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$H_2$ optimal decoupling of previewed signals in the discrete-time case

# $H_2$ OPTIMAL DECOUPLING OF PREVIEWED SIGNALS IN THE DISCRETE–TIME CASE

GIOVANNI MARRO, DOMENICO PRATTICHIZZO AND ELENA ZATTONI

The synthesis of a feedforward unit for $H_2$ optimal decoupling of measurable or previewed signals in discrete-time linear time-invariant systems is considered. It is shown that an $H_2$ optimal compensator can be achieved by connecting a finite impulse response (FIR) system and a stable dynamic unit. To derive the FIR system convolution profiles an easily implementable computational scheme based on pseudoinversion (possibly nested to avoid computational constraints) is proposed, while the dynamic unit is derived by solving a standard LQR problem, in general cheap or singular.

## 1. INTRODUCTION

The aim of this paper is to present an algorithmic framework for the solution of $H_2$ optimal decoupling of previewed or measurable signals, i.e. the problem of minimizing the effect at the output of a signal which can either be known in advance by a certain amount of time or can be completely given a priori. In very recent years many papers and books have been written, dealing with the $H_2$ optimal control problem, see e.g. [11, 12, 33, 34, 37]. However, to the best of our knowledge, the possibility of taking advantage of either a partial or a complete preview of the signal to be decoupled by adding a suitable feedforward compensator to the feedback control scheme has not been considered yet.

The decoupling problem is a classical control problem typically treated with the geometric approach tools: the unaccessible disturbance localization was first approached in [2] and independently in [40]. A few years later the localization of measurable signals was investigated in [7]. Extensions were provided in [38] and [20], while the dual problem, i.e. unknown input observation with differentiators, had been already solved with stability in [5], one year before. In this context, see also [9, 10]. Instead, more recently, the problem was extended to include also the case of previewed signals for dealing with cases where the stabilizability condition is not satisfied and a preaction steering the state along the unstable zero dynamics is therefore mandatory: see e.g. [1] and [26], where efficient algorithms for discrete-time MIMO systems were presented. However, if the geometric conditions

which guarantee perfect decoupling are not satisfied, then an approach which aims at minimizing some norm of the decoupling error is mandatory. In particular, an $H_2$ optimal decoupling appears to be convenient for its simplicity from the computational viewpoint. Furthermore, an exactly solvable decoupling problem reduces to an $H_2$ optimal decoupling problem that is cost-free solvable.

In this paper, the control target is achieved through a feedforward compensator unit which consists of the cascade of a dynamic system and a finite impulse response system. This latter choice is innovative. In fact, within control theory, FIR systems have usually been used for filtering rather than for control purposes, see e.g. [31] and the references therein. As far as the algorithm for designing the FIR system gain matrix is concerned, it is worth noticing that it is based on pseudoinversion techniques provided with a mean to overcome the dimensionality constraint intrinsic to the use of such techniques.

The interest of this work from a practical standpoint is mainly due to the close connection existing between decoupling and perfect tracking, this latter also extensively studied in the literature ([13, 16, 17, 32, 35, 36]). As recently pointed out in [26], any perfect tracking (or right-inversion) problem can be recast as a signal decoupling problem. Hence, $H_2$ optimal decoupling includes $H_2$ optimal tracking as a special case. In the light of this achievement, the attempt to extend to decoupling the advantages (widely exploited in tracking) of preview and preaction comes naturally. In fact, it is well known that perfect or almost perfect tracking can be achieved also in the non-minimum phase case if the signal to be tracked is known in advance. See, for instance, [15] and [19] for the infinite horizon nonlinear and linear case, respectively, while refer to [18] and to [23] for two different approaches to the receding horizon SISO case.

The results presented within the signal decoupling problem also apply to the dual setting, i. e. $H_2$ optimal observation (with a possible delay) of a linear function of the state in the presence of unknown inputs.

Throughout this paper, $\mathbb{R}$ stands for the field of real numbers; sets, vector spaces and subspaces are denoted by script capitals like $\mathcal{V}$, matrices and linear maps by slanted capitals like $A$, the image and the null space of $A$ by $\operatorname{im} A$ and $\ker A$ respectively, the trace by $\operatorname{tr} A$, the transpose by $A'$, the pseudo-inverse by $A^{\#}$ and, finally, the spectrum by $\sigma(A)$.


## 2. RECALLS ON LQ OPTIMAL CONTROL WITH CONSTRAINED FINAL STATE

This section recalls some results on the solution of the finite-horizon linear quadratic optimal control problem with both the initial and the final states assigned. This problem has been widely investigated in the regular case, namely under the assumption that the matrix (usually denoted by $R$) weighting the control input in the cost function is positive definite: see for instance [8, 14, 21]. However, in order to solve the $H_2$ optimal decoupling problem, which is the object of this paper, the solution of the above cited problem with $R=0$ is required. In [27], the authors presented a solution based on pseudoinversion to the problem with both a terminal cost in

the performance index and a non-stiff constraint on the terminal state under the assumption of $R$ not necessarily invertible. In this section, the algorithm presented in [27] is modified to cope with a stiff terminal constraint.

Consider the discrete time-invariant linear system

$$
\begin{aligned}
x(k+1) &= A\,x(k) + B\,u(k)\,, \quad x(0) = x_0\,, \\
y(k) &= C\,x(k) + D\,u(k)\,,
\end{aligned}
\tag{1}
$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, $y \in \mathbb{R}^q$, $k \in [0,\ N-1]$, and with terminal state sharply assigned, i.e.

$$
x(N) = x_1\,.
\tag{2}
$$

Assume that the matrix $[B'\ D']$ is of full rank, the pair $(A, B)$ is controllable and the final time instant $N$ is greater than or equal to the system controllability index. Furthermore, system (1) is assumed to be left-invertible, i.e.

$$
\hat{\mathcal{V}}^* \cap \hat{\mathcal{S}}^* = \{0\}\,,
$$

where $\hat{\mathcal{V}}^*$ denotes the maximum $(\hat{A}, \operatorname{im}\hat{B})$-controlled invariant contained in $\ker\hat{C}$ and $\hat{\mathcal{S}}^*$ the minimum $(\hat{A}, \ker\hat{C})$-conditioned invariant containing $\operatorname{im}\hat{B}$, with $(\hat{A}, \hat{B}, \hat{C}) = (A, B, C)$ if $D = 0$ or

$$
\hat{A} := \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}, \quad \hat{B} := \begin{bmatrix} B \\ D \end{bmatrix}, \quad \hat{C} := \begin{bmatrix} 0 & I_q \end{bmatrix},
$$

if $D \neq 0$. This corresponds to add a unit delay at the output, as suggested in [6]. The left invertibility assumption guarantees uniqueness of the optimal control sequence, but can easily be removed by re-stating the problem as shown in Remark 2.

The discrete-time finite-horizon linear quadratic optimal control problem with the terminal state sharply assigned can be stated as follows.

**Problem 1.** Consider system (1) and find a control sequence $u(k)$, $k \in [0, N-1]$, such that the cost function

$$
J := \sum_{k=0}^{N-1} y(k)'y(k)
\tag{3}
$$

is minimized under the constraint (2).

A solution to Problem 1, also working for $D'D$ not necessarily positive definite, can be obtained by simple algebraic manipulations, provided that the problem is suitably re-stated. To this aim, it is convenient to introduce the following notation for the sequences of the control inputs and of the controlled outputs,

$$
u_N := \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(N-1) \end{bmatrix}, \quad y_N := \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N-1) \end{bmatrix},
\tag{4}
$$

respectively. The system equations written for each $k \in [0, N-1]$ yield the following relation between $u_N$ and $y_N$,

$$y_N = A_N\, x_0 + B_N\, u_N \,, \tag{5}$$

where

$$A_N := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{N-1} \end{bmatrix}, \quad B_N := \begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{N-2}B & CA^{N-3}B & \cdots & D \end{bmatrix}. \tag{6}$$

The constraint (2) on the terminal state can be written as a linear function of $u_N$ and $x_0$ as

$$x_1 = A^N x_0 + L_N\, u_N \,, \tag{7}$$

where

$$L_N := \begin{bmatrix} A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix}, \tag{8}$$

Finally, the cost function (3) can be written as the square of the Euclidean norm of $y_N$, i.e.

$$J = \|y_N\|_2^2 \,. \tag{9}$$

Hence, Problem 1 can be recast as follows

**Problem 2.** Find $u_N$ minimizing (9) with $y_N$ given by (5) under the constraint (7).

The following Theorem 1 provides the solution of Problem 2. Then, the solution of Problem 1 can be retrieved from the former by means of equations (4).

**Theorem 1.** A control input sequence vector $u_N^o$ solving Problem 2 and the corresponding controlled output sequence vector $y_N^o$ are respectively given by

$$u_N^o = T_N\, x_0 + V_N\, x_1 \,, \tag{10}$$
$$y_N^o = C_N\, x_0 + D_N\, x_1 \,, \tag{11}$$

where

$$T_N := -\left(I - K\, (B_N K)^{\#}\, B_N\right) L_N^{\#} A^N - K\, (B_N K)^{\#}\, A_N \,, \tag{12}$$

$$V_N := \left(I - K\, (B_N K)^{\#}\, B_N\right) L_N^{\#} \,, \tag{13}$$

$$C_N := \left(I - B_N K\, (B_N K)^{\#}\right) \left(A_N - B_N L_N^{\#} A^N\right) \,, \tag{14}$$

$$D_N := \left(I - B_N K\, (B_N K)^{\#}\right) B_N L_N^{\#} \,, \tag{15}$$

with $K$ denoting a basis matrix for $\ker L_N$.

Proof. The constraint (7) can be solved with respect to $u_N$ as

$$u_N = L_N^\# \left( x_1 - A^N x_0 \right) + K\nu \, ,\tag{16}$$

where $\nu$ parameterizes the solutions in $\ker L_N$. From (5) and (16), it ensues

$$y_N = \left( A_N - B_N L_N^\# A^N \right) x_0 + B_N L_N^\# x_1 + B_N K\nu \, .\tag{17}$$

The expression of $\nu$ ensuring the minimum Euclidean norm of $y_N$ is

$$\nu = -\left( B_N K \right)^\# \left( A_N - B_N L_N^\# A^N \right) x_0 - \left( B_N K \right)^\# B_N L_N^\# x_1 + H\gamma \, ,\tag{18}$$

where $\gamma$ parameterizes the solutions in $\ker\left( B_N K \right)$, whose basis matrix is denoted by $H$, clearly without affecting the cost. In other words, $\gamma$ is a free parameter of our solution and it does not influence the value of the cost. Hence, from (17) and (18), it follows

$$y_N = \left( I - B_N K (B_N K)^\# \right)\left( A_N - B_N L_N^\# A^N \right) x_0 + \left( I - B_N K (B_N K)^\# \right) B_N L_N^\# x_1 \, ,$$

i.e. $y_N^o$ is given by (11) with (14) and (15). Finally, from (16) and (18), it ensues

$$u_N = -\left( \left( I - K(B_N K)^\# B_N \right) L_N^\# A^N + K(B_N K)^\# A_N \right) x_0 \\ + \left( I - K(B_N K)^\# B_N \right) L_N^\# x_1 + KH\gamma \, ,$$

i.e. $u_N^o$ is given by (10) with (12), (13) and $\gamma = 0$ (an arbitrary value for $\gamma$). □

**Corollary 1.** The optimal value $J^o$ of the cost function can be written as a quadratic function of the initial state $x_0$ and the terminal state $x_1$:

$$J^o = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}' \begin{bmatrix} C_N' C_N & C_N' D_N \\ D_N' C_N & D_N' D_N \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} .\tag{19}$$

Thus, the optimal control sequence and cost are computed as functions of $x_0$ and $x_1$ by means of pseudoinversion-based procedures. Optimality is guaranteed by the application of pseudoinversion to the equation derived from (17) assuming $y_N = 0$. In fact, this minimizes the value of $\|y_N\|_2^2$, which actually is the expression of the cost $J$, see eq. (9). The main drawback of such procedures is the fact that they become unfeasible for large values of $N$. In [27] a recursive approach is proposed coping with large control time intervals by solving a sequence of *nested* problems.

## 3. *H₂* OPTIMAL DECOUPLING OF PREVIEWED SIGNALS

Consider the system

$$x(k+1) = A\,x(k) + B\,u(k) + H\,h(k)\, ,\tag{20}$$

$$y(k) = C\,x(k) + D\,u(k) + G\,h(k)\, ,\tag{21}$$

with state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^p$, previewed or measured input $h \in \mathbb{R}^s$, controlled output $y \in \mathbb{R}^q$. Assume that the matrices $[B' \; D']$ and $[H' \; G']$ are of full rank. Refer to the block diagram in Figure 1, where $\Sigma_c$ denotes a linear controller exploiting the $N$-step preview of signal $h(k)$. It is worth noticing that the preview interval $N$ is taken into account by the delay block shown in the figure, so that the overall system having $h_p(k) = h(k+N)$ as input and $y(k)$ as output is causal. Denote by $W(z)$ the transfer function matrix of the overall system from $h_p(k)$ to $y(k)$ and by $w(k)$ the corresponding impulse response matrix. Then, the $H_2$ optimal decoupling problem of the previewed signal $h(k)$ consists in deriving a regulator $\Sigma_c$ that minimizes

$$
\begin{aligned}
\|W\|_2 &= \left( \frac{1}{2\pi} \mathrm{tr} \left[ \int_{-\pi}^{\pi} W(e^{j\omega}) W^*(e^{j\omega}) \, \mathrm{d}\omega \right] \right)^{\frac{1}{2}} \\
&= \left( \mathrm{tr} \left[ \sum_{k=0}^{\infty} w(k) w'(k) \right] \right)^{\frac{1}{2}} \\
&= \left( \sum_{k=0}^{\infty} \sum_{j=1}^{s} \sum_{i=1}^{q} w_{ij}^2(k) \right)^{\frac{1}{2}} .
\end{aligned}
\tag{22}
$$

From (22) it ensues that an equivalent statement of the problem is to find a linear system $\Sigma_c$ whose impulse response consists of the sequences $u_j(k)$, $j = 1, \ldots, s$, $k = 0, 1, \ldots$, minimizing

$$
\sum_{k=0}^{\infty} y_j'(k) \, y_j(k), \quad j = 1, \ldots, s,
$$

where $y_j(k)$, $j = 1, \ldots, s$, denotes the output generated by the input signal $h_{p,j}(k)$, equal to the $j$th vector of the natural basis of $\mathbb{R}^s$ at the time instant $k = 0$ and equal to zero for $k \neq 0$.

We briefly recall some geometric results derived in [26]. First, consider the condition

$$
\mathrm{im} \, \hat{H} \subseteq \hat{\mathcal{V}}^* + \hat{\mathcal{S}}^*, \tag{23}
$$

with $\hat{H} := H$ and $\hat{\mathcal{V}}^*$ and $\hat{\mathcal{S}}^*$ referred to the triple $(A, B, C)$ if both $D$ and $G$ are null matrices, or $\hat{H} := [H' \; G']'$ and $\hat{\mathcal{V}}^*$ and $\hat{\mathcal{S}}^*$ referred to the extended triple $(\hat{A}, \hat{B}, \hat{C})$ if not. It guarantees perfect decoupling if system (20, 21) is minimum phase with respect to input $u$. This is the case considered in [38] and, in this case, only a relative-degree preaction is required. On the other hand, if the system is non-minimum phase, condition (23) enables perfect decoupling only if the preaction time $N$ approaches infinity. Almost perfect decoupling is achieved when the preaction time is large enough with respect to the time constant of the unstable zero closest to the unit circle. In the above-mentioned cases $\|W\|_2$ is zero or almost zero. These cases were presented and discussed in [26].

If condition (23) is not met or the system is non-minimum phase and the available preaction time is not large enough, $H_2$ optimality is a convenient resort.
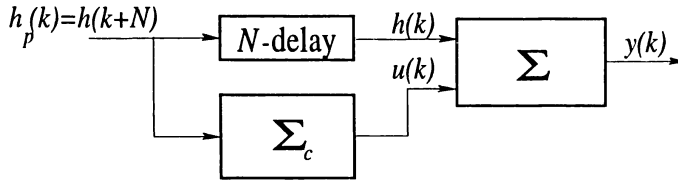
**Fig. 1.** Decoupling of a measurable or previewed signal.

## 4. $H_2$ OPTIMAL DECOUPLING AS A COMPOSED LQ OPTIMAL CONTROL PROBLEM

In order to solve the optimal decoupling problem stated in the previous section, the linear controller $\Sigma_c$ introduced in Figure 1 is specified here in its inner structure (see also Figure 2). It consists of a FIR system

$$v(k) = \sum_{\ell=0}^{N} \Phi(\ell) h_p(k - \ell) \,, \tag{24}$$

which accounts for the feedforward action (preaction) based on the preview of signal $h(k)$, and of a dynamic unit $\Sigma_1$ satisfying equation (20) with state feedback $K$, i. e. evolving according to

$$\tilde{x}(k + 1) = A_K \, \tilde{x}(k) + B \, v(k) + H \, h(k) \,, \tag{25}$$

with $A_K := A + BK$. The matrix $K$ is the optimal state feedback of the infinite-horizon Kalman regulator problem referred to system (1) with the optimal cost

$$c_\infty = \sum_{k=0}^{\infty} y'(k) \, y(k) \,. $$

This latter problem can be cheap or singular depending on the output matrices $C$ and $D$. The standard routines solving the cheap or singular Kalman regulator problem, like function *dare. m* in Matlab 5 or that proposed in [29] (that, unlike dare.m, also applies to non-left invertible systems), provide the optimal infinite-time cost matrix $S_\infty$ weighting the initial state, other than the optimal feedback matrix $K$. The optimal cost is expressed by the quadratic function

$$c_\infty = x_2' \, S_\infty \, x_2 \,, \tag{26}$$

where $x_2$ represents the generic initial state. The matrix $S_\infty$ will be used in the following algebraic manipulation to account for the $H_2$ cost from time $k = N + 1$ to
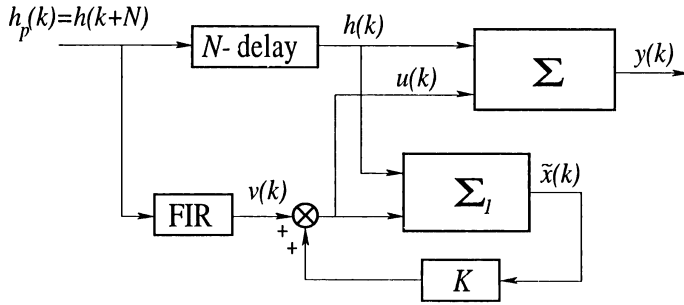
**Fig. 2.** Structure of the feedforward unit $\Sigma_c$.

$k = \infty$. For the sake of simplicity we assume $N \geq \nu$. This assumption can be relaxed, as it will be pointed out in Remark 1.

Refer to Figure 2 and suppose that an impulse $h(k) = \bar{h}\,\delta(k - N)$ (which will occur at $k = N$), is known in advance at time $k = 0$, i.e., in the setting of Figure 2, $h_p(k) = \bar{h}\,\delta(k)$. Then the $H_2$ optimal control problem with previewed signal consists in joining an LQ optimal control problem from $k = 0$ to $k = N$ with constrained final state of the type considered in Section 2 and a standard infinite horizon LQR problem from $k = N + 1$, while taking into account the occurrence of the impulse. Figure 3 represents a typical state trajectory for this problem.

Let the state of system $\Sigma_1$ and the control input at $k = N$ be referred to as $\tilde{x}_1$ and $u_1$, respectively, and the state at $k = N + 1$ as $\tilde{x}_2$, so that, according to (25),

$$\tilde{x}_2 = A_K\,\tilde{x}_1 + B\,u_1 + H\,\bar{h}\,. \tag{27}$$

By using equations (19), (26) and (27), it is an easy matter to verify that the overall cost is expressed by

$$c(\zeta) = \zeta'\,M_1\,\zeta + 2\,M_2\,\zeta + M_3\,, \tag{28}$$

with $\zeta := [\tilde{x}_1'\ u_1']'$ and

$$M_1 = \begin{bmatrix} D_N'D_N + C'C + A_K'S_\infty A_K & C'D + A_K'S_\infty B \\ D'C + B'S_\infty A_K & D'D + B'S_\infty B \end{bmatrix}, \tag{29}$$

$$M_2 = \begin{bmatrix} \bar{h}'\,G'C + \bar{h}'\,H'S_\infty A_K & \bar{h}'\,G'D + \bar{h}'\,H'S_\infty B' \end{bmatrix}, \tag{30}$$

$$M_3 = \bar{h}'\,G'G\,\bar{h} + \bar{h}'\,H'S_\infty H\,\bar{h}\,, \tag{31}$$

where $D_N$ is assumed to refer to the quadruple $(A_K, B, C, D)$. The optimal values of $\tilde{x}_1$ and $u_1$ are simply obtained by differentiating (28) with respect to the unknown parameter $\zeta$. Therefore,

$$\zeta^o = \begin{bmatrix} \tilde{x}_1^o \\ u_1^o \end{bmatrix} = -M_1^{\#}M_2\,. \tag{32}$$

The generic previewed signal $h_p(k)$ can be $H_2$-optimally decoupled through the FIR system (24) whose gain matrices are computed through the following steps:

1. Substitute $\bar{h}$ with the $s \times s$ identity matrix $I_s$ in (30) and (31).

2. From equation (32), evaluate the optimal parameter matrices

$$X_1^o \in \mathbb{R}^{n \times s}, \quad U_1^o \in \mathbb{R}^{p \times s}. \tag{33}$$

3. By using (10) with $x_0 = 0_{n \times s}$, $x_1 = X_1^o$, compute the gain matrices of the FIR system as

$$\Phi(j) = \begin{bmatrix} \phi(1) & \phi(2) & \cdots & \phi(N) \end{bmatrix} \begin{bmatrix} V_N X_1^o \\ U_1^o \end{bmatrix}, \tag{34}$$

with

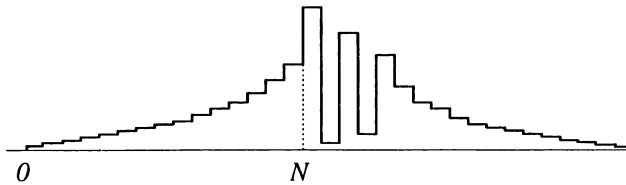$$\begin{cases} \phi(i) = 0_s & \text{for } i \neq j, \\ \phi(i) = I_s & \text{for } i = j. \end{cases}$$



**Fig. 3.** A typical optimal state trajectory for an $N$-step previewed impulse signal.

**Remark 1.** The assumption $N \geq \nu$ can be easily removed by suitably constraining the intermediate state $x_1$ to lie on the $N$-step reachable subspace. Let $R_N$ be a basis matrix of the column space of $L_N$ in (8). Solve the optimal reduced-dimension problem obtained by assuming

$$\zeta = \begin{bmatrix} R_N & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} \lambda \\ u_1 \end{bmatrix}$$

in equation (28), thus simply replacing the unknown variable $x_1$ with $\lambda$, which parameterizes the optimal state vector $x_1$ on the $N$-step reachable subspace. Compute the matrix solution $[\Lambda^{o\prime} \ U_1^{o\prime}]'$ obtained with $\bar{h} = I_s$ as before, and finally replace $X_1^o$ with $R_N \Lambda^o$ in equation (34).

Although from a theoretical point of view the left-invertibility assumption is not mandatory, it is usually introduced to simplify computational procedures, since most of the available routines for the solution of cheap/singular LQR problems only work with left-invertible systems. This assumption can easily be removed by using the geometric argument described in the following remark.

**Remark 2.** (Extension to non left-invertible systems) If the quadruple $(A, B, C, D)$ is not left-invertible, the previous procedure should be applied to $(\bar{A}, \bar{B}, C, D)$, with

1. $\bar{A} := A + B\bar{F}$, where $\bar{F}$ is a state feedback matrix such that $(A + B\bar{F})\mathcal{V}^* \subseteq \mathcal{V}^*$ and all the elements of $\sigma(A + B\bar{F})|_{\mathcal{R}_{\mathcal{V}^*}}$, that are arbitrarily assignable, are stable. $\mathcal{R}_{\mathcal{V}^*}$ denotes the reachable subspace of $\mathcal{V}^*$, computable as $\mathcal{R}_{\mathcal{V}^*} = \mathcal{V}^* \cap \mathcal{S}^*$.

2. $\bar{B} := B\bar{U}$, where $\bar{U}$ is a basis matrix of the subspace $\bar{\mathcal{U}} := (B^{-1}\mathcal{V}^*)^{\perp}$, the orthogonal complement of the inverse image of $\mathcal{V}^*$ with respect to $B$.

Let $\bar{u}(k)$ and $\bar{x}(k)$ be the optimal sequences of controls and states referring to $(\bar{A}, \bar{B}, C, D)$. The corresponding control sequences for $(A, B, C, D)$ are computed as $u(k) = \bar{U}\bar{u}(k) + \bar{F}\bar{x}(k)$.
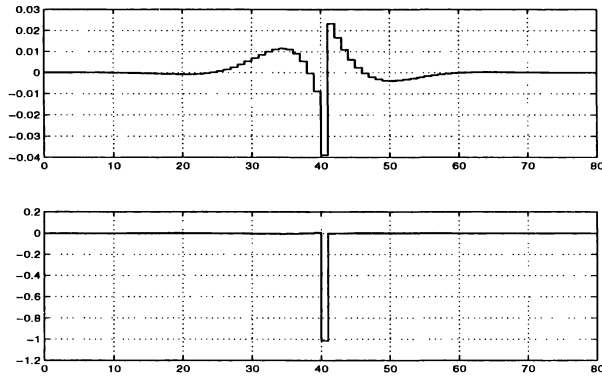


**Fig. 4.** Convolution profiles for the inputs.

## 5. A NUMERICAL EXAMPLE

Let us assume in system (1) the following matrices:

$$A = \begin{bmatrix} 0.5 & 1 & -0.4 & 0 \\ 0.1 & 0.7 & 0 & -0.5 \\ 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0.6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 0 \\ 1 \\ 0.1 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$
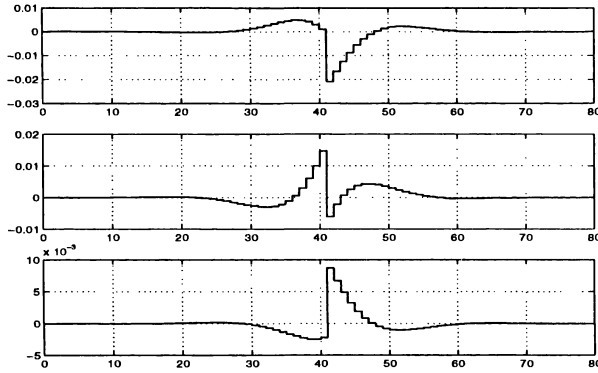
**Fig. 5.** $H_2$-optimally decoupled outputs.

The characterizing controlled and conditioned invariants are

$$\hat{\mathcal{V}}^* = \mathrm{im}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right), \quad \hat{\mathcal{S}}^* = \mathrm{im}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\right).$$

The system is left-invertible, since $\hat{\mathcal{V}}^* \cap \hat{\mathcal{S}}^* = \{0\}$. Condition (23) is not satisfied, so that, although the plant is minimum-phase (it has no invariant zeros), preaction improves the minimum $H_2$ norm. Preaction clearly appears in the plots of the optimal inputs and outputs. Figure 4 shows the convolution profiles $u_1(k)$ and $u_2(k)$ which optimally decouple a previewed unit impulse $h(k) = \delta(k - N)$ occurring at $k = 40$, while Figure 5 shows the corresponding optimal responses $y_1(k)$, $y_2(k)$ and $y_3(k)$.

## 6. CONCLUDING REMARKS

It has been shown that $H_2$-optimal decoupling of an $N$-step previewed signal (that for $N = 0$ reduces to $H_2$-optimal decoupling of a measurable signal) can be achieved by a feedforward unit consisting of a FIR system and a stable dynamic unit. This latter ensures optimality from the strict mathematical viewpoint, but in practice can also be replaced by a FIR system realizing the same (truncated) impulse response or computed with the algorithm presented in Section 2 and final state set to zero. In fact, it can be shown that in both cases the error uniformly approaches zero as the impulse response interval of the FIR system increases. The results obtained in this paper can be directly applied to the dual problem, $H_2$ optimal unknown-input observation of a linear function of the state with $N$-step postknowledge. This duality

was analyzed in [26], where the geometric conditions ensuring zero or almost zero $H_2$ norm were derived and an algorithm for computing the input convolution profiles for the zero-cost case was presented.

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