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KOLMOGOROV COMPLEXITY AND PROBABILITY MEASURES

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Classes of strings (infinite sequences resp.) with a specific flow of Kolmogorov complexity are introduced. Namely, lower bounds of Kolmogorov complexity are prescribed to strings (initial segments of infinite sequences resp.) of specified lengths. Dependence of probabilities of the classes on lower bounds of Kolmogorov complexity is the main theme of the paper. Conditions are found under which the probabilities of the classes of the strings are close to one. Similarly, conditions are derived under which the probabilities of the classes of the sequences equal one.

It is shown that there are lower bounds of Kolmogorov complexity such that the studied classes of the strings are of probability close to one, classes of the sequences are of probability one, both with respect to almost all probability measures used in practice.

A variant of theorem on infinite oscillations is derived.

1. INTRODUCTION

The presented paper starts a series of papers dealing with applications of Kolmogorov complexity, mainly in probability theory and statistics. The results of the paper will be applied in the next papers of the series.

The papers will be devoted to distinguishing of probability measures by means of Kolmogorov complexity, controlling of the flow of Kolmogorov complexity of infinite sequences, pseudorandom generators and Monte–Carlo methods, statistical models testing, the law of the iterated logarithm.

Infinite sequences with a specific flow of Kolmogorov complexity are considered in the paper. Namely, a sequence is of this type if the Kolmogorov complexities of its initial segments of specified lengths are bounded from below by given lower bounds. Here is the difference from classical approach. Classical results prescribe lower bounds of Kolmogorov complexity for all lengths of initial segments of infinite sequences under consideration [1, 9, 10]. Classical approach deals with all the lengths of initial segments, we deal with specified lengths.

Dependence of probability of our classes on the lower bounds of Kolmogorov complexity is the main theme of the paper. Classical approach usually deals with the lower bounds close to the lengths of strings under consideration (with some exceptions like Theorem 4.1 in [1] is). Our lower bounds may increase slowly (like in Example 1), or they may be close to the lengths of strings under consideration, or they may vary somewhere between the above two types of bounds.

Our approach was motivated by theorems on infinite oscillations. They deal with upper bounds of Kolmogorov complexity of some initial segments of infinite sequences. If we choose lower bounds of Kolmogorov complexity for the initial segments of the same lengths and ascribe no lower bounds for the other ones, then we grasp the resulting class of infinite sequences in a way more adequate for further applications. Moreover, including of the other lower bounds may affect the classes in an inappropriate way. Our approach enables us to control the flow of Kolmogorov complexity of infinite sequences in a paper of our series.

Specific classes of infinite sequences were introduced in [8]. Each of the classes depends on a sequence of real numbers. This sequence determines lower bounds of Kolmogorov complexity of sequences from the class. Namely, an infinite sequence is places into the class iff the Kolmogorov complexity of almost all of its initial segments equals or exceeds the corresponding lower bounds. Here "almost all" means "up to a finite number of cases".

Kolmogorov complexity theory was originated by Kolmogorov in [5]. A similar approach to the program size complexity was initiated independently by Solomonoff [13] and Chaitin [2]. Exposition of the theory can be found e. g. in [1], Chapter 4, for detailed explanation with a wide range of applications see [9].

The paper is organized as follows.

The concept of Kolmogorov complexity is outlined in Section 1. Classes of strings (sequences resp.) with a specific flow of Kolmogorov complexity are defined in Section 2.

Probabilities of the classes introduced in Section 2 are considered in Section 3. Dependence of the probabilities on the lower bounds of Kolmogorov complexity is analyzed. It is shown that the classes are of probability close to one (equal one resp.) for almost all probability measures used in practice. The special case of the Lebesgue measure is treated in Section 4.

A mild generalization of famous Martin–Löf's result on infinite oscillations [11] is derived in Section 5.

Basic results of the paper concern a relationship between Kolmogorov complexity and probability measures (Sections 3 and 4) and infinite oscillations (Section 5).

NOTATION

The following notation is used in the paper.

The set $\{0, 1, 2, 3, \ldots\}$ of natural numbers is denoted by N. The symbols n, t denote natural numbers.

The symbol Σ denotes a finite alphabet of cardinality $c \geq 2$. The symbol Σ^* denotes the set of all strings over Σ, $l(x)$ denotes the length of a string x. The symbol Σ^n denotes the set of all strings over Σ having the length n.

We interpret strings as natural numbers too. Namely, we arrange strings into a lexicographical order, say x_0, x_1, x_2, ..., and interpret each string x_n as the natural number n.

The set of all (infinite) sequences over Σ is denoted by Σ^∞. The symbol S_n denotes the initial segment of a sequence S having the length n. Consider a set $\mathcal{X}$ of sequences. The symbol $S\mathcal{X}$ denotes the set of all initial segments of the sequences form $\mathcal{X}$, i. e. $S\mathcal{X} = \{S_n | S \in \mathcal{X} \,\&\, n \in N\}$.

The symbol Ψ denotes a universal Kolmogorov algorithm (see [1], p. 309) with inputs from the set $\Sigma^* \times N$ and with outputs in the set Σ^* (this universal Kolmogorov algorithm can be replaced by a partial recursive function computed by a universal Turing machine in the present paper).

We consider the σ-field of subsets of Σ^∞ generated by the set of cylinders. The symbol P denotes a probability measure on Σ^∞, while P_n denotes the corresponding marginal probability measure on Σ^n. Hence

$$P_n\{x\} = P\{S \in \Sigma^\infty \mid S_n = x\}$$

holds for each string $x \in \Sigma^n$. If P is the Lebesgue measure and $x \in \Sigma^n$, then we have $P_n\{x\} = c^{-n}$.

In addition to conventional notation we introduce the following operation. Assume that $A \subseteq \Sigma^*$ is a set of strings. We define

$$\begin{aligned} A * \Sigma^\infty/_n \quad &:= \quad (A * \Sigma^\infty) \cap \Sigma^n \\ &= \quad \text{the set of all initial segments of sequences from } A * \Sigma^\infty \\ & \qquad \text{having the length } n. \end{aligned}$$

Here $*$ denotes the operation of concatenation. We assume that the operations $\cup$ (set union) and $* \ldots /_n$ are of the same priority.

2. KOLMOGOROV COMPLEXITY

A concept of (conditional) Kolmogorov complexity is briefly outlined in the section. A simple lemma is stated characterizing the number of strings of bounded Kolmogorov complexity.

Let us start with a definition of (conditional) Kolmogorov complexity.

Definition 1.1. Let ϕ be a partial mapping from $\Sigma^* \times \Sigma^*$ to Σ^*. For each x, $w \in \Sigma^*$, the Kolmogorov complexity is defined by

$$K_\phi(x|w) = \inf\{l(p) \mid p \in \Sigma^* \;\&\; \phi(p, w) = x\}.$$

The string w represents our prior information about the string x. We do not assume that the mapping ϕ is computable at this moment. The reason is to distinguish the results based on computability assumptions from the more general ones.

The number of strings of bounded Kolmogorov complexity is estimated in

Lemma 1.1. Let ϕ be a partial mapping from $\Sigma^* \times \Sigma^*$ to Σ^*. Assume that w is a string, f is a nonnegative real number. Then we have

$$\operatorname{card}\{x \in \Sigma^* \mid K_\phi(x|w) < f\} \leq \frac{c^{f+1} - 1}{c - 1}.$$

Proof. The set $\{x \in \Sigma^* | K_\phi(x|w) < f\}$ is the union of the sets $X_n := \{x \in \Sigma^* | K_\phi(x|w) = n\}$ over all $0 \leq n < f$. It suffices to show that each set X_n contains at most c^n members. Assume that X_n contains j members, say $x_1, x_2, \ldots, x_j$.

We want to show that $j \leq c^n$. If $x \in \Sigma^n$ satisfies $K_\phi(x|w) = n$, then there is at least one $p(x) \in \Sigma^n$ such that $\phi(p(x), w) = x$ is true, as follows from Definition 1.1. Clearly, all $p(x_1), p(x_2), \ldots, p(x_j)$ are different members of Σ^n. Hence $j \leq \operatorname{card} \Sigma^n = c^n$. □

3. STRINGS AND SEQUENCES WITH A SPECIFIC FLOW OF KOLMOGOROV COMPLEXITY

Classes of strings and sequences with a specific flow of Kolmogorov complexity are introduced in the section. Their properties are derived in the foregoing sections.

Our considerations will be parameterized by the following entities.

i. A partial mapping ϕ from $\Sigma^* \times \Sigma^*$ to Σ^*.

ii. A sequence $\mathbf{w} = \langle w_0, w_1, w_2, \ldots \rangle$ of strings from Σ^*.

iii. A sequence $\mathbf{f} = \langle f_0, f_1, f_2, \ldots \rangle$ of nonnegative real numbers.

iv. A sequence $\mathcal{N} = \langle n_0, n_1, n_2, \ldots \rangle$ of different naturals.

Any number n_i represents the length of a string or strings under consideration. Let x be a string of the length n_i. Then the string w_i represents our prior information about x, the number f_i represents a lower bound of the corresponding Kolmogorov complexity.

We fix the partial mapping ϕ, our prior information $\mathbf{w}$ and the lengths $\mathcal{N}$ of the strings under consideration. We shall vary the lover bounds $\mathbf{f}$ of Kolmogorov complexity.

Namely, we consider strings satisfying the following two conditions. The length of our string is specified in the sequence $n_0, n_1, n_2, \ldots$. The Kolmogorov complexity of the string is bounded from below by the corresponding lower bound. Classes of such strings are introduced in

Definition 2.1. Consider a string $x \in \Sigma^*$. The string is called $(\phi, \mathbf{w}, \mathbf{f}, \mathcal{N})$-complex iff there is some $i \in N$ such that we have $l(x) = n_i$ and

$$K_\phi(x|w_i) \geq f_i.$$

The set of all (ϕ, $\mathbf{w}$, $\mathbf{f}, \mathcal{N}$)-complex strings is denoted by

$$\mathrm{Cstr}^{\phi,\mathbf{w}}_{\mathbf{f},\mathcal{N}}.$$

Clearly, the class of all (ϕ, $\mathbf{w}$, $\mathbf{f}, \mathcal{N}$)-complex strings having the length n_i equals

$$\mathrm{Cstr}^{\phi,\mathbf{w}}_{\mathbf{f},\mathcal{N}} \cap \Sigma^{n_i}.$$

We investigate probabilities of such classes below.

Classes of sequences with a specific flow of Kolmogorov complexity were studied e. g. in [7] and [8]. Loosely speaking, a sequence is of this nature if the Kolmogorov complexities of its long initial segments are bounded from below by prescribed lower bounds. We consider a slightly more general type of sequences. Namely, the Kolmogorov complexities of (sufficiently long) initial segments of specified lengths are bounded from below by prescribed lower bounds. The sequences are introduced in

Definition 2.2. Consider a sequence $S \in \Sigma^\infty$. The sequence is called (ϕ, $\mathbf{w}$, $\mathbf{f}, \mathcal{N}$)-complex iff

$$\exists t\ \forall i \geq t : K_\phi(S_{n_i}|w_i) \geq f_i.$$

The class of all (ϕ, $\mathbf{w}$, $\mathbf{f}, \mathcal{N}$)-complex sequences is denoted by

$$\mathrm{Cseq}^{\phi,\mathbf{w}}_{\mathbf{f},\mathcal{N}}.$$

Let S be an (ϕ, $\mathbf{w}$, $\mathbf{f}, \mathcal{N}$)-complex sequence. If i is sufficiently large, then the Kolmogorov complexity $K_\phi(S_{n_i}|w_i)$ of the initial segment S_{n_i} is bounded from below by f_i.

The class of all (ϕ, $\mathbf{w}$, $\mathbf{f}, \mathcal{N}$)-complex sequences is the union of the classes introduced in

Definition 2.3. Consider a sequence $S \in \Sigma^\infty$. The sequence is called (ϕ, $\mathbf{w}$, $\mathbf{f}, \mathcal{N}$, t)-complex iff

$$\forall i \geq t : K_\phi(S_{n_i}|w_i) \geq f_i.$$

The class of all (ϕ, $\mathbf{w}$, $\mathbf{f}, \mathcal{N}$, t)-complex sequences is denoted by

$$\mathrm{Cseq}^{\phi,\mathbf{w}}_{\mathbf{f},\mathcal{N},t}.$$

Clearly, the sets $\mathrm{Cseq}^{\phi,\mathbf{w}}_{\mathbf{f},\mathcal{N},t}$ constitute a nondecreasing sequence of sets. Their union equals $\mathrm{Cseq}^{\phi,\mathbf{w}}_{\mathbf{f},\mathcal{N}}$, hence we have

$$\mathrm{Cseq}^{\phi,\mathbf{w}}_{\mathbf{f},\mathcal{N},t} \nearrow_t \mathrm{Cseq}^{\phi,\mathbf{w}}_{\mathbf{f},\mathcal{N}}. \tag{3.1}$$

Assume that $i \geq t$. Then the initial segments of the $(\phi, \mathbf{w}, \mathbf{f}, \mathcal{N}, t)$-complex sequences having the length n_i are $(\phi, \mathbf{w}, \mathbf{f}, \mathcal{N})$-complex strings. Therefore, we have

$$x \in \mathcal{S}\mathrm{Cseq}^{\phi,\mathbf{w}}_{\mathbf{f},\mathcal{N},t} \cap (\cup_{i=0}^{\infty}\Sigma^{n_i}) \Longrightarrow x \in \mathrm{Cstr}^{\phi,\mathbf{w}}_{\mathbf{f},\mathcal{N}} \quad \text{a.s.} \tag{3.2}$$

"Almost surely" means "up to a finite number of cases". Clearly, if $i < t$, then Kolmogorov complexity of the initial segments of the length n_i may be "too low". The number of such strings is finite. Here the "almost surely" in (3.2) arises.

4. KOLMOGOROV COMPLEXITY AND PROBABILITY MEASURES

We investigate probabilities of the classes introduced in the previous section. A dependence of the probabilities on the sequence $\mathbf{f}$ of the lower bounds of Kolmogorov complexity is the main theme of the section.

Before going ahead, we introduce a notation and prove a lemma.

Let P be a probability measure on Σ^∞. For each n, f we define

$$\Pi_{nf}(P) := \max\left\{P_n(X) \mid X \subseteq \Sigma^n \ \&\ \mathrm{card}\,(X) \leq \frac{c^{f+1}-1}{c-1}\right\}. \tag{4.1}$$

Here f denotes a nonnegative real number.

Consider the strings of bounded Kolmogorov complexity, i.e. the strings x satisfying $K_\phi(x|w) < f$. The probability of such strings having the length n is bounded from above by $\Pi_{nf}(P)$, as is shown in

Lemma 3.1. Let P be a probability measure on Σ^∞. Assume that w is a string, f is a nonnegative real.

Then we have

$$P_n\{x \in \Sigma^n \mid K_\phi(x|w) < f\} \leq \Pi_{nf}(P).$$

Proof. We have $\mathrm{card}\,\{x \in \Sigma^n \mid K_\phi(x|w) < f\} \leq \frac{c^{f+1}-1}{c-1}$ according to Lemma 1.1, which together with (4.1) proves our lemma. □

Probability of the class of $(\phi, \mathbf{w}, \mathbf{f}, \mathcal{N})$-complex strings having the length n_i is considered now. A lower bound of such probabilities is derived in the following proposition. A simple condition is given under which the probabilities converge to one.

Proposition 3.1. Let P be a probability measure on Σ^∞.

a) For each i natural we have

$$P_{n_i}(\mathrm{Cstr}^{\phi,\mathbf{w}}_{\mathbf{f},\mathcal{N}} \cap \Sigma^{n_i}) \geq 1 - \Pi_{n_i f_i}(P). \tag{4.2}$$

b) If $\lim_{i\to\infty} \Pi_{n_i f_i}(P) = 0$, then

$$\lim_{i\to\infty} P_{n_i}(\mathrm{Cstr}^{\phi,\mathbf{w}}_{\mathbf{f},\mathcal{N}} \cap \Sigma^{n_i}) = 1. \tag{4.3}$$

Proof. We have

$$P_{n_i}\left(\mathrm{Cstr}^{\phi,\mathbf{w}}_{\mathbf{f},\mathcal{N}} \cap \Sigma^{n_i}\right) = 1 - P_{n_i}\left(\Sigma^{n_i} \setminus \mathrm{Cstr}^{\phi,\mathbf{w}}_{\mathbf{f},\mathcal{N}}\right)$$
$$= 1 - P_{n_i}\left(x \in \Sigma^{n_i} \,|\, K_\phi(x|w_i) < f_i\right) \qquad (4.4)$$

according to the definition of the set $\mathrm{Cstr}^{\phi,\mathbf{w}}_{\mathbf{f},\mathcal{N}}$. At the same time, Lemma 3.1 gives

$$P_{n_i}\left(x \in \Sigma^{n_i} \,|\, K_\phi(x|w_i) < f_i\right) \le \Pi_{n_i f_i}(P),$$

which, together with (4.4) gives (4.2).
Part b) of our proposition immediately follows from its part a). □

Probability of the set of (ϕ, $\mathbf{w}$, $\mathbf{f}, \mathcal{N}$, t)-complex sequences is considered in the following proposition. A lower bound of the probability is found. A simple condition is stated under which the probability of the class of (ϕ, $\mathbf{w}$, $\mathbf{f}, \mathcal{N}$)-complex sequences equals one.

Proposition 3.2. Let P be a probability measure on Σ^∞.
a) For each t natural we have

$$P(\mathrm{Cseq}^{\phi,\mathbf{w}}_{\mathbf{f},\mathcal{N},t}) \ge 1 - \Sigma^{\infty}_{i=t}\Pi_{n_i f_i}(P). \qquad (4.5)$$

b) If

$$\Sigma^{\infty}_{i=0}\Pi_{n_i f_i}(P) < \infty, \qquad (4.6)$$

then we have

$$P(\mathrm{Cseq}^{\phi,\mathbf{w}}_{\mathbf{f},\mathcal{N}}) = 1. \qquad (4.7)$$

Proof. a) It suffices to prove that the set

$$\mathcal{X}_t := \Sigma^\infty \setminus \mathrm{Cseq}^{\phi,\mathbf{w}}_{\mathbf{f},\mathcal{N},t}$$

is measurable and

$$P(\mathcal{X}_t) \le \Sigma^{\infty}_{i=t}\,\Pi_{n_i f_i}(P) \qquad (4.8)$$

takes place. Clearly, we have

$$\mathcal{X}_t = \cup^{\infty}_{i=t}\,\{S \in \Sigma^\infty \,|\, K_\phi(S_{n_i}|w_i) < f_i\}. \qquad (4.9)$$

The sets $\{S \in \Sigma^\infty \,|\, K_\phi(S_n|w) < f\}$ are cylinders, hence they are measurable, i. e. the set $\mathcal{X}_t$ is measurable too.
It holds

$$P(\mathcal{X}_t) \le \Sigma^{\infty}_{i=t}\,P\{S \in \Sigma^\infty \,|\, K_\phi(S_{n_i}|w_i) < f_i\}, \qquad (4.10)$$

as follows from (4.9). Moreover, for each n we have

$$\begin{aligned} P\{S \in \Sigma^\infty \mid K_\phi(S_n|w) < f\} &= P_n\{S_n \mid S \in \Sigma^\infty \,\&\, K_\phi(S_n|w) < f\} \\ &= P_n\{x \in \Sigma^n \mid K_\phi(x|w) < f\}. \end{aligned}$$

Applying the last result to (4.10) we obtain the inequality

$$P(\mathcal{X}_t) \;\leq\; \Sigma_{i=t}^\infty P_{n_i}\{x \in \Sigma^{n_i} \mid K_\phi(x|w_i) < f_i\},$$

which, together with Lemma 3.1 proves (4.8).

b) The set $\mathrm{Cseq}_{\mathbf{f},\mathcal{N}}^{\phi,\mathbf{w}}$ equals $\Sigma^\infty \setminus \cup_{t=0}^\infty \mathcal{X}_t$, hence it is measurable. We have (3.1), so that it suffices to prove that $\lim_{t\to\infty} P(\mathrm{Cseq}_{\mathbf{f},\mathcal{N},t}^{\phi,\mathbf{w}}) = 1$ is true. The last written equality follows from (4.6) and (4.5). □

Before going on, we introduce a notation and prove a simple lemma.
Assume that P is a probability measure on Σ^∞. We define

$$\pi_n(P) := \max_{x\in\Sigma^n} P_n\{x\}. \tag{4.11}$$

Clearly, $\pi_n(P)$ is the probability of a most probable string of the length n (recall that there may be several most probable strings).

Lemma 3.2. Let P be a probability measure on Σ^∞, f and g be nonnegative reals. If $\pi_n(P) \leq g$, then we have $\Pi_{nf}(P) < 2\cdot g\cdot c^f$.

Proof. We have $\Pi_{nf}(P) \leq \pi_n(P)\cdot \frac{c^{f+1}-1}{c-1}$ by (4.1) and (4.11). Moreover $2 \leq c$, hence $\frac{c^{f+1}-1}{c-1} < 2\cdot c^f$ is true. Therefore, we always have

$$\Pi_{nf}(P) < 2\cdot \pi_n(P)\cdot c^f. \tag{4.12}$$

If $\pi_n(P) \leq g$, then we have $\Pi_{nf}(P) < 2\cdot g\cdot c^f$. □

Estimates of the probabilities of interest, i.e. of the probabilities $P_{n_i}(\mathrm{Cstr}_{\mathbf{f},\mathcal{N}}^{\phi,\mathbf{w}} \cap \Sigma^{n_i})$ and $P(\mathrm{Cseq}_{\mathbf{f},\mathcal{N},t}^{\phi,\mathbf{w}})$ can be obtained from the lemma by means of the relations (4.2) and (4.5). Consider a sequence $g_0, g_1, g_2, \ldots$ of positive reals. Then we have

$$\pi_{n_i}(P) \leq g_i \;\Longrightarrow\; P_{n_i}(\mathrm{Cstr}_{\mathbf{f},\mathcal{N}}^{\phi,\mathbf{w}} \cap \Sigma^{n_i}) > 1 - 2\cdot g_i\cdot c^{f_i}, \tag{4.13}$$

$$\forall i \geq t:\, \pi_{n_i}(P) \leq g_i \;\Longrightarrow\; P(\mathrm{Cseq}_{\mathbf{f},\mathcal{N},t}^{\phi,\mathbf{w}}) > 1 - 2\cdot \Sigma_{i=t}^\infty \cdot g_i\cdot c^{f_i}. \tag{4.14}$$

There is a natural question which classes of probability measures are covered by the results obtained in Propositions 3.1 and 3.2. More precisely, we search for classes of probability measures such that each probability measure in the class satisfies our basic relations $\lim_{i\to\infty} P_{n_i}(\mathrm{Cstr}_{\mathbf{f},\mathcal{N}}^{\phi,\mathbf{w}} \cap \Sigma^{n_i}) = 1$ and $P(\mathrm{Cseq}_{\mathbf{f},\mathcal{N}}^{\phi,\mathbf{w}}) = 1$, i.e. (4.3) and (4.7).

Theorem 3.1. Assume that $\mathcal{P}$ is a class of probability measures on Σ^∞, $g_0, g_1, g_2, \ldots$ are positive reals.

Suppose that the relation $\pi_{n_i}(P) \leq g_i$ holds almost surely for each probability measure $P \in \mathcal{P}$.

a) If $\lim_{i\to\infty} g_i = 0$ is true, then there is a sequence $\mathbf{f}$ of lower bounds such that $\lim_{i\to\infty} f_i = \infty$ takes place and $\lim_{i\to\infty} P_{n_i}(\mathrm{Cstr}_{\mathbf{f},\mathcal{N}}^{\phi,\mathbf{w}} \cap \Sigma^{n_i}) = 1$ is true for each probability measure $P \in \mathcal{P}$.

b) If $\Sigma_{i=0}^{\infty} g_i < \infty$ is true, then there is a sequence $\mathbf{f}$ of lower bounds such that $\lim_{i\to\infty} f_i = \infty$ takes place and $P(\mathrm{Cseq}_{\mathbf{f},\mathcal{N}}^{\phi,\mathbf{w}}) = 1$ is true for each probability measure $P \in \mathcal{P}$.

Proof. Let $\mathcal{P}$ be a class of probability measures over Σ^∞, $P \in \mathcal{P}$.

a) Put $f_i = -\frac{1}{2} \cdot \log_c g_i$ if $g_i \leq 1$, $f_i = 0$ otherwise. Clearly, $\lim_{i\to\infty} f_i = \infty$. Moreover, $2 \cdot g_i \cdot c^{f_i} = 2 \cdot \sqrt{g_i}$ holds almost surely. Hence $2 \cdot g_i \cdot c^{f_i} \to_i 0$ is true and (4.3) follows form (4.13).

b) Put $i_0 = 0$. There are $0 < i_1 < i_2 < i_3 < \ldots$ natural such that $\Sigma_{i=i_k}^{\infty} g_i \leq c^{-2k}$ takes place for all $k = 1, 2, 3, \ldots$. Put $f_i = k$ for each $i = i_k, \ldots, i_{k+1} - 1$ and for any $k = 0, 1, 2, \ldots$. Clearly, $\lim_{i\to\infty} f_i = \infty$. Moreover, there is an index $k_0 \in N \setminus \{0\}$ such that $\pi_{n_i}(P) \leq g_i$ is true for all $i \geq i_{k_0}$. It suffices to prove that

$$\Sigma_{i=i_{k_0}}^{\infty} 2 \cdot g_i \cdot c^{f_i} < \infty$$

holds, as follows from (4.14). We have

$$\Sigma_{i=i_{k_0}}^{\infty} 2 \cdot g_i \cdot c^{f_i} = 2 \cdot \Sigma_{k=k_0}^{\infty} \Sigma_{i=i_k}^{i_{k+1}-1} g_i \cdot c^{f_i}.$$

Further on, we have

$$\begin{aligned} \Sigma_{i=i_k}^{i_{k+1}-1} g_i \cdot c^{f_i} &= c^k \cdot \Sigma_{i=i_k}^{i_{k+1}-1} g_i \\ &< c^k \cdot c^{-2k}, \end{aligned}$$

as both $f_i = k$ is true for $i = i_k, \ldots, i_{k+1} - 1$ and $\Sigma_{i=i_k}^{\infty} g_i \leq c^{-2k}$ takes place. Therefore, we have $\Sigma_{i=i_{k_0}}^{\infty} 2 \cdot g_i \cdot c^{f_i} < 2 \cdot \Sigma_{k=k_0}^{\infty} c^{-k} < \infty$ which finishes the proof. □

If a single probability measure P is considered, then our basic relations $\lim_{i\to\infty} P_{n_i}(\mathrm{Cstr}_{\mathbf{f},\mathcal{N}}^{\phi,\mathbf{w}} \cap \Sigma^{n_i}) = 1$ and $P(\mathrm{Cseq}_{\mathbf{f},\mathcal{N}}^{\phi,\mathbf{w}}) = 1$ are satisfied under simple and mild conditions, as is shown in

Corollary 3.1. Let P be a probability measure on Σ^∞.

a) If $\lim_{i\to\infty} \pi_{n_i}(P) = 0$ is true, then there is a sequence $\mathbf{f}$ of lower bounds such that both $\lim_{i\to\infty} f_i = \infty$ and $\lim_{i\to\infty} P_{n_i}(\mathrm{Cstr}_{\mathbf{f},\mathcal{N}}^{\phi,\mathbf{w}} \cap \Sigma^{n_i}) = 1$ are true.

b) If $\Sigma_{i=0}^{\infty} \pi_{n_i}(P) < \infty$ is true, then there is a sequence $\mathbf{f}$ of lower bounds such that both $\lim_{i\to\infty} f_i = \infty$ and $P(\mathrm{Cseq}_{\mathbf{f},\mathcal{N}}^{\phi,\mathbf{w}}) = 1$ are true.

Proof. Put $g_i = \pi_{n_i}(P)$ for each i natural. Apply Theorem 3.1. □

We slightly reformulate Theorem 3.1 to the form appropriate for practical applications.

Corollary 3.2. Assume that $\mathcal{P}$ is a class of probability measures on Σ^∞, $h_0, h_1, h_2, \dots$ are positive reals.

Suppose that $\pi_{n_i}(P)$ is of $O(h_i)$ type[1] for each probability measure $P \in \mathcal{P}$.

a) If $\lim_{i\to\infty} h_i = 0$ is true, then there is a sequence $\mathbf{f}$ of lower bounds such that $\lim_{i\to\infty} f_i = \infty$ takes place and $\lim_{i\to\infty} P_{n_i}(\mathrm{Cstr}_{\mathbf{f},\mathcal{N}}^{\phi,\mathbf{w}} \cap \Sigma^{n_i}) = 1$ is true for each probability measure $P \in \mathcal{P}$.

b) If $\Sigma_{i=0}^{\infty} h_i < \infty$ is true, then there is a sequence $\mathbf{f}$ of lower bounds such that $\lim_{i\to\infty} f_i = \infty$ takes place and $P(\mathrm{Cseq}_{\mathbf{f},\mathcal{N}}^{\phi,\mathbf{w}}) = 1$ is true for each probability measure $P \in \mathcal{P}$.

Proof. a) It suffices to find a sequence $g_0, g_1, g_2, \dots$ of positive reals such that we have both $g_i \to_i 0$ and $g_i/h_i \to_i \infty$. Then $C(P) \le g_i/h_i$ holds almost surely, hence $\pi_{n_i}(P) \le C(P) \cdot h_i \le g_i$ holds almost surely too and the proof follows directly from Theorem 3.1a. The desirable g_i's can be found easily; we can take $g_i = \sqrt{h_i}$ for each i natural.

b) It suffices to find a sequence $g_0, g_1, g_2, \dots$ of positive reals such that we have both $\Sigma_{i=0}^{\infty} g_i < \infty$ and $g_i/h_i \to_i \infty$. Then $\pi_{n_i}(P) \le g_i$ holds almost surely and the proof follows directly from Theorem 3.1b.

Put $i_0 = 0$. There are $0 < i_1 < i_2 < i_3 < \dots$ natural such that $\Sigma_{i=i_k}^{\infty} h_i \le c^{-2k}$ takes place for all $k = 1, 2, 3, \dots$. Put $g_i = c^{-k}$ for each $i = i_k, \dots, i_{k+1} - 1$ and for any $k = 0, 1, 2, \dots$. We have $g_i/h_i \ge 2^k$ for any $i = i_k, \dots, i_{k+1} - 1$ and for any $k \ge 1$ natural, hence $g_i/h_i \to_i \infty$. We have $\Sigma_{i=0}^{\infty} g_i = \Sigma_{i=0}^{i_0-1} g_i + \Sigma_{k=1}^{\infty} \Sigma_{i=i_k}^{i_{k+1}-1} h_i \cdot g_i/h_i$, so that $\Sigma_{i=0}^{\infty} g_i \le i_1 + \Sigma_{k=1}^{\infty} c^{-2k} \cdot c^k < \infty$. □

The following example shows that the results of this section cover almost all probability measures used in practice. Corollary 3.2 is used as a tool.

Example 1. The aim of the example is to show that our basic relations

$$\lim_{i\to\infty} P_{n_i}(\mathrm{Cstr}_{\mathbf{f},\mathcal{N}}^{\phi,\mathbf{w}} \cap \Sigma^{n_i}) = 1,$$

$$P(\mathrm{Cseq}_{\mathbf{f},\mathcal{N}}^{\phi,\mathbf{w}}) = 1$$

are true for almost all probability measure used in practice. Moreover, there is a single sequence $\mathbf{f}$ of lower bounds with

$$\lim_{i\to\infty} f_i = \infty$$

[1] Hence $\pi_{n_i}(P) \le C(P) \cdot h_i$ is true almost surely, where $C(P)$ is a constant depending on P.

such that our basic relations are valid for all the probability measures just mentioned.

It is worth mentioning that strings represent members of a product sample space here.

Consider a probability measure P. Recall that $\pi_n(P)$ is the probability of a most probable string of the length n. As a rule, $\pi_n(P)$ converges to zero exponentially with the length of the strings. This is true for ergodic measures. *The opinion of the authors is that $\pi_n(P)$ is of $O(n^{-2})$ type for almost all probability measures used in statistical practice.*

Consider a class of probability measures, say $\mathcal{P}$.

a) For the sake of safety assume, that $\pi_n(P)$ is of $O(1/\ln\ln n)$ type for each probability measure P from our class. Then there is a single sequence $\mathbf{f}$ of lower bounds satisfying $\lim_{i\to\infty} f_i = \infty$ and such that our basic relation $\lim_{i\to\infty} P_{n_i}(\mathrm{Cstr}_{\mathbf{f},\mathcal{N}}^{\phi,\mathbf{w}} \cap \Sigma^{n_i}) = 1$ is true, as follows from Corollary 3.2a. For instance, we can take

$$f_n = \log_c \ln\ln\ln n \quad \forall n \geq 3814280\ , \tag{4.15}$$

as can be proved easily by means of (4.13). At the same time, almost all probability measures used in practice lie in our class $\mathcal{P}$.

b) For the sake of safety assume, that $\pi_n(P)$ is of $O(n^{-1}\ln^{-3/2} n)$ type for each probability measure P from our class. Then there is a single sequence $\mathbf{f}$ of lower bounds satisfying $\lim_{i\to\infty} f_i = \infty$ and such that our basic relation $P(\mathrm{Cseq}_{\mathbf{f},\mathcal{N}}^{\phi,\mathbf{w}}) = 1$ is true, as follows from Corollary 3.2b. Of course, $\lim_{i\to\infty} P_{n_i}(\mathrm{Cstr}_{\mathbf{f},\mathcal{N}}^{\phi,\mathbf{w}} \cap \Sigma^{n_i}) = 1$ holds too. For instance, we can take

$$f_n = \log_c(\ln^{1/3} n) \quad \forall n \geq 3, \tag{4.16}$$

as can be proved easily by means of (4.14). Once more, almost all probability measures used in practice lie in our class $\mathcal{P}$.

c) Let us turn to nondegenerated ergodic measures. If a single ergodic measure P is considered, then our basic relations are fulfilled through

$$f_n = \varepsilon \cdot n \quad \forall\ n, \tag{4.17}$$

where $0 < \varepsilon < 1$ depends on our ergodic measure. If $\mathcal{P}$ is the class of all nondegenerated ergodic measures, then our basic relations are fulfilled through

$$f_n = n/\ln\ln n \quad \forall\ n \geq 2, \tag{4.18}$$

as $\varepsilon \cdot n \leq n/\ln\ln n$ holds almost surely for any $0 < \varepsilon < 1$.

To cover a wide class of probability measures by the results of this section, the sequence $\mathbf{f}$ of lower bounds should tend to infinity slowly, e. g. like the sequences (4.15) and (4.16) do.

For narrower classes of probability measures, like for ergodic measures, the sequence $\mathbf{f}$ of lower bounds is usually of the form $n/$"something slowly nondecreasing", e. g. like the sequences (4.17) and (4.18) are.

5. KOLMOGOROV COMPLEXITY AND THE LEBESGUE MEASURE

The Lebesgue measure, denoted by P in the section, plays a significant role in probability theory and statistics. We derive conditions under which our basic relations $\lim_{i\to\infty} P_{n_i}(\mathrm{Cstr}\,^{\phi,\mathbf{w}}_{\mathbf{f},\mathcal{N}} \cap \Sigma^{n_i}) = 1$ and $P(\mathrm{Cseq}\,^{\phi,\mathbf{w}}_{\mathbf{f},\mathcal{N}}) = 1$ are true for the case of the Lebesgue measure.

The probabilities of the classes of $(\phi, \mathbf{w}, \mathbf{f}, \mathcal{N})$-complex strings having the length n_i converge to one under a simple and general conditions, as is shown in

Proposition 4.1. Let P be the Lebesgue measure on Σ^∞.

a) For each i natural we have

$$P_{n_i}(\mathrm{Cstr}\,^{\phi,\mathbf{w}}_{\mathbf{f},\mathcal{N}} \cap \Sigma^{n_i}) \geq 1 - 2 \cdot c^{f_i - n_i}.$$

b) If $\lim_{i\to\infty} n_i - f_i = \infty$ takes place, then we have

$$\lim_{i\to\infty} P_{n_i}(\mathrm{Cstr}\,^{\phi,\mathbf{w}}_{\mathbf{f},\mathcal{N}} \cap \Sigma^{n_i}) = 1.$$

Proof. a) Clearly, $P_n\{x\} = c^{-n}$ is true for each string x of the length n, as P is the Lebesgue measure. So that we have $\pi_{n_i}(P) = c^{-n_i}$ for each i natural. We take $g_i := \pi_{n_i}(P) = c^{-n_i}$ in (4.13) and obtain the desirable result.
Part b) of our proposition immediately follows from its part a). □

A simple condition under which the probability of the class of $(\phi, \mathbf{w}, \mathbf{f}, \mathcal{N})$-complex sequences equals one is stated in

Proposition 4.2. Let P be the Lebesgue measure on Σ^∞.

a) For each t natural we have

$$P(\mathrm{Cseq}\,^{\phi,\mathbf{w}}_{\mathbf{f},\mathcal{N},t}) > 1 - 2 \cdot \Sigma_{i=t}^{\infty} c^{f_i - n_i}.$$

b) If $\Sigma_{i=0}^{\infty} c^{f_i - n_i} < \infty$, then we have

$$P(\mathrm{Cseq}\,^{\phi,\mathbf{w}}_{\mathbf{f},\mathcal{N}}) = 1.$$

Proof. a) We take $g_i := \pi_{n_i}(P) = c^{-n_i}$, like in the proof of the previous proposition, and apply it to (4.14).
Part b) follows from the part a). □

Proposition 4.2b was proved by Martin–Löf for the special case when all the lengths of initial segments of infinite sequences are considered, i.e. for the case when $n_i = i$ holds for all i natural (see [11]).

6. ON INFINITE OSCILLATIONS

The aim of this section is to state and prove a result on infinite oscillations closely related to the original and well known result of Martin–Löf [11].

The classes of $(\Psi, \mathcal{N}, \mathbf{f}, \mathcal{N})$-complex sequences are considered in the foregoing text. In other words, Kolmogorov complexity is defined by means of a universal Kolmogorov algorithm Ψ, i.e. $\phi = \Psi$ takes place. Our prior information about a string equals the length of the string, i.e. $\mathbf{w} = \mathcal{N}$ holds.

Let us recall a result on infinite oscillations due to Martin–Löf [11] and Katseff [4]. It reads (in our notation):

Assume that $n_i = i$ holds for each i natural. Moreover, let h_0, h_1 $h_2, \dots$ be a recursive sequence of naturals such that $h_i \leq f_i \leq i$ takes place for each i. If

$$\Sigma_{i=0}^{\infty} c^{h_i - i} = \infty,$$

then

$$K_\Psi(S_i|i) < f_i \tag{6.1}$$

is true for infinitely many $i \in N$, i.e. the class $\mathrm{Cseq}_{\mathbf{f},\mathcal{N}}^{\Psi,\mathcal{N}}$ is empty.

We need an information on indexes i satisfying (6.1) for purposes of the next papers in our series. For this reason we prove a result resembling the Martin–Löf's theorem. The core of our construction is closely related to both Calude's [1], pp. 361–367 and Katseff's [4] ones.

We start with

Lemma 5.1. Let $n_i \leq n$ and let $A_i \subseteq \Sigma^{n_i}$ hold for each $i = 1, 2, \dots, p$, where p is a natural number.

If $A_1 * \Sigma^\infty/_n$, $A_2 * \Sigma^\infty/_n$, $\dots$, $A_p * \Sigma^\infty/_n$ are pairwise disjoint sets, then we have

$$\Sigma_{i=1}^{p} c^{-n_i} \operatorname{card} A_i \leq 1. \tag{6.2}$$

Moreover, equality takes place in (6.2) if and only if

$$\cup_{i=1}^{p} A_i * \Sigma^\infty/_n = \Sigma^n. \tag{6.3}$$

Proof. Clearly,

$$\operatorname{card}\,(A_i * \Sigma^\infty/_n) = c^{n-n_i} \cdot \operatorname{card} A_i \tag{6.4}$$

holds for each $i = 1, 2, \dots, p$. Further on, we have

$$\begin{aligned} 1 &= c^{-n} \operatorname{card} \Sigma^n \\ &\geq c^{-n} \operatorname{card}\,(\cup_{i=1}^{p} A_i * \Sigma^\infty/_n). \end{aligned} \tag{6.5}$$

The sets $A_1 * \Sigma^\infty/_n, \dots, A_p * \Sigma^\infty/_n$ are pairwise disjoint, hence

$$\begin{aligned} c^{-n} \operatorname{card}\,(\cup_{i=1}^{p} A_i * \Sigma^\infty/_n) &= c^{-n} \Sigma_{i=1}^{p} c^{n-n_i} \operatorname{card} A_i \\ &= \Sigma_{i=1}^{p} c^{-n_i} \operatorname{card} A_i \end{aligned} \tag{6.6}$$

takes place, as follows from (6.4). Using (6.5) and (6.6) we find that (6.2) is true. Finally, the equality takes place in (6.2) iff the equality is valid in (6.5), i.e. iff (6.3) holds. □

Let us proceed to the main topic of the section, a theorem on infinite oscillations.

Theorem 5.1. Assume that both sequences $\mathcal{N}$ and $\mathbf{f}$ are recursive sequences of naturals. Moreover, let $f_i \le n_i$ hold for each $i \in N$. Finally, let $q_0, q_1, q_2, \ldots$ be a recursive and increasing sequence of naturals.

Then there is a constant C such that for each $U \in \Sigma^\infty$ and $m \in N$ the following holds: if

$$\Sigma_{i=q_m}^{q_{m+1}-1} c^{f_i - n_i} \ge 1 \tag{6.7}$$

takes place, then there is $i \in \{q_m, q_m + 1, \ldots, q_{m+1} - 1\}$ such that we have

$$K_\Psi(U_{n_i} \mid n_i) < f_i + C. \tag{6.8}$$

Proof. Let us describe the core of our proof. We find a recursive sequence $A_0, A_1, A_2, \ldots$ of finite sets of strings such that:

i. The cardinality of each set A_i does not exceed the prescribed value c^{f_i}.

ii. The set $\cup_{i=q_m}^{q_{m+1}-1} A_i$ contains an initial segment of any string in some Σ^n. Here $m \in N$.

Hence if U is a sequence from Σ^∞, then at least one of its initial segments lies in some of the sets $A_{q_m}, A_{q_m+1}, \ldots, A_{q_{m+1}-1}$. This gives us the desired upper bound of Kolmogorov complexity of the segment.

1. First of all, we construct a recursive sequence $A_0, A_1, A_2, \ldots$ of sets such that:
   a) Both $A_i \subseteq \Sigma^{n_i}$ and card $A_i \le c^{f_i}$ hold for each $i \in N$.
   b) If $m \in N$ is fixed, then the sets $A_{q_m} * \Sigma^\infty/_n, A_{q_m+1} * \Sigma^\infty/_n, \ldots, A_{q_{m+1}-1} * \Sigma^\infty/_n$ are pairwise disjoint, where $n = \max\{n_i | i = q_m, q_m+1, \ldots, q_{m+1}-1\}$.

2. Let m be fixed in the rest of the proof. We denote

$$\begin{aligned} r &:= q_m, \\ s &:= q_{m+1} - 1, \\ n &:= \max\{n_i \mid i = r, r+1, \ldots, s\}. \end{aligned}$$

   Suppose that $r \le i \le s$ and that the sets $A_r, A_{r+1}, \ldots, A_{i-1}$ have been constructed. We put

$$E_i := \{x \in \Sigma^{n_i} \mid \cup_{k=r}^{i-1} A_k * \Sigma^\infty/_n \cap \{x\} * \Sigma^\infty/_n = \emptyset\}.$$

   Hence E_i contains all the strings from Σ^{n_i} which are not the initial segments of strings from the set $\cup_{k=r}^{i-1} A_k * \Sigma^\infty/_n$.

We consider two cases. If $\operatorname{card} E_i < c^{f_i}$, we put $A_i = E_i$; then $\operatorname{card} A_i < c^{f_i}$ holds. If $\operatorname{card} E_i \geq c^{f_i}$, then A_i contains the first c^{f_i} sequences from E_i (first with respect to a specified lexicographical order); hence $\operatorname{card} A_i = c^{f_i}$ holds. Therefore, a) and b) are true in both cases.

We started with $i = r$, i. e. with the empty sequence $A_r, \ldots, A_{r-1}$. Thus $\cup_{k=r}^{r-1} A_k = \emptyset$ and $E_r = \Sigma^{n_r}$ are true, so that $\operatorname{card} E_r \geq c^{f_r}$ holds. Therefore, we have $\operatorname{card} A_r = c^{f_r}$.

3. We prove that a), b) and $\Sigma_{i=r}^{s} c^{f_i - n_i} \geq 1$ (i. e. (6.7)) imply

$$\cup_{k=r}^{s} A_k * \Sigma^{\infty}/_n = \Sigma^n. \tag{6.9}$$

It means that any sequence from Σ^{∞} has an initial segment in some of the sets $A_r, A_{r+1}, \ldots, A_s$. We distinguish two cases.

First, let $\operatorname{card} A_i = c^{f_i}$ hold for each $i = r, r+1, \ldots, s$. Then

$$\rho := \Sigma_{i=r}^{s} c^{-n_i} \operatorname{card} A_i = \Sigma_{i=r}^{s} c^{f_i - n_i} \geq 1$$

is true, as follows from (6.7). The inequality $\rho \leq 1$ follows from a), b) and Lemma 5.1. Hence we have $\rho = 1$ and (6.9) follows from Lemma 5.1.

Second, let $i \in \{r+1, r+2, \ldots, s\}$ be such that $\operatorname{card} A_i < c^{f_i}$ is true (recall that we have $\operatorname{card} A_r = c^{f_r}$). Then

$$(\cup_{k=r}^{i-1} A_k * \Sigma^{\infty}/_n) \cup (E_i * \Sigma^{\infty}/_n) = \Sigma^n$$

and $A_i = E_i$ are true, so that (6.9) holds.

4. The sequence $A_0, A_1, A_2, \ldots$ is obviously recursive, hence there is a constant C such that the inequality

$$K_{\Psi}(x|n_i) < \log_c(\operatorname{card} A_i) + C$$

holds for each $i \in N$ and $x \in A_i$. Moreover, $\log_c(\operatorname{card} A_i) \leq f_i$ is true by a). So that

$$K_{\Psi}(x|n_i) < f_i + C \tag{6.10}$$

takes place for each $i \in N$ and $x \in A_i$.

5. Let $U \in \Sigma^{\infty}$. Then $U_n \in \Sigma^n$, so that there is $i \in \{r, r+1, \ldots, s\}$ such that $U_n \in A_i * \Sigma^{\infty}/_n$, hence we have $U_{n_i} \in A_i$. Therefore, (6.8) follows from (6.10) with $x = U_{n_i}$. □

The Martin–Löf's result on infinite oscillations stated above is a special case of

Corollary 5.1. Assume that both sequences $\mathcal{N}$ and $\mathbf{h} = \langle h_0, h_1, h_2, \ldots \rangle$ are recursive sequences of naturals. Moreover, let $h_i \le f_i \le n_i$ hold for each $i \in N$.

If

$$\Sigma_{i=0}^{\infty} c^{h_i - n_i} = \infty, \tag{6.11}$$

then the class $\mathrm{Cseq}_{\mathbf{f},\mathcal{N}}^{\Psi,\mathcal{N}}$ is empty.

Proof. We can find a recursive sequence $f'_0, f'_1, f'_2, \ldots$ of naturals such that $f'_i \le h_i$ holds for each i natural,

$$\lim_{i \to \infty} h_i - f'_i = \infty$$

takes place and

$$\Sigma_{i=0}^{\infty} c^{f'_i - n_i} = \infty$$

is valid.

Then we find a recursive and increasing sequence $q_0, q_1, q_2, \ldots$ of naturals such that

$$\Sigma_{i=q_m}^{q_{m+1}-1} c^{f'_i - n_i} \ge 1$$

is true for each m natural.

Let $U \in \Sigma^\infty$. Applying Theorem 5.1 we find that

$$K_\Psi(U_{n_i} \,|\, n_i) < f'_i + C$$

holds for infinitely many i's, where C is a constant. Hence $K_\Psi(U_{n_i} \,|\, n_i) < f_i$ is valid for infinitely many i's, which finishes the proof. □

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