ has a measurable graph.

We denote the set of all compact subsets of $\mathbb{R}^p$ by C^p and the α-neighborhood of X by $U_\alpha X := \{x \in \mathbb{R}^p | \inf_{y\in X} d(x,y) < \alpha\}$. By $\overline{U}_\alpha X$ we denote the closure of $U_\alpha X$.

The lower semicontinuous approximation a. s. and the epi-upper approximation a. s. possess an helpful equivalent description which enables their extension 'in probability' sense. For that purpose we establish two auxiliary sets.

Definition 3.11. For a couple of functions $f, g|\ \mathbb{R}^p \times \Omega \to \overline{\mathbb{R}}$ we abbreviate

$$\mathcal{D}_{l,\varepsilon}(f,g,X;\omega) := \left(\operatorname{clo}\operatorname{Epi} f(\cdot,\omega)\cap[\overline{U}_{\frac{1}{l}}X\times\mathbb{R}]\right) \setminus U_\varepsilon\left(\operatorname{Epi} g(\cdot,\omega)\cap[X\times\mathbb{R}]\right), \tag{3.16}$$

$$\mathcal{H}_\varepsilon(f,g,X;\omega) := (\operatorname{clo}\operatorname{Epi} g(\cdot,\omega)\cap[X\times\mathbb{R}]) \setminus U_\varepsilon \operatorname{Epi}\ f(\cdot,\omega). \tag{3.17}$$

Briefly, the set $\mathcal{D}_{l,\varepsilon}(f,g,X;\omega)$ contains each cluster point of the epi-graph of f with argument in $\overline{U}_{\frac{1}{l}}X$, but with distance at least ε from the epi-graph of g restricted to X. The set $\mathcal{H}_\varepsilon(f,g,X;\omega)$ contains each cluster point of the epi-graph of g with argument in X, but with distance at least ε from the epi-graph of f.

To avoid any misunderstanding, let us note that "lim sup", "lim inf" and "lim" for sets are used in the sense of Kuratowski. The limits in the set-theoretical sense will be denoted by "Limsup", "Liminf" and "Lim".

Theorem 3.1. Let $f_0(\cdot,\omega)$ be l.s.c. on X for almost all ω. Then

$$\left(f_n \xrightarrow[X]{\text{l-a.s.}} f_0\right) \tag{3.18}$$

$$\Leftrightarrow \quad \left(P\left\{\omega : \begin{array}{c} \forall x_0 \in X\ \ \forall (x_n)_{n\in\mathbb{N}} \text{ with } x_n \to x_0 : \\ \liminf_{n\to+\infty} f_n(x_n,\omega) \geq f_0(x_0,\omega) \end{array}\right\} = 1\right), \tag{3.19}$$

$$\Leftrightarrow \quad \left(P\left\{\omega : \begin{array}{c} \limsup_{\substack{n\to+\infty\\ l\to+\infty}} (\operatorname{Epi} f_n(\cdot,\omega)\cap[U_{\frac{1}{l}}X\times\mathbb{R}]) \subset \\ \subset (\operatorname{Epi} f_0(\cdot,\omega)\cap[X\times\mathbb{R}]) \end{array}\right\} = 1\right), \tag{3.20}$$

$$\Leftrightarrow \quad \left(P\left\{\omega : \begin{array}{c} \limsup_{\substack{n\to+\infty\\ l\to+\infty}} (\operatorname{clo}\operatorname{Epi} f_n(\cdot,\omega)\cap[\overline{U}_{\frac{1}{l}}X\times\mathbb{R}]) \subset \\ \subset (\operatorname{Epi} f_0(\cdot,\omega)\cap[X\times\mathbb{R}]) \end{array}\right\} = 1\right), \tag{3.21}$$

$$\Leftrightarrow \quad \left(\forall\varepsilon>0\ \ \forall K\in C^{p+1}: \lim_{\substack{n\to+\infty\\ l\to+\infty}} P\left\{\bigcup_{\substack{m\geq n\\ s\geq l}} \{\omega : \mathcal{D}_{s,\varepsilon}(f_m,f_0,X;\omega)\cap K\neq\emptyset\}\right\} = 0\right). \tag{3.22}$$

Proof. The equivalence between (3.18) and (3.19) follows by (3.3), (3.5) and (3.7).

Assuming X to be a closed set we obtain

$$\begin{aligned}\text{Epi } EL_* f_n(\cdot,\ \omega) \cap [X \times \mathbb{R}] &= \limsup_{\substack{n\to+\infty\\ l\to+\infty}} \text{ Epi } f_n(\cdot,\ \omega) \cap [U_{\frac{1}{l}} X \times \mathbb{R}] \\ &= \limsup_{\substack{n\to+\infty\\ l\to+\infty}} \text{ clo Epi } f_n(\cdot,\ \omega) \cap [\overline{U}_{\frac{1}{l}} X \times \mathbb{R}] \quad \forall \omega \in \Omega\end{aligned}$$

and, consequently, (3.19) $\Leftrightarrow$ (3.20) $\Leftrightarrow$ (3.21).

To prove the equivalence of (3.21) and (3.22) we make use of the corollary 4.11(b) and Theorem 4.10(b') in [18]. These two relations are giving a chain of equivalences

$$\left(\limsup_{\substack{n\to+\infty\\ l\to+\infty}} G_{nl} \subset G_0 \right) \Leftrightarrow \left(\forall \varepsilon > 0 : \lim_{\substack{n\to+\infty\\ l\to+\infty}} (G_{nl} \setminus U_\varepsilon G_0) = \emptyset \right)$$

$$\Leftrightarrow \left(\forall \varepsilon \in \mathbb{Q}_+ \forall x \in \mathbb{Q}^{p+1} \forall r \in \mathbb{Q}_+ : \exists n_0 \exists l_0 \forall n \geq n_0 \forall l \geq l_0 : \ (G_{nl} \setminus U_\varepsilon G_0) \cap \overline{U}_r\{x\} = \emptyset \right)$$

provided G_{nl}, G_0 are closed subsets of $\mathbb{R}^{p+1}$. The symbols $\mathbb{Q}^{p+1}$ and $\mathbb{Q}_+$ denote the rational numbers of $\mathbb{R}^{p+1}$ and $\mathbb{R}_+$, respectively.

The sets clo Epi $f_n(\cdot,\ \omega) \cap [\overline{U}_{\frac{1}{l}} X \times \mathbb{R}]$ are closed by definition and the set Epi $f_0(.,\omega) \cap [X \times \mathbb{R}]$ is closed a. s. since f_0 is l.s.c. on the closed set X.

Thus, we have

$$(3.21) \Leftrightarrow$$

$$\Leftrightarrow \left(P\left\{ \omega : \begin{array}{l} \forall \varepsilon \in \mathbb{Q}_+ \ \forall x \in \mathbb{Q}^{p+1} \ \forall r \in \mathbb{Q}_+ : \exists n_0(\omega) \ \exists l_0(\omega) \\ \forall n \geq n_0(\omega) \ \forall l \geq l_0(\omega) : \ \mathcal{D}_{l,\varepsilon}(f_n, f_0, X; \omega) \cap \overline{U}_r\{x\} = \emptyset \end{array} \right\} = 1 \right)$$

$$\Leftrightarrow \left(P\left\{ \omega : \begin{array}{l} \exists \varepsilon \in \mathbb{Q}_+ \ \exists x \in \mathbb{Q}^{p+1} \ \exists r \in \mathbb{Q}_+ : \forall n_0(\omega) \ \forall l_0(\omega) \\ \exists n \geq n_0(\omega) \ \exists l \geq l_0(\omega) : \ \mathcal{D}_{l,\varepsilon}(f_n, f_0, X; \omega) \cap \overline{U}_r\{x\} \neq \emptyset \end{array} \right\} = 0 \right)$$

$$\Leftrightarrow \left(P\{\} \bigcup_{\varepsilon\in\mathbb{Q}_+} \bigcup_{x\in\mathbb{Q}^{p+1}} \bigcup_{r\in\mathbb{Q}_+} \bigcap_{n_0,l_0\in\mathbb{N}} \bigcup_{\substack{n\geq n_0\\ l\geq l_0}} \{\omega : \mathcal{D}_{l,\varepsilon}(f_n, f_0, X; \omega) \cap \overline{U}_r\{x\} \neq \emptyset\} = 0 \right).$$

The multifunction $\omega \mapsto \mathcal{D}_{l,\varepsilon}(f_n, f_0, X; \omega)$ is possessing measurable graphs since the functions f_n and f_0 are measurable. According to Proposition 8.4.4. in [6], the set $\{\omega : \mathcal{D}_{l,\varepsilon}(f_n, f_0, X; \omega) \cap \overline{U}_r\{x\} \neq \emptyset\}$ belongs to $\mathcal{A}$ because we assume a complete probability space. Therefore, we can prolong the chain of equivalences.

$$\Leftrightarrow \left(\begin{array}{l} \forall \varepsilon \in \mathbb{Q}_+ \ \forall x \in \mathbb{Q}^{p+1} \ \forall r \in \mathbb{Q}_+ : \\ \qquad \lim\limits_{\substack{n\to+\infty\\ l\to+\infty}} P\left\{ \bigcup\limits_{\substack{m\geq n\\ s\geq l}} \{\omega : \mathcal{D}_{s,\varepsilon}(f_m, f_0, X; \omega) \cap \overline{U}_r\{x\} \neq \emptyset\} \right\} = 0 \end{array} \right)$$

$\Leftrightarrow$ (3.22). □

Continuity of the function f_0 simplifies the statement (3.22).

Proposition 3.1. Let $f_0(\cdot,\ \omega)$ be continuous on X for almost all ω. Then $(f_n \xrightarrow[X]{\text{l-a.s.}} f_0) \Leftrightarrow$

$$\left(P \left\{ \omega : \begin{array}{l} \forall K \in C^p : \\ \liminf\limits_{\substack{n\to+\infty \\ l\to+\infty}} \inf\limits_{x\in X\cap K} \left(\inf\limits_{y\in U_{\frac{1}{l}}\{x\}} f_n(y,\omega) - f_0(x,\omega) \right) \geq 0 \end{array} \right\} = 1 \right) \tag{3.23}$$

Proof. The formula (3.23) evidently implies the statement (3.19). We have to show the reverse implication, only.

Let $\omega \in \Omega$ be fixed and such that $f_0(\cdot,\omega)$ is u.s.c. on X and suppose that there are $\varepsilon > 0$, $K \in C^p$ and sequences $(n_k)_{k\in\mathbb{N}}$, $(l_k)_{k\in\mathbb{N}}$, $(x_k)_{k\in\mathbb{N}}$, $(y_k)_{k\in\mathbb{N}}$ with $x_k \in X \cap K$, $y_k \in U_{\frac{1}{l_k}}\{x_k\}$ and $f_{n_k}(y_k,\ \omega) - f_0(x_k,\ \omega) < -\varepsilon$. W.l.o.g. we can assume that $x_k \to x_0 \in X \cap K$. Since the function $f_0(\cdot,\ \omega)$ is u.s.c. on X, we receive

$$\liminf_{k\to+\infty} f_{n_k}(y_k,\ \omega) \leq \liminf_{k\to+\infty} f_0(x_k,\ \omega) - \varepsilon \leq \limsup_{k\to+\infty} f_0(x_k,\ \omega) - \varepsilon \leq f_0(x_0,\ \omega) - \varepsilon$$

and, therefore, $EL_* f_n(.,\omega)(x_0) < f_0(x_0,\omega)$. □

Proposition 3.2. Let $f_0(\cdot,\ \omega)$ be continuous at X for almost all ω. Then $(f_n \xrightarrow[X]{\text{l-a.s.}} f_0) \Leftrightarrow$

$$\left(P \left\{ \omega : \forall K \in C^p \quad \liminf_{\substack{n\to+\infty \\ l\to+\infty}} \inf_{x\in U_{\frac{1}{l}} X\cap K} (f_n(x,\omega) - f_0(x,\omega)) \geq 0 \right\} = 1 \right). \tag{3.24}$$

Proof. We shall show that (3.19) is equivalent with (3.24).

i) Let $\omega \in \Omega$ be fixed and such that $f_0(\cdot,\omega)$ is u.s.c. at X and suppose that there are $\varepsilon > 0$, $K \in C^p$ and sequences $(n_k)_{k\in\mathbb{N}}$, $(l_k)_{k\in\mathbb{N}}$, $(x_k)_{k\in\mathbb{N}}$ with $x_k \in U_{\frac{1}{l_k}} X \cap K$ and $f_{n_k}(x_k,\ \omega) - f_0(x_k,\ \omega) < -\varepsilon$. W.l.o.g. we can assume that $x_k \to x_0 \in X \cap K$. Since the function $f_0(\cdot,\ \omega)$ is u.s.c. at X, we receive

$$\liminf_{k\to+\infty} f_{n_k}(x_k,\ \omega) \leq \liminf_{k\to+\infty} f_0(x_k,\ \omega) - \varepsilon \leq \limsup_{k\to+\infty} f_0(x_k,\ \omega) - \varepsilon \leq f_0(x_0,\ \omega) - \varepsilon.$$

ii) Now, suppose that $f_0(\cdot,\ \omega)$ is l.s.c. at X for a fixed $\omega \in \Omega$ and that there are an $x_0 \in X$, a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \to x_0$, and an $\varepsilon > 0$ such that

$$\liminf_{n\to+\infty} f_n(x_n,\ \omega) \leq f_0(x_0,\ \omega) - \varepsilon.$$

Because the function $f_0(\cdot,\ \omega)$ is l.s.c. at X we obtain

$$\begin{aligned} \liminf_{n\to+\infty} (f_n(x_n,\ \omega) - f_0(x_n,\ \omega)) &\leq \liminf_{n\to+\infty} f_n(x_n,\ \omega) + \limsup_{n\to+\infty}(-f_0(x_n,\ \omega)) \\ &\leq f_0(x_0,\ \omega) - \varepsilon - \liminf_{n\to+\infty} f_0(x_n,\ \omega) \leq -\varepsilon. \end{aligned}$$

Consequently for $K := \overline{U}_1\{x_0\}$, we receive $\inf\limits_{x \in U_{\frac{1}{t}} X \cap K} (f_n(x,\, \omega) - f_0(x,\, \omega)) \leq -\varepsilon$.

□

It is well-known that continuous convergence is equivalent to uniform convergence on compact sets if the limit function is continuous [9], Propositions 3.1 and 3.2 reflect this fact in our setting.

Theorem 3.2. Without any additional assumption

$$\left(f_n \xrightarrow[X]{\text{epi-u-a.s.}} f_0\right) \tag{3.25}$$

$$\Leftrightarrow \quad \left(P\left\{\omega : \begin{array}{c} \forall x_0 \in X \;\; \exists (x_n)_{n\in\mathbb{N}} \text{ with } x_n \to x_0 : \\ \limsup\limits_{n\to+\infty} f_n(x_n,\, \omega) \leq f_0(x_0,\, \omega) \end{array}\right\} = 1\right) \tag{3.26}$$

$$\Leftrightarrow \quad \left(P\left\{\omega : \liminf_{n\to+\infty} \operatorname{Epi} f_n(\cdot,\, \omega) \supset \operatorname{Epi} f_0(\cdot,\omega) \cap [X \times \mathbb{R}]\right\} = 1\right) \tag{3.27}$$

$$\Leftrightarrow \quad \left(P\left\{\omega : \liminf_{n\to+\infty} \operatorname{clo}\operatorname{Epi} f_n(\cdot,\, \omega) \supset \operatorname{clo}\operatorname{Epi} f_0(\cdot,\omega) \cap [X \times \mathbb{R}]\right\} = 1\right) \tag{3.28}$$

$$\Leftrightarrow \quad \left(\forall \varepsilon > 0 \;\; \forall K \in C^{p+1} : \lim_{n\to+\infty} P\left\{\bigcup_{m\geq n} \{\omega : \mathcal{H}_\varepsilon(f_n, f_0, X; \omega) \cap K \neq \emptyset\}\right\} = 0\right). \tag{3.29}$$

Proof. The equivalence between (3.25) and (3.26) follows by (3.4), (3.6) and (3.9).

The statements (3.26) and (3.27) are equivalent because of
$\operatorname{Epi} \mathrm{EL}^* f_n(\cdot,\, \omega) \cap [X \times \mathbb{R}] = \liminf\limits_{n\to+\infty} \operatorname{Epi} f_n(\cdot,\, \omega) \cap [X \times \mathbb{R}] \quad \forall_\omega \in \Omega$.

Employing the corollary 4.11(a) in [18] and Theorem 4.10(b') in [18] for G_n arbitrary subsets of $\mathbb{R}^{p+1}$, we receive the following chain of equivalences

$$\left(\liminf_{n\to+\infty} G_n \supset G_0\right) \Leftrightarrow \left(\liminf_{n\to+\infty} \operatorname{clo} G_n \supset \operatorname{clo} G_0\right)$$
$$\Leftrightarrow \left(\forall \varepsilon > 0 : \lim_{n\to+\infty} (\operatorname{clo} G_0 \setminus U_\varepsilon \operatorname{clo} G_n) = \emptyset\right)$$
$$\Leftrightarrow \left(\forall \varepsilon > 0 : \lim_{n\to+\infty} (\operatorname{clo} G_0 \setminus U_\varepsilon G_n) = \emptyset\right)$$
$$\Leftrightarrow \left(\forall x \in \mathbb{Q}^{p+1} \;\; \forall r \in \mathbb{Q}_+ : \exists n_0 \;\; \forall n \geq n_0 : \;\; (\operatorname{clo} G_0 \setminus U_\varepsilon G_n) \cap \overline{U}_r\{x\} = \emptyset\right).$$

Consequently, (3.27) ⇔ (3.28) ⇔ (3.29) follows by considerations similar to those in the proof of Theorem 3.1. □

Note that the equivalent characterization (3.26) implies that pointwise convergence a. s. of $(f_n)_{n\in\mathbb{N}}$ to f_0 at x_0 is sufficient for $f_n \xrightarrow[\{x_0\}]{\text{epi-u-a.s.}} f_0$.

Under additional assumptions the lower semicontinuous approximation a. s. and the epi-upper approximation a. s. imply convergences of optimal values and optimal solutions.

Theorem 3.3. Let $\{f_n,\ n \in \mathbb{N}_0\}$ be a family of functions $f_n|\ \mathbb{R}^p \times \Omega \to \overline{\mathbb{R}}$ and $\Delta \subset \mathbb{R}^p$ be such that $\inf\limits_{x\in\Delta} f_0(x,\omega) = \varphi\left(f_0(.,\omega)\right)$ for almost all $\omega \in \Omega$.

If $f_n \xrightarrow[\Delta]{\text{epi-u-a.s.}} f_0$ then $\limsup\limits_{n\to+\infty} \varphi\left(f_n(.,\omega)\right) \le \varphi\left(f_0(.,\omega)\right)$ for almost all $\omega \in \Omega$.

Proof. Let $\omega \in \Omega$ such that $\inf\limits_{x\in\Delta} f_0(x,\omega) = \varphi\left(f_0(.,\omega)\right)$ and $\delta > \varphi\left(f_0(.,\omega)\right)$.
Then there is $\hat{x} \in \Delta$ such that $f_0(\hat{x},\omega) < \delta$. Hence,

$$\sup_{V\in\mathcal{N}(\hat{x})}\ \limsup_{n\to+\infty}\ \inf_{x\in V} f_n(x,\omega) \le f_0(\hat{x},\omega) < \delta \quad \text{since } f_n \xrightarrow[\Delta]{\text{epi-u-a.s.}} f_0.$$

Consequently, $\limsup\limits_{n\to+\infty} \varphi\left(f_n(.,\omega)\right) \le \varphi\left(f_0(.,\omega)\right)$. □

Theorem 3.4. Let $\{f_n,\ n \in \mathbb{N}_0\}$ be a family of functions $f_n|\ \mathbb{R}^p \times \Omega \to \overline{\mathbb{R}}$.
Let a compact $K \in C^p$ be such that

$$\liminf_{n\to+\infty}\ \inf_{x\notin K} f_n(x,\omega) \ge \varphi\left(f_0(.,\omega)\right) \quad \text{for almost all } \omega \in \Omega$$

If $f_n \xrightarrow[K]{\text{l-a.s.}} f_0$ then for almost all $\omega \in \Omega$ we have

$$\liminf_{n\to+\infty} \varphi\left(f_n(.,\omega)\right) \ge \varphi\left(f_0(.,\omega)\right).$$

Proof. Let $\omega \in \Omega$ such that $\liminf_{n\to+\infty} \inf_{x\notin K} f_n(x,\omega) \ge \varphi\left(f_0(.,\omega)\right)$.
Assume $\delta < \varphi\left(f_0(.,\omega)\right)$.
Then for each $x \in K$, we have a neighborhood $V_x \in \mathcal{N}(x)$ such that

$$\liminf_{n\to+\infty}\ \inf_{y\in V_x} f_n(y,\omega) > \delta \quad \text{since } f_n \xrightarrow[K]{\text{l-a.s.}} f_0.$$

Of course, $\bigcup_{x\in K} V_x \supset K$. We assume K to be a compact and, therefore, there is a finite subset $I \subset K$ such that $\bigcup_{x\in I} V_x \supset K$.
Thus,

$$\liminf_{n\to+\infty}\ \inf_{y\in K} f_n(y,\omega) \ge \min_{x\in I}\ \liminf_{n\to+\infty}\ \inf_{y\in V_x} f_n(y,\omega) > \delta$$

and, hence,

$$\begin{aligned}\liminf_{n\to+\infty} \varphi\left(f_n(.,\omega)\right) &= \liminf_{n\to+\infty} \min\left\{\inf_{y\in K} f_n(y,\omega),\ \inf_{y\notin K} f_n(y,\omega)\right\}\\ &\ge \min \liminf_{n\to+\infty}\ \inf_{y\in K} f_n(y,\omega), \liminf_{n\to+\infty}\ \inf_{y\notin K} f_n(y,\omega) > \delta.\end{aligned}$$

Consequently, $\liminf\limits_{n\to+\infty} \varphi\left(f_n(\cdot,\omega)\right) \ge \varphi\left(f_0(\cdot,\omega)\right)$. □

Theorem 3.5. Let $\{f_n,\ n \in \mathbb{N}_0\}$ be a family of functions $f_n|\ \mathbb{R}^p \times \Omega \to \overline{\mathbb{R}}$ and $\alpha_n|\ \Omega \to \mathbb{R}$, $n \in \mathbb{N}_0$, $\limsup\limits_{n\to+\infty} \alpha_n \le \alpha_0$ a. s.

Let a compact $K \in C^p$ be such that

$$\liminf_{n\to+\infty} \inf_{x\notin K} f_n(x,\omega) > \varphi\left(f_0(.,\omega)\right) \quad \text{for almost all } \omega \in \Omega$$

If $f_n \xrightarrow[K]{\text{l-a.s.}} f_0$ then for almost all $\omega \in \Omega$ we have

$$\liminf_{n\to+\infty} \varphi\left(f_n(.,\omega)\right) \ge \varphi\left(f_0(.,\omega)\right)$$

and if $\liminf\limits_{n\to+\infty} \inf\limits_{x\notin K} f_n(x,\omega) > \alpha_0(\omega)$ then

$$\lim_{n\to+\infty} \operatorname{excess}(\Psi\left(f_n(.,\omega);\alpha_n(\omega)\right)\Psi\left(f_0(.,\omega);\alpha_0(\omega)\right)) = 0$$

while $\Psi\left(f_n(.,\omega);\alpha_n(\omega)\right) \subset K$ is a compact for all $n \in \mathbb{N}$ large enough.

Proof. The relation between optimal values follows from Theorem 3.4.

We have to consider the level sets, only.

Let $\omega \in \Omega$ such that $\liminf_{n\to+\infty} \inf_{x\notin K} f_n(x,\omega) > \alpha_0(\omega) \ge \varphi\left(f_0(.,\omega)\right)$.

Assume $\delta_0 \in \mathbb{R}$ fulfilling $\liminf_{n\to+\infty} \inf_{x\notin K} f_n(x,\omega) > \delta_0 > \alpha_0(\omega)$.

Then there is $n_0 \in \mathbb{N}$ such that $\inf_{x\notin K} f_n(x,\omega) > \delta_0$ for each $n \ge n_0$.

Hence $\text{level}_\delta(f_n(.,\omega),\delta) \subset K$ for each $\delta \in (\alpha_0(\omega),\delta_0)$, $n \ge n_0$.

Consequently for each $n \ge n_0$, $\Psi\left(f_n(.,\omega);\alpha_n(\omega)\right) \subset K$ and is a compact being a closed set by definition.

According to our agreement that $\operatorname{excess}(\emptyset,B) = \operatorname{excess}(\emptyset,\emptyset) = 0$, we need only treat the case $\alpha_n(\omega) \ge \varphi\left(f_n(.,\omega)\right)\ \forall\ n \in \mathbb{N}$, i. e. $\Psi\left(f_n(.,\omega);\alpha_n(\omega)\right) \ne \emptyset$.

Let $x_n \in \Psi\left(f_n(.,\omega);\alpha_n(\omega)\right)$ be such that

$$d_p\left(x_n,\Psi\left(f_0(.,\omega);\alpha_0(\omega)\right)\right) > \operatorname{excess}(\Psi\left(f_n(.,\omega);\alpha_n(\omega)\right)\Psi\left(f_0(.,\omega);\alpha_0(\omega)\right)) - 2^{-n}.$$

For $n \ge n_0$, x_n belongs in the compact K. Therefore, there is a convergent subsequence $\lim\limits_{k\to+\infty} x_{n_k} = \hat{x}$.

By the definition there are $y_n \in \mathbb{R}^p$ such that

$$\lim_{n\to+\infty} d_p\left(y_n,x_n\right) = 0 \quad \text{and} \quad \limsup_{n\to+\infty} f_n(y_n,\omega) \le \alpha_0(\omega).$$

Therefore, $\alpha_0(\omega) \ge \liminf\limits_{k\to+\infty} f_{n_k}(y_{n_k},\omega) \ge f_0(\hat{x},\omega)$ since $f_n \xrightarrow[K]{\text{l-a.s.}} f_0$.

Thus, we conclude $\lim\limits_{n\to+\infty} \operatorname{excess}\Psi\left(f_n(.,\omega);\alpha_n(\omega)\right)\Psi\left(f_0(.,\omega);\alpha_0(\omega)\right) = 0.$ □

Theorem 3.6. Let $\{f_n,\ n \in \mathbb{N}_0\}$ be a family of functions $f_n \,|\, \mathbb{R}^p \times \Omega \to \overline{\mathbb{R}}$, $\alpha_n \,|\, \mathbb{R}^p \times \Omega \to \mathbb{R}$ for $n \in \mathbb{N}_0$ and $\Delta \subset \mathbb{R}^p$ be such that $\alpha_n \geq \varphi(f_n)$ a.s. for each $n \in \mathbb{N}_0$, $\limsup\limits_{n\to+\infty} \alpha_n \leq \alpha_0$ a.s. and $\int_{x\in\Delta} f_0(x) = \varphi(f_0(.,\omega))$ for almost all $\omega \in \Omega$.

Let a compact $K \in C^p$ be such that

$$\liminf_{n\to+\infty} \inf_{x\notin K} f_n(x,\omega) > \varphi(f_0(.,\omega)) \quad \text{for almost all } \omega \in \Omega, \tag{3.30}$$

$$f_n \xrightarrow[K]{\text{l-a.s.}} f_0 \quad \text{and} \quad f_n \xrightarrow[\Delta]{\text{epi-u-a.s.}} f_0. \tag{3.31}$$

Then for almost all $\omega \in \Omega$ we have

$$\liminf_{n\to+\infty} \varphi(f_n(.,\omega)) = \varphi(f_0(.,\omega))$$

and if $\liminf\limits_{n\to+\infty} \int_{x\notin K} f_n(x,\omega) > \alpha_0(\omega)$ then

$$\lim_{n\to+\infty} \operatorname{excess}(\Psi(f_n(.,\omega);\alpha_n(\omega))\Psi(f_0(.,\omega);\alpha_0(\omega))) = 0$$

while $\Psi(f_n(.,\omega);\alpha_n(\omega)) \subset K$ is a compact for all $n \in \mathbb{N}$ large enough.

Proof. The statement is a direct combination of Theorems 3.3 and 3.6. □

Let us note that Theorems 3.3, 3.4, 3.5 and 3.6 extend Proposition 7.30 and Theorem 7.33 in [18]; for more general setting see Theorem 5.3.6 in [5].

4. CONVERGENCE IN PROBABILITY OF RANDOM FUNCTIONS

Now we shall consider one-sided approximations "in probability". To avoid any misunderstanding, let us repeat the note that "lim sup", "lim inf" and "lim" for sets are used in the sense of Kuratowski and the limits in the set-theoretical sense are denoted by "Limsup", "Liminf" and "Lim".

Definition 4.1. The sequence $(f_n)_{n\in\mathbb{N}}$ is said to be

i) a lower semicontinuous approximation in probability to f_0 at X (notation $f_n \xrightarrow[X]{\text{l-prob}} f_0$) if

$$\forall \varepsilon > 0\ \forall K \in C^{p+1} : \lim_{\substack{n\to+\infty \\ l\to+\infty}} P\{\omega : \mathcal{D}_{l,\varepsilon}(f_n, f_0, X;\omega) \cap K \neq \emptyset\} = 0,$$

ii) an upper semicontinuous approximation in probability to f_0 at X (notation $f_n \xrightarrow[X]{\text{u-prob}} f_0$) if $(-f_n \xrightarrow[X]{\text{l-prob}} -f_0)$,

iii) an epi-upper approximation in probability to f_0 at X (notation $f_n \xrightarrow[X]{\text{epi-u-prob}} f_0$) if

$$\forall \varepsilon > 0\ \forall K \in C^{p+1} : \lim_{n\to+\infty} P\{\omega : \mathcal{H}_\varepsilon(f_n, f_0, X;\omega) \cap K \neq \emptyset\} = 0.$$

Continuous convergence or epi-convergence in probability can be defined combining (i) and (ii) or (i) and (iii), respectively. The condition (3.22) and hence $f_n \xrightarrow[X]{\text{l-a.s.}} f_0$ imply $f_n \xrightarrow[X]{\text{l-prob}} f_0$ as well as (3.29) and hence $f_n \xrightarrow[X]{\text{epi-u-a.s.}} f_0$ imply $f_n \xrightarrow[X]{\text{epi-u-prob}} f_0$.

The following lemma gives further insight into the relation between the approximations almost surely and in probability. It offers the possibility to derive stability assertions 'in probability' from the a. s. case.

Lemma 4.1. Let $f_0(\cdot,\, \omega)$ be l.s.c. on X for almost all ω. Then

$$\left(\begin{array}{l}\text{Each subsequence of } (f_n)_{n\in\mathbb{N}} \text{ contains}\\ \text{a subsequence } (f_{n_k})_{k\in\mathbb{N}} \text{ with } f_{n_k} \xrightarrow[X]{\text{l-a.s.}} f_0.\end{array}\right) \Rightarrow (f_n \xrightarrow[X]{\text{l-prob}} f_0).$$

Proof. Suppose that $(f_n)_{n\in\mathbb{N}}$ fails to be a lower semicontinuous approximation in probability to f_0 on X. Hence there are $\varepsilon > 0$, $K \in C^{p+1}$, $\alpha > 0$ and subsequences $(f_n)_{n\in\tilde{N}\subset\mathbb{N}}$, $(l_n)_{n\in\tilde{N}}$, $\lim\limits_{\substack{n\to+\infty\\ n\in\tilde{N}}} l_n = \infty$ with

$$P\{\omega :\, \mathcal{D}_{l_n,\varepsilon}(f_n, f_0, X;\omega) \cap K \neq \emptyset\} > \alpha \;\; \forall n \in \tilde{N}.$$

Consequently, $P\left\{\operatorname*{Limsup}\limits_{\substack{k\to+\infty\\ l\to+\infty}} \{\omega :\, \mathcal{D}_{l,\varepsilon}(f_{n_k}, f_0, X;\omega) \cap K \neq \emptyset\}\right\} \geq \alpha$, for each subsequence $(n_k)_{k\in\mathbb{N}} \subset \tilde{N}$ and, thus, the sequence $(f_n)_{n\in\tilde{N}}$ cannot contain a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ with $f_{n_k} \xrightarrow[X]{\text{l-a.s.}} f_0$. □

Assuming a bit stronger condition than lower semicontinuous approximation in probability, we are able to prove the reverse statement.

Lemma 4.2. Let $f_0(\cdot,\, \omega)$ be l.s.c. on X for almost all ω. If there is $l_0 \in \mathbb{N}$ such that $\forall\varepsilon > 0\;\forall K \in C^{p+1} : \lim\limits_{n\to+\infty} P\{\omega : \mathcal{D}_{l_0,\varepsilon}(f_n, f_0, X;\omega) \cap K \neq \emptyset\} = 0$ then

$$\left(\begin{array}{l}\text{Each subsequence of } (f_n)_{n\in\mathbb{N}} \text{ contains}\\ \text{a subsequence } (f_{n_k})_{k\in\mathbb{N}} \text{ with } f_{n_k} \xrightarrow[X]{\text{l-a.s.}} f_0.\end{array}\right).$$

Proof. Consider a subsequence $(f_n)_{n\in\tilde{N}\subset N}$ of $(f_n)_{n\in\mathbb{N}}$.
For every $k\in\mathbb{N}$ we find an $\tilde{n}_k\in\tilde{N}$ such that, for each $n \geq \tilde{n}_k \quad n\in\tilde{N}$,

$$P\left\{\omega :\, \mathcal{D}_{l_0,\frac{1}{2^k}}(f_n, f_0, X;\omega) \cap \overline{U}_k\{0\} \neq \emptyset\right\} < \tfrac{1}{2^k}.$$

Let $n_1 = \tilde{n}_1$ and $n_k := \max\{n_{k-1} + 1,\, \tilde{n}_k\}$. Let us denote $\hat{N} := \{n_1, n_2, \ldots\}$.

For fixed $\varepsilon > 0$ and $K \in C^{p+1}$, we obtain

$$\sum_{k=1}^{\infty} P\{\omega : \mathcal{D}_{l_0,\varepsilon}(f_{n_k}, f_0, X;\omega) \cap K \neq \emptyset\} < \infty,$$

since

$$\{\omega : \mathcal{D}_{l_0,\varepsilon}(f_{n_k}, f_0, X; \omega) \cap K \neq \emptyset\} \subset \left\{\omega : \mathcal{D}_{l_0,\frac{1}{2^k}}(f_{n_k}, f_0, X; \omega) \cap \overline{U}_k\{0\} \neq \emptyset\right\}$$

for each k sufficiently large.

That, by the Borel-Cantelli-Lemma, implies

$$P\left\{\operatorname*{Limsup}_{k\to+\infty}\{\omega : \mathcal{D}_{l_0,\varepsilon}(f_{n_k}, f_0, X; \omega) \cap K \neq \emptyset\}\right\} = 0.$$

Consequently, $\forall \varepsilon > 0\ \forall K \in C^{p+1}$:

$$\begin{aligned} 0 &= \lim_{k\to+\infty} P\left\{\bigcup_{j\geq k}\{\omega : \mathcal{D}_{l_0,\varepsilon}(f_{n_j}, f_0, X; \omega) \cap K \neq \emptyset\}\right\} \\ &= \lim_{k\to+\infty} P\left\{\bigcup_{\substack{m\geq n_k,\ m\in\hat{N} \\ s\geq l_0}}\{\omega : \mathcal{D}_{s,\varepsilon}(f_m, f_0, X; \omega) \cap K \neq \emptyset\}\right\} \\ &\geq \lim_{\substack{n\to+\infty, n\in\hat{N} \\ l\to+\infty}} P\left\{\bigcup_{\substack{m\geq n,\ m\in\hat{N} \\ s\geq l}}\{\omega : \mathcal{D}_{s,\varepsilon}(f_m, f_0, X; \omega) \cap K \neq \emptyset\}\right\}. \end{aligned}$$

That means $f_{n_k} \xrightarrow[X]{\text{l-a.s.}} f_0$. □

Unfortunately, in general, the reverse to Lemma 4.1 fails.

Example. Let $p = 1$, $X = \{0\}$ and $A_{n,v}$, $n, v \in \mathbb{N}$ be such random events that $P\{A_{n,v}\} = 2^{-v}$, $A_{n,v} \cap A_{n,w} = \emptyset$ whenever $v \neq w$, $\bigcup_{v=1}^{+\infty} A_{n,v} = \Omega$ and the collections $\{A_{n,v}, v \in \mathbb{N}\}$, $n \in \mathbb{N}$ are independent. Consider random l.s.c. functions

$$f_0 \equiv 0 \quad \text{and} \quad f_n(t,\omega) = \begin{cases} -1 & \text{if } t \geq \frac{1}{v}, \omega \in A_{n,v}, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } n \in \mathbb{N}.$$

Hence taking $\omega \in A_{n,v}$, we receive

$$\begin{aligned} &\operatorname{Epi} f_n(\cdot,\ \omega) \cap [U_{\frac{1}{l}} X \times \mathbb{R}] \\ &\quad = \left(-\frac{1}{l}, \frac{1}{\max\{v,l\}}\right) \times [0,+\infty) \cup \left[\frac{1}{\max\{v,l\}}, \frac{1}{l}\right) \times [-1,+\infty), \\ &U_\varepsilon(\operatorname{Epi} f_0(\cdot,\ \omega) \cap [X \times \mathbb{R}]) \ = \ U_\varepsilon(\{0\} \times [0,+\infty)). \end{aligned}$$

Therefore, for each $\frac{1}{l} \leq \varepsilon < 1$ and $K \in C^2$, $K \supset [-1,1]^2$ we have

$$\{\omega \in \Omega : D_{n,l,\varepsilon}(\omega) \cap K \neq \emptyset\} = \{\omega \in \Omega : D_{n,l,\varepsilon}(\omega) \cap [-1,1]^2 \neq \emptyset\} = \bigcup_{v=l+1}^{+\infty} A_{n,v}.$$

Then,

$$P\{\omega \in \Omega : D_{n,l,\varepsilon}(\omega) \cap K \neq \emptyset\} \leq \sum_{v=l+1}^{+\infty} \frac{1}{2^v} = \frac{1}{2^l} \xrightarrow[l\to+\infty]{} 0.$$

Therefore, $f_n \xrightarrow[X]{\text{l-prob}} f_0$.

Let f_{n_k}, $k \in \mathbb{N}$ be a subsequence of our functions. Then for a fixed $l \in \mathbb{N}$, $\frac{1}{l} \leq \varepsilon < 1$, the sets

$$\{\omega \in \Omega : D_{n_k,l,\varepsilon}(\omega) \cap K \neq \emptyset\} = \bigcup_{v=l+1}^{+\infty} A_{n_k,v} \ , \quad k \in \mathbb{N}$$

are independent with common positive probability. Hence according to the Borel–Cantelli lemma,

$$P\left\{\operatorname*{Limsup}_{k\to+\infty} \{\omega \in \Omega : D_{n_k,l,\varepsilon}(\omega) \cap K \neq \emptyset\}\right\} = 1.$$

Therefore, the convergence $f_{n_k} \xrightarrow[X]{\text{l-a.s.}} f_0$ cannot be true for any subsequence.

Lemma 4.3. Always, we have

$$(f_n \xrightarrow[X]{\text{epi-u-prob}} f_0) \Leftrightarrow \left(\begin{array}{l} \text{Each subsequence of } (f_n)_{n\in\mathbb{N}} \text{ contains} \\ \text{a subsequence } (f_{n_k})_{k\in\mathbb{N}} \text{ with } f_{n_k} \xrightarrow[X]{\text{epi-u-a.s.}} f_0. \end{array} \right).$$

Proof.

i) Let $f_n \xrightarrow[X]{\text{epi-u-prob}} f_0$ and consider a subsequence $(f_n)_{n\in\tilde{N}\subset N}$ of $(f_n)_{n\in\mathbb{N}}$. For every $k\in\mathbb{N}$ we find an $\tilde{n}_k \in \tilde{N}$ such that, for $n \geq \tilde{n}_k \quad n\in\tilde{N}$,

$$P\left\{\omega : \mathcal{H}_{\frac{1}{2^k}}(f_n, f_0, X; \omega) \cap \overline{U}_k\{0\} \neq \emptyset\right\} < \tfrac{1}{2^k}.$$

Let $n_1 = \tilde{n}_1$ and $n_k := \max\{n_{k-1}+1,\ \tilde{n}_k\}$.

For $\varepsilon > 0$, $K \in C^{p+1}$, we obtain $\sum\limits_{k=1}^{\infty} P\{\omega : \mathcal{H}_\varepsilon(f_{n_k}, f_0, X; \omega) \cap K \neq \emptyset\} < \infty$, since $\{\omega : \mathcal{H}_\varepsilon(f_{n_k}, f_0, X; \omega) \cap K \neq \emptyset\} \subset \left\{\omega : \mathcal{H}_{\frac{1}{2^k}}(f_{n_k}, f_0, X; \omega) \cap \overline{U}_k\{0\} \neq \emptyset\right\}$ for each k sufficiently large. That, by the Borel-Cantelli-Lemma, implies

$$P\left\{\operatorname*{Limsup}_{k\to+\infty} \{\omega \ : \ \mathcal{H}_\varepsilon(f_{n_k}, f_0, X; \omega) \cap K \neq \emptyset\}\right\}$$

$$= \lim_{k\to+\infty} P\left\{\bigcup_{j\geq k} \{\omega \ : \ \mathcal{H}_\varepsilon(f_{n_j}, f_0, X; \omega) \cap K \neq \emptyset\}\right\} = 0,$$

which is (3.29) and, therefore, $f_{n_k} \xrightarrow[X]{\text{epi-u-a.s.}} f_0$.

ii) Suppose that $(f_n)_{n\in\mathbb{N}}$ fails to be an epi-upper approximation in probability to f_0 on X. Hence there are $\varepsilon > 0$, $K \in C^{p+1}$, $\alpha > 0$ and subsequence $(f_n)_{n\in\tilde{N}\subset\mathbb{N}}$ with $P\{\omega : \mathcal{H}_\varepsilon(f_n, f_0, X; \omega) \cap K \neq \emptyset\} > \alpha \ \ \forall n \in \tilde{N}$.

Consequently,

$$P\left\{\operatorname*{Limsup}_{k\to+\infty}\{\omega:\ \mathcal{H}_\varepsilon(f_{n_k},f_0,X;\omega)\cap K\neq\emptyset\}\right\}\geq\alpha$$

for each subsequence $(n_k)_{k\in\mathbb{N}}\subset\tilde{N}$ and, thus, the sequence $(f_n)_{n\in\tilde{N}\subset\mathbb{N}}$ cannot contain a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ with $f_{n_k}\xrightarrow[X]{\text{epi-u-a.s.}}f_0$. □

Let us close the section with the relation between stability of stochastic optimization problem and our concept of approximations in probability. For random variables with values in the extended real line we define lower and upper approximation in probability.

Definition 4.2. Let X_n, $n\in\mathbb{N}_0$ be a sequence of random variables with values in $\overline{\mathbb{R}}$. We say that X_n is a lower semicontinuous approximation in probability to X_0 whenever for each $\varepsilon>0$, $\delta\in\mathbb{R}$

$$\lim_{n\to+\infty}P\{X_n<X_0-\varepsilon,X_0\in\mathbb{R}\}=0,\quad\lim_{n\to+\infty}P\{X_n<\delta,X_0=+\infty\}=0,\quad(4.1)$$

and is an upper semicontinuous approximation in probability to X_0 whenever for each $\varepsilon>0$, $\delta\in\mathbb{R}$

$$\lim_{n\to+\infty}P\{X_n>X_0-\varepsilon,X_0\in\mathbb{R}\}=0,\quad\lim_{n\to+\infty}P\{X_n>\delta,X_0=-\infty\}=0.\quad(4.2)$$

We will use the notation $X_n\xrightarrow{\text{l-prob}}X_0$ and $X_n\xrightarrow{\text{u-prob}}X_0$, respectively.

If both approximations take place in the same time we speak on convergence in probability denoted by $X_n\xrightarrow{\text{prob}}X_0$.

Evidently, $X_n\xrightarrow{\text{u-prob}}X_0\iff -X_n\xrightarrow{\text{l-prob}}-X_0$. Lower semicontinuous approximation in probability can be equivalently described in several ways.

Lemma 4.4. Let X_n, $n\in\mathbb{N}_0$ be a sequence of random variables with values in $\overline{\mathbb{R}}$. Then the following statements are equivalent

$$X_n\xrightarrow{\text{l-prob}}X_0\quad(4.3)$$

$$\iff\lim_{n\to+\infty}P\{X_n\leq\delta,X_0\geq\delta+\varepsilon\}=0\quad\forall\delta\in\mathbb{R}\ \forall\varepsilon>0\quad(4.4)$$

$$\iff\lim_{n\to+\infty}P\{X_n\leq\delta,X_0>\delta\}=0\quad\forall\delta\in\mathbb{R}\quad(4.5)$$

$$\iff\lim_{n\to+\infty}P\{[X_n,X_0-\varepsilon]\cap K\neq\emptyset\}=0\quad\forall\varepsilon>0\ \forall K\in C.\quad(4.6)$$

Proof.

iii) (4.3) implies (4.4) because

$$P\{X_n \le \delta, X_0 \ge \delta + \varepsilon\} = P\{X_n \le \delta \le X_0 - \varepsilon\}$$

$$\le P\left\{X_n < X_0 - \frac{\varepsilon}{2}, X_0 \in \mathbb{R}\right\} + P\{X_n \le \delta, X_0 = +\infty\}.$$

iv) (4.4) implies (4.5) because

$$P\{X_n \le \delta, X_0 > \delta\} \le P\{X_n \le \delta, X_0 \ge \delta + \varepsilon\} + P\{\delta < X_0 < \delta + \varepsilon\}.$$

v) Let $K \in C$. Then there is some $I \in \mathbb{N}$ such that $K \subset [-I\varepsilon, I\varepsilon]$. Hence,

$$P\{[X_n, X_0 - \varepsilon] \cap K \ne \emptyset\} \le P\{X_n \le I\varepsilon, X_0 \ge -(I-1)\varepsilon, X_n \le X_0 - \varepsilon\}$$

$$\le \sum_{i=-I}^{I-1} P\{X_n \le I\varepsilon, X_n \le X_0 - \varepsilon, i\varepsilon < X_0 \le (i+1)\varepsilon\} + P\{X_n \le I\varepsilon, I\varepsilon < X_0\}$$

$$\le \sum_{i=-I}^{I} P\{X_n \le i\varepsilon, i\varepsilon < X_0\}.$$

Consequently, (4.5) implies (4.6).

vi) For $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$ we obtain

$$P\{X_n < X_0 - \varepsilon, X_0 \in \mathbb{R}\} \le P\{X_0 \in \mathbb{R} - [\alpha, \beta]\} + P\{X_n < X_0 - \varepsilon, X_0 \in [\alpha, \beta]\}$$

$$\le P\{X_0 \in \mathbb{R} - [\alpha, \beta]\} + P\{[X_n, X_0 - \varepsilon] \cap [\alpha - \varepsilon, \beta] \ne \emptyset\},$$

and

$$P\{X_n < \delta, X_0 = +\infty\} \le P\{[X_n, X_0] \cap \{\delta\} \ne \emptyset\}.$$

Therefore, (4.6) implies (4.3). □

Let us note that we can read "$[X_n(\omega), X_0(\omega) - \varepsilon] = \mathcal{D}_{l,\varepsilon}(X_n, X_0, \,.\,; \omega)$" and "$[X_0(\omega), X_n(\omega) - \varepsilon] = \mathcal{H}_\varepsilon(X_n, X_0, \,.\,; \omega) = -\mathcal{D}_{l,\varepsilon}(-X_n, -X_0, \,.\,; \omega) - \varepsilon$". Therefore, we did not introduce any epi-upper approximation in probability for a sequence of random variables since that notion would coincide with the upper semicontinuous approximation in probability.

Under additional assumptions the lower semicontinuous approximation in probability and the epi-upper approximation in probability imply a convergence of optimal values and optimal solutions.

Theorem 4.1. Let $\{f_n,\ n \in \mathbb{N}_0\}$ be a family of functions $f_n|\ \mathbb{R}^p \times \Omega \to \overline{\mathbb{R}}$ and $\Delta \in C^p$. If

$$\int_{x\in\Delta} f_0(x,\,.\,) = \varphi\,(f_0) \quad \text{a.s. and} \quad f_n \xrightarrow[\Delta]{\text{epi-u-prob}} f_0 \tag{4.7}$$

then

$$\varphi\,(f_n) \xrightarrow{\text{u-prob}} \varphi\,(f_0)\,. \tag{4.8}$$

Proof. Let $\delta \in \mathbb{R}$ and $\varepsilon > 0$. Then we have the inclusion

$$\begin{aligned}&\{\omega\ :\ \varphi\,(f_n(.,\omega)) \geq \delta + 2\varepsilon, \varphi\,(f_0(.,\omega)) < \delta\} \cap \left\{\omega\ :\ \int_{x\in\Delta} f_0(x,\omega) = \varphi\,(f_0(.,\omega))\right\}\\ &\quad \subset \{\omega\ :\ \forall x \in \mathbb{R}^p\ f_n(x,\omega) \geq \delta + 2\varepsilon, \exists y \in \Delta\ f_0(y,\omega) < \delta\}\\ &\quad \subset \{\omega\ :\ \mathcal{H}_\varepsilon(f_n, f_0, \Delta; \omega) \cap (\Delta \times [\delta, \delta+\varepsilon]) \neq \emptyset\}\,.\end{aligned}$$

Therefore, (4.7) implies (4.8) since $\Delta \times [\delta, \delta+\varepsilon]$ is a compact. □

Theorem 4.2. Let $\{f_n,\ n \in \mathbb{N}_0\}$ be a family of functions $f_n|\ \mathbb{R}^p \times \Omega \to \overline{\mathbb{R}}$ and $K \in C^p$. If

$$\inf_{x\notin K} f_n(x,\,.\,) \xrightarrow{\text{l-prob}} \varphi\,(f_0) \quad \text{and} \quad f_n \xrightarrow[K]{\text{l-prob}} f_0 \tag{4.9}$$

then

$$\varphi\,(f_n) \xrightarrow{\text{l-prob}} \varphi\,(f_0)\,. \tag{4.10}$$

Proof. Let $\delta \in \mathbb{R}$ and $\varepsilon > 0$. Then we have the inclusion

$$\begin{aligned}&\left\{\omega\ :\ \inf_{x\in K} f_n(x,\omega) \leq \delta, \varphi\,(f_0(.,\omega)) > \delta + 2\varepsilon\right\}\\ &\quad \subset \{\omega\ :\ \exists x \in K\ f_n(x,\omega) \leq \delta, \forall y \in K\ f_0(y,\omega) > \delta + 2\varepsilon\}\\ &\quad \subset \{\omega\ :\ \mathcal{D}_{l,\varepsilon}(f_n, f_0, K; \omega) \cap (K \times [\delta, \delta+\varepsilon]) \neq \emptyset\} \quad \forall\ l \in \mathbb{N}.\end{aligned}$$

Therefore, (4.9) implies (4.10). □

Theorem 4.3. Let $\{f_n,\ n \in \mathbb{N}_0\}$ be a family of functions $f_n|\ \mathbb{R}^p \times \Omega \to \overline{\mathbb{R}}$, $\alpha_n|\ \Omega \to \overline{\mathbb{R}}$, $n \in \mathbb{N}_0$, $\limsup\limits_{n\to+\infty} \alpha_n \leq \alpha_0$ a.s., and $K \in C^p$. If

$$\lim_{n\to+\infty} P\left\{\omega\ :\ \inf_{x\notin K} f_n(x,\omega) > \max\left\{\varphi\,(f_0(.,\omega))\,, \alpha_0(\omega)\right\}\right\} = 1 \tag{4.11}$$

and

$$f_n \xrightarrow[K]{\text{l-prob}} f_0 \tag{4.12}$$

then

$$\varphi\,(f_n) \xrightarrow{\text{l-prob}} \varphi\,(f_0) \tag{4.13}$$

and

$$\operatorname{excess}(\Psi\left(f_n;\alpha_n\right)\Psi\left(f_0;\alpha_0\right)) \xrightarrow{\text{prob}} 0 \tag{4.14}$$

while

$$\lim_{n\to+\infty} P\left\{\omega\ :\ \Psi\left(f_n(.,\omega);\alpha_n(\omega)\right)\subset K \text{ is a compact}\right\} = 1. \tag{4.15}$$

Proof.

i) The relation (4.13) follows from Theorem 4.2 since (4.11) and (4.12) imply (4.9).

ii) Evidently, (4.11) implies (4.15).

iii) Let $\varepsilon > 0$, $A < B$, $\varepsilon, A, B \in \mathbb{R}$.
Then for all $l \in \mathbb{N}$ we obtain

$$\begin{aligned} &P\left\{\omega\ :\ \begin{array}{l} \operatorname{excess}(\Psi\left(f_n(.,\omega);\alpha_n(\omega)\right)\Psi\left(f_0(.,\omega);\alpha_0(\omega)+2\varepsilon\right)) > \varepsilon, \\ A \le \alpha_0(\omega) \le B, \alpha_n(\omega) < \alpha_0(\omega)+\varepsilon, \\ \Psi\left(f_n(.,\omega);\alpha_n(\omega)\right)\subset K, \end{array}\right\} \\ \le\ & P\left\{\omega\ :\ \mathcal{D}_{l,\varepsilon}(f_n,f_0,K;\omega)\cap(K\times[A+\varepsilon,B+\varepsilon])\neq\emptyset\ \right\} \end{aligned}$$

The assumptions of the theorem are giving

$$\lim_{n\to+\infty} P\left\{\omega\ :\ \begin{array}{l} \operatorname{excess}(\Psi\left(f_n(.,\omega);\alpha_n(\omega)\right)\Psi\left(f_0(.,\omega);\alpha_0(\omega)+2\varepsilon\right)) > \varepsilon, \\ A \le \alpha_0(\omega) \le B, \end{array}\right\} = 0.$$

Since ε, A, B can be arbitrary chosen, we are receiving (4.14). □

Theorem 4.4. Let $\{f_n,\ n \in \mathbb{N}_0\}$ be a family of functions $f_n|\ \mathbb{R}^p\times\Omega\to\overline{\mathbb{R}}$, $\alpha_n|\ \Omega\to\overline{\mathbb{R}}$, $n\in\mathbb{N}_0$, $\limsup\limits_{n\to+\infty}\alpha_n \le \alpha_0$ a.s., and $K,\Delta\in C^p$. If

$$\lim_{n\to+\infty} P\left\{\omega\ :\ \inf_{x\notin K} f_n(x,\omega) > \max\left\{\varphi\left(f_0(.,\omega)\right),\alpha_0(\omega)\right\}\right\} = 1, \tag{4.16}$$

$$\int_{x\in\Delta} f_0(x,\omega) = \varphi\left(f_0,\omega\right) \text{ for almost all } \omega\in\Omega, \tag{4.17}$$

$$f_n \xrightarrow[\Delta]{\text{epi-u-prob}} f_0 \quad\text{and}\quad f_n \xrightarrow[K]{\text{l-prob}} f_0 \tag{4.18}$$

then

$$\varphi\left(f_n\right) \xrightarrow{\text{prob}} \varphi\left(f_0\right) \tag{4.19}$$

and

$$\operatorname{excess}(\Psi\left(f_n;\alpha_n\right)\Psi\left(f_0;\alpha_0\right)) \xrightarrow{\text{prob}} 0 \tag{4.20}$$

while

$$\lim_{n\to+\infty} P\left\{\omega\ :\ \Psi\left(f_n(.,\omega);\alpha_n(\omega)\right)\subset K \text{ is a compact}\right\} = 1. \tag{4.21}$$

Proof. The statement is a direct combination of Theorems 4.1 and 4.3. □

5. POINTWISE CONDITIONS

Verification of the lower semicontinuous approximation or/and the epi-upper approximation almost surely or in probability could be rather complex at first glance. Fortunately in applications, we can often employ "pointwise approach" and the results from [27], section V. Let us repeat them in this special section.

Definition 5.1. Let $x_0 \in \mathbb{R}^p$ be fixed. By $f_n \xrightarrow[\{x_0\}]{p\text{-l-a.s.}} f_0$ we abbreviate the following property:

$$\forall \varepsilon > 0 \quad \exists U\{x_0\} \in C^p : P\left\{\omega : \liminf_{n\to+\infty} \inf_{x\in U\{x_0\}} f_n(x,\omega) < f_0(x_0) - \varepsilon\right\} = 0 \qquad (5.1)$$

and $f_n \xrightarrow[\{x_0\}]{p\text{-l-prob}} f_0$ stands for

$$\forall \varepsilon > 0 \quad \exists U\{x_0\} \in C^p : \lim_{n\to+\infty} P\left\{\omega : \inf_{x\in U\{x_0\}} f_n(x,\omega) < f_0(x_0) - \varepsilon\right\} = 0. \qquad (5.2)$$

The additional letter "p" stands to point at pointwise approach.

Then we have the following relations, cf. Theorem 9 in [27].

Proposition 5.1. Let f_0 be l.s.c. on X. Then

i) $(\forall x_0 \in X : f_n \xrightarrow[\{x_0\}]{p\text{-l-a.s.}} f_0) \Longrightarrow (f_n \xrightarrow[X]{\text{l-a.s.}} f_0)$.

ii) $(\forall x_0 \in X : f_n \xrightarrow[\{x_0\}]{p\text{-l-prob}} f_0) \Longrightarrow (f_n \xrightarrow[X]{\text{l-prob}} f_0)$.

The conditions (5.1) and (5.2) have the advantage to be "pointwise" conditions, therefore, it will be sufficient to show that (5.1) and (5.2) are satisfied at each $x_0 \in X$ in order to show that $f_n \xrightarrow[X]{\text{l-a.s.}} f_0$ and $f_n \xrightarrow[X]{\text{l-prob}} f_0$, respectively. Note, however, that for instance the condition

$$\forall x_0 \in X \quad \forall (x_n) \text{ with } x_n \to x_0 : P\left\{\omega : \liminf_{n\to+\infty} f_n(x_n,\omega) < f_0(x_0)\right\} = 0 \qquad (5.3)$$

is not sufficient for $f_n \xrightarrow[X]{\text{l-prob}} f_0$, even if f_0 is continuous.

Example 5.1. Let $p = 1$, $\Omega = [0,1]$, $\mathcal{A} = \Sigma_{[0,1]}$ the σ-field of Borel subsets of $[0,1]$, $X = [0,1]$, and P the Lebesgue measure on $[0,1]$. Suppose that

$$f_0(x) = 1 \quad \forall x \in [0,1] \quad \text{and} \quad f_n(x,\omega) = \begin{cases} 0 & \text{if } x = \omega + \frac{1}{n} \\ 1 & \text{otherwise.} \end{cases}$$

Then the condition 5.3 is satisfied, but

$$P\left\{\omega : [\mathrm{Epi}\, f_n(\cdot,\omega) \setminus U_{\frac{1}{2}}\mathrm{Epi}\, f_0(\cdot,\omega)] \cap [0,1]\times[0,1] \neq \emptyset\right\} = 1 - \frac{1}{n},$$

hence $(f_n)_{n\in\mathbb{N}}$ fails to be a lower semicontinuous approximation to f_0. □

Often in applications, the epi-upper approximation must be really checked for a single point, only. For that, we can use a well known relation that pointwise convergence implies the upper part of epi-convergence. Hence, Proposition 5.2 is obvious.

Proposition 5.2. If $(x_n)_{n\in\mathbb{N}}$ is a sequence with $x_n \to x_0$ then

(i) $\left(f_n(x_n,\cdot) \xrightarrow{\text{u-a.s.}} f_0(x_0)\right) \Rightarrow \left(f_n \xrightarrow[\{x_0\}]{\text{epi-u-a.s.}} f_0\right)$,

(ii) $\left(f_n(x_n,\cdot) \xrightarrow{\text{u-prob}} f_0(x_0)\right) \Rightarrow \left(f_n \xrightarrow[\{x_0\}]{\text{epi-u-prob}} f_0\right)$.

ACKNOWLEDGEMENT

The authors are grateful to the referees for helpful and fruitful comments, remarks and suggestions.

(Received May 28, 1999.)

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