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ESTIMATION OF VARIANCES IN A HETEROSCEDASTIC RCA(1) MODEL

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The paper concerns with a heteroscedastic random coefficient autoregressive model (RCA) of the form $X_t = b_t X_{t-1} + Y_t$. Two different procedures for estimating $\sigma_t^2 = EY_t^2, \sigma_b^2 = Eb_t^2$ or $\sigma_B^2 = E(b_t - Eb_t)^2$, respectively, are described under the special seasonal behaviour of σ_t^2. For both types of estimators strong consistency and asymptotic normality are proved.

1. INTRODUCTION

A random coefficient autoregressive model (RCA) is defined as $X_t = b_t X_{t-1} + Y_t$, where $\{b_t\}$ are random coefficients with $Eb_t = \beta$ and $\{Y_t\}$ is an error process. The primary aim of many authors is to estimate an unknown parameter β under various sets of assumptions and derive asymptotic properties of such estimators (see for example [1, 3, 5, 7, 9] and [10]). Next to β, remaining parameters (such as $E(b_t - \beta)^2 = \sigma_B^2$ or $EY_t^2 = \sigma_t^2$) are also unknown very frequently and hence should be estimated too. It is for example useful for estimating asymptotic variance of OLS or WLS estimators of β since their asymptotic distribution depends on these unknown parameters (see [5]). Moreover, WLS and CWLS estimators (for definition of CWLS see [3] or [4]) depend on these parameters directly. Hence, in case they are unknown, they must be firstly estimated and then replaced by their estimates. These all are reasons why to estimate these nuisance parameters.

A standard least squares procedure for estimation of σ_B^2 and σ_t^2 is well described in [10] under the assumption that processes $\{Y_t\}$ and $\{b_t\}$ are mutually independent and consist of independent and identically distributed random variables. This technique was generalized for example in [3] for a RCA model where processes $\{Y_t\}$ and $\{b_t\}$ are correlated. In [9] the author deals with a heteroscedastic RCA(1) model but this procedure is then applied only to the case of constant variances $\sigma_t^2 = \sigma^2$ for all t. In general both processes $\{Y_t\}$, $\{b_t\}$ are allowed to be non-stationary. Maximum likelihood procedure is another approach of estimation σ_B^2 and σ_t^2. This technique in a homoscedastic case is described for example in [10].

In this paper we will generalize the standard least squares procedure for a heteroscedastic RCA(1) model with a special seasonal pattern of σ_t^2. Moreover, we will describe an alternative approach of estimation of σ_B^2 and σ_t^2. Under both approaches we will prove strong consistency and asymptotic normality of given estimators. At the end of the paper we numerically compare estimates from both procedures. The main theoretical results of this paper are substantially based on the fact that OLS estimator of β in a heteroscedastic RCA(1) model is strongly consistent and asymptotically normal that is shortly proved in [5]. Full versions of the proofs and all auxiliary lemmas can be found in [4] or [8]. Generalization of these results for RCA(1) processes containing martingale differences is given in [7].

2. MODEL DEFINITION

Let us suppose that the behaviour of the process $\{X_t\}$ is described by the RCA(1) model

$$X_t = b_t X_{t-1} + Y_t, \quad t = 1, \ldots, n \tag{1}$$

where X_0 is a random variable with $EX_0 = 0, 0 < EX_0^2 = \sigma_0^2 < \infty$, $Y_t, t = 1, \ldots, n$ are random variables with $EY_t = 0 \,\forall\, t, 0 < EY_t^2 = \sigma_t^2 < \infty$ that are independent of X_0 and $b_t, t = 1, \ldots, n$ are random variables with $Eb_t = \beta \,\forall\, t, 0 < Eb_t^2 = \sigma_b^2 < \infty$ $\forall\, t$ that are independent of X_0 and of $\{Y_t\}$.

Model (1) can be rewritten into the form of a fixed coefficient AR(1) model:

$$X_t = \beta X_{t-1} + B_t X_{t-1} + Y_t = \beta X_{t-1} + u_t, \tag{2}$$

where $u_t = B_t X_{t-1} + Y_t$ and $B_t = b_t - \beta$. To keep a unified notation let us denote $\sigma_B^2 := EB_t^2$, so the equation $\sigma_B^2 = \sigma_b^2 - \beta^2$ holds. Further, let us define the system of σ-fields $\mathcal{F} = \{\mathcal{F}_t\}$ in the following way: $\mathcal{F}_0 = \sigma(X_0)$, $\mathcal{F}_t = \sigma(X_0, Y_1, B_1, \ldots, Y_t, B_t)$ for $t = 1, 2 \ldots$

In [5] we concerned with estimation of the unknown parameter β in model (2) under assumption of known variances σ_t^2 and σ_B^2. But in practice these parameters are mainly unknown and have to be estimated. In a fully general form of $EY_t^2 = \sigma_t^2$ this problem is unsolvable since there is more parameters than observations in the model. In the sequel we will focus on a special structure of σ_t^2 behaving according to the following model:

$$EY_t^2 = \sigma_t^2 = \sigma_Y^{2[i]} \quad \text{for} \quad t \in I_i := \{i, k+i, \ldots, n-k+i\},\ i = 1, \ldots, k, \tag{3}$$

where k is a given fixed number such that $1 \le k \le N < n$. Without loss of generality we can suppose that $n = mk$ where $m \in \mathbb{N}$. A constant N plays a role of a reasonable upper bound such that m is a sufficiently large number of observations for regression estimation. Due to the time shift, in the following it will be useful to define the set $I_0 := \{0, k, 2k, \ldots, n-k\}$.

This model describes seasonal behaviour of variances σ_t^2 with a period k. In our opinion this pattern is reasonable and useful generalization of a homoscedastic assumption that can be used in a real time series analysis. Moreover, it satisfies a condition $\frac{1}{n}\sum_{t=1}^n \sigma_t^2 \overset{n\to\infty}{\longrightarrow} \sigma^2 > 0$ for $\sigma^2 = \frac{1}{k}\sum_{i=1}^k \sigma_Y^{2[i]}$. This condition was introduced in [5] and is crucial for proving strong consistency and asymptotic normality of the OLS and WLS estimator of β. On the other hand it is not as restrictive as a condition $\sigma_n^2 \overset{n\to\infty}{\longrightarrow} \sigma^2 > 0$ that is assumed in [9].

Further, this model significantly reduces number of all unknown parameters in model (1) to $k+2$ and hence they can be already estimated.

Agreement: For simplicity we will use the following abbreviations: SLLN–MD for strong law of large numbers for martingale differences (see Theorem 20.11 in [2]), SLLN–MX for strong law of large numbers for mixingales (see Theorem 20.16 in [2]) and CLT–MD for central limit theorem for martingale differences (see Theorem VI.4.12 in [11]).

3. STANDARD APPROACH

3.1. Estimation procedure

Let us suppose that the starting value X_0 and observations $X_1, \dots, X_n$ are available. The standard approach of estimating σ_t^2 and σ_B^2 is based on estimated OLS residuals $\hat{u}_t := X_t - \hat{\beta}X_{t-1}$, where $\hat{\beta}$ is the OLS estimator of β defined as

$$\hat{\beta} = \frac{\sum_{t=1}^n X_t X_{t-1}}{\sum_{t=1}^n X_{t-1}^2}. \tag{4}$$

Since

$$E(u_t^2|\mathcal{F}_{t-1}) = \sigma_t^2 + \sigma_B^2 X_{t-1}^2 \quad \text{a.s.} \tag{5}$$

holds for unobservable u_t, it looks reasonable to get estimators of interest by minimizing $\sum_{t=1}^n (\hat{u}_t^2 - \sigma_t^2 - \sigma_B^2 X_{t-1}^2)^2$. For a seasonal heteroscedasticity given by (3) it is equivalent to an OLS procedure in the following regression model:

$$\hat{\boldsymbol{U}}^2 = \boldsymbol{D}\boldsymbol{\sigma}_Y^2 + \boldsymbol{X}^2\sigma_B^2 + \boldsymbol{\zeta},$$

where $\hat{\boldsymbol{U}}^2 = (\hat{u}_1^2, \dots, \hat{u}_n^2)', \boldsymbol{X}^2 = (X_0^2, X_1^2, \dots, X_{n-1}^2)'$ are vectors of input values, $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_n)$ is a vector of errors, $\boldsymbol{\sigma}_Y^2 = (\sigma_Y^{2[1]}, \dots, \sigma_Y^{2[k]})'$ and σ_B^2 are unknown coefficients and $\boldsymbol{D} = \boldsymbol{i}_{m\times 1} \otimes \boldsymbol{I}_{k\times k}, \boldsymbol{i} = (1, 1, \dots, 1)'$ are fixed matrices. By solving normal equations we can easily derive that OLS estimators of unknown coefficients are given by

$$\hat{\sigma}_B^2 = (\boldsymbol{X}^{2'}\boldsymbol{M_D}\boldsymbol{X}^2)^{-1}\boldsymbol{X}^{2'}\boldsymbol{M_D}\hat{\boldsymbol{U}}^2, \tag{6}$$

$$\hat{\boldsymbol{\sigma}}_Y^2 = (\boldsymbol{D}'\boldsymbol{D})^{-1}\boldsymbol{D}'(\hat{\boldsymbol{U}}^2 - \boldsymbol{X}^2\hat{\sigma}_B^2), \tag{7}$$

where $\boldsymbol{M_D} = \boldsymbol{I} - \boldsymbol{D}(\boldsymbol{D}'\boldsymbol{D})^{-1}\boldsymbol{D}' = \boldsymbol{I} - \frac{1}{m}(\boldsymbol{i}\boldsymbol{i}' \otimes \boldsymbol{I})$. After some algebra, expression (6) can be rewritten into the form:

$$\hat{\sigma}_B^2 = \frac{\sum_{i=1}^k \sum_{t\in I_i} \hat{u}_t^2 \left(X_{t-1}^2 - \overline{X^2}^{[i-1]}\right)}{\sum_{i=1}^k \sum_{t\in I_i} \left(X_{t-1}^2 - \overline{X^2}^{[i-1]}\right)^2}, \tag{8}$$

where $\overline{X^2}^{[i]} = \frac{1}{m}\sum_{t\in I_i} X_t^2$. Due to a special structure of a matrix $\boldsymbol{D}$ the vector estimator $\hat{\boldsymbol{\sigma}}_{\boldsymbol{Y}}^2$ can be decomposed into k scalar estimators

$$\hat{\sigma}_Y^{2[i]} = \overline{\hat{u}^2}^{[i]} - \overline{X^2}^{[i-1]}\hat{\sigma}_B^2, \quad i = 1,\dots,k, \tag{9}$$

where $\overline{\hat{u}^2}^{[i]} = \frac{1}{m}\sum_{t\in I_i}\hat{u}_t^2$.
The estimator of the second moment σ_b^2 can be then obtained as $\hat{\sigma}_b^2 = \hat{\sigma}_B^2 + \hat{\beta}^2$.

3.2. Strong consistency

In order to prove strong consistency of given estimators we have to impose stronger conditions than in case of strong consistency of $\hat{\beta}$ (see [5]). Let us assume:

A0: $\{Y_t\}$ is a process of independent random variables, $\{b_t\}$ is a process of independent and identically distributed random variables,

A1: $E|X_0|^{4+\delta} < \infty$ and $\omega_t := E|Y_t|^{4+\delta} \le K < \infty\,\forall\, t$ and for some $\delta > 0$,

A2: $\omega_b := E|b_t|^{4+\delta} < 1$ for some $\delta > 0$,

A3: $\frac{1}{n}\sum_{t=1}^n EY_t^4 \overset{n\to\infty}{\longrightarrow} \mu_{Y,4}$.

Remark 3.1. The assumption of identically distributed $\{b_t\}$ is not necessary but it technically simplifies all proofs. Analogous techniques can be applied under more general moment conditions.

As mentioned in the introduction, the main results of this paper are substantially based on the asymptotic properties of the OLS estimator $\hat{\beta}$ given by (4). Hence, for better readability of the text, these properties are summarized in the next two theorems together with their shortened proofs. Detailed proofs can be found in [4] or [8], for a generalized case of RCA(1) model with martingale differences they are given also in [7].

Theorem 3.1. Under Assumptions A0 – A2, $\hat{\beta} \overset{n\to\infty}{\underset{\text{a.s.}}{\longrightarrow}} \beta$ holds.

Proof. Combining (2) and (4) we get

$$\hat{\beta} - \beta = \left(\frac{1}{n}\sum_{t=1}^n X_{t-1}u_t\right)\left(\frac{1}{n}\sum_{t=1}^n X_{t-1}^2\right)^{-1}.$$

In the first step it is shown that $\frac{1}{n}\sum_{t=1}^n X_{t-1}u_t \overset{n\to\infty}{\underset{\text{a.s.}}{\longrightarrow}} 0$. This arises from the fact that $\{X_{t-1}u_t\}$ is an $\mathcal{F}_t$-martingale difference sequence (see Lemma 3.3 in [8]) and from SLLN–MD. Further, it can be proved that the sequence $\{X_t^2 - EX_t^2, \mathcal{F}_t\}$ is an $L_{1+\varepsilon}$-mixingale of an arbitrary size for some $\varepsilon > 0$ (see Lemma 3.4. in [8]). This fact together with SLLN–MX yields that $\frac{1}{n}\sum_{t=1}^n X_{t-1}^2 \overset{n\to\infty}{\underset{\text{a.s.}}{\longrightarrow}} \frac{\sigma^2}{1-\sigma_b^2} > 0$, where $\sigma^2 = \frac{1}{k}\sum_{i=1}^k \sigma_Y^{2[i]}$, which concludes the proof. □

Theorem 3.2. Under Assumptions A0 – A3, the asymptotic distribution of $\sqrt{n}(\hat{\beta}-\beta)$ is $N\left(0,\Delta\left(\frac{1-\sigma_b^2}{\sigma^2}\right)^2\right)$, where $\Delta = \sigma_B^2\frac{6\sigma_b^2\overline{\sigma}^2+\mu_{Y,4}}{1-\mu_{b,4}}+\overline{\sigma}^2, \overline{\sigma}^2 = \lim_{n\to\infty}\frac{1}{n}\sum_{t=1}^n \sigma_t^2 EX_{t-1}^2$ and $\mu_{b,4} = Eb_t^4$.

Proof. The proof is based on analyzing asymptotic behaviour of the expression

$$\sqrt{n}\left(\hat{\beta}-\beta\right) = \left(\frac{1}{s_n}\sum_{t=1}^n X_{t-1}u_t\right)\left(\sqrt{\frac{n}{s_n^2}}\frac{1}{n}\sum_{t=1}^n X_{t-1}^2\right)^{-1},$$

where $s_n^2 := \sum_{t=1}^n E\left(X_{t-1}^2u_t^2\right)$. Firstly, it can be derived that $\frac{1}{n}s_n^2 \xrightarrow{n\to\infty} \Delta$ holds. Thus, in the rest of the proof it is sufficient to show that $\frac{1}{s_n}\sum_{t=1}^n X_{t-1}u_t$ has the asymptotic distribution $N(0,1)$. CLT–MD and SLLN–MX are useful in this case (see the proof of Theorem 3.3 in [8]). □

Remark 3.2. In cited papers, Theorem 3.1 is proved under weaker version of Assumptions A1 and A2 for moments of order $2+\delta$ instead of $4+\delta$. Since later on in this paper it will be used only in cases where moments of order $4+\delta$ are required, it is formulated in this form.

Auxiliary lemmas

Lemma 3.1. Assumptions A0 – A2 imply that $E|X_t|^{4+\delta} \le C < \infty \quad \forall t$.

Proof. Firstly, the process $\{X_t\}$ can be expressed in the form $X_t = \sum_{j=0}^t c_{t,j-1}Y_{t-j}$, where $Y_0 := X_0, c_{t,j} := \prod_{i=0}^j b_{t-i}$ and $c_{t,-1} := 1$. Further, applying Minkowski's inequality for $p = 4+\delta$ on this expression we get:

$$\left(E|X_t|^{4+\delta}\right)^{\frac{1}{4+\delta}} = \left(E\left|\sum_{j=0}^t c_{t,j-1}Y_{t-j}\right|^{4+\delta}\right)^{\frac{1}{4+\delta}} \le \sum_{j=0}^t \left(E|c_{t,j-1}Y_{t-j}|^{4+\delta}\right)^{\frac{1}{4+\delta}}$$

$$= \sum_{j=0}^t\left(E\left(\prod_{i=0}^{j-1}|b_{t-i}|^{4+\delta}\right)\omega_{t-j}\right)^{\frac{1}{4+\delta}} \le K^{\frac{1}{4+\delta}}\sum_{j=0}^t\left(\omega_b^{\frac{1}{4+\delta}}\right)^j \le \frac{C}{1-\omega_b^{\frac{1}{4+\delta}}} \le C',$$

where C and C' denote general positive constants. □

Lemma 3.2. Let $\{Z_t\}$ be a martingale difference sequence with respect to $\mathcal{Z}_t = \sigma(Z_1,\ldots,Z_t)$ ($\mathcal{Z}_t$-m.d.s.), then $\{T_t^{[i]}\}$ where $T_t^{[i]} := Z_{tk+i}$ are martingale differences with respect to $\mathcal{T}_t^{[i]} := \mathcal{Z}_{tk+i}$ for $i = 1,\ldots,k$.

Proof. $\mathcal{T}_t^{[i]}$-measurability and L_1 integrability of $\{T_t^{[i]}\}$ are obvious. Further, for $i = 1,\ldots,k$ we get

$$E\left(T_t^{[i]}\middle|\mathcal{T}_{t-1}^{[i]}\right) = E(Z_{tk+i}|\mathcal{Z}_{(t-1)k+i}) = E\left[E(Z_{tk+i}|\mathcal{Z}_{tk+i-1})\middle|\mathcal{Z}_{tk+i-k}\right] = 0 \quad \text{a.s.} \ \square$$

In the sequel let us use the following notation: $\Pi_j := \lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} EX_t^j$, $\tau_j \overset{\text{a.s.}}{:=} \lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} X_t^j, \Pi_j^{[i]} := \lim_{m\to\infty} \frac{1}{m}\sum_{t\in I_i} EX_t^j, \tau_j^{[i]} \overset{\text{a.s.}}{:=} \lim_{m\to\infty} \frac{1}{m}\sum_{t\in I_i} X_t^j$.

Lemma 3.3. Under Assumptions A0 – A2, limits $\Pi_j^{[i]}$ and $\tau_j^{[i]}$ for $j = 1,2$ exist for all i. If moreover A3 holds, then $\Pi_4^{[i]}$ and $\tau_4^{[i]}$ exist. Furthermore, $\tau_j^{[i]} = \Pi_j^{[i]}$ holds a.s. for $j = 1,2,4$ and $i = 1,\dots,k$.

Proof. One possibility is explicit derivation of each expression separately for $i = 1,\dots,k$. Alternatively, the limits of interest can be obtained as limits of the solutions of a system of k linear equations. In case of $\Pi_j^{[i]}$ the system arises when summing the equation $EX_t^j = E(b_tX_{t-1} + Y_t)^j$ over $t \in I_1,\dots,I_k$ and dividing by m. In case of $\tau_j^{[i]}$ the same is done for the equality $X_t^j = (b_tX_{t-1}+Y_t)^j$. Convergence of all redundant terms to zero is ensured by SLLN–MD.

This procedure for $j = 1,2$ is demonstrated in the next example. Both systems of equations are constructed, all limits of the solution are derived and convergence of all redundant terms is explained. For $j = 4$, the procedure is analogous. $\sqsubset$

Remark 3.3. Alternatively, the proof of Lemma 3.3 for $j = 1,2$ can be based on mixingales theory. In [8] it is shown that the sequences $\{S_t^{[i]}, \Sigma_t^{[i]}\}$ and $\{S_t^{[i]^2} - ES_t^{[i]^2}, \Sigma_t^{[i]}\}$, where $S_t^{[i]} := X_{tk+i}$ and $\Sigma_t^{[i]} := \mathcal{F}_{tk+i}$ are all $L_{1+\varepsilon}$-mixingales (see Lemma 3.13 in [8]). Hence, equalities $\tau_j^{[i]} = \Pi_j^{[i]}$ a.s for $j = 1,2$ directly yield also from SLLN–MX (see Lemma 3.14 in [8]).

Corollary. Under assumptions of Lemma 3.3, limits Π_j, τ_j for $j = 1,2,4$ exist and they are given as $\Pi_j = \frac{1}{k}\sum_{i=1}^{k} \Pi_j^{[i]}$ and $\tau_j = \frac{1}{k}\sum_{i=1}^{k} \tau_j^{[i]}$.

Example. It trivially holds that $\tau_1^{[i]} = \Pi_1^{[i]} = 0$ a.s for all i. In case of $j = 2$ we get the two following systems:

$$\boldsymbol{t} = \sigma_b^2 \boldsymbol{t}_{-1} + \boldsymbol{\sigma}_Y^2 + \boldsymbol{B}_m + \boldsymbol{C}_m \qquad \boldsymbol{p} = \sigma_b^2 \boldsymbol{p}_{-1} + \boldsymbol{\sigma}_Y^2 + \boldsymbol{A}_m$$

where $\boldsymbol{t} = \left(t^{[1]},\dots,t^{[k]}\right)'$, $\boldsymbol{t}_{-1} = \left(t^{[k]}, t^{[1]}\dots,t^{[k-1]}\right)'$, $t^{[i]} = \frac{1}{m}\sum_{t\in I_i} X_t^2$,
$\boldsymbol{p} = E\boldsymbol{t}, \boldsymbol{p}_{-1} = E\boldsymbol{t}_{-1}$, $\boldsymbol{B}_m = \left(B_m^{[1]},\dots,B_m^{[k]}\right)'$,
$B_m^{[i]} = \frac{1}{m}\sum_{t\in I_i}\left[(b_t^2 - \sigma_b^2)X_{t-1}^2 + 2X_{t-1}b_tY_t + \left(Y_t^2 - \sigma_Y^{2[i]}\right)\right]$, $\boldsymbol{C}_m = (C_m, 0,\dots,0)'$,
$C_m = \sigma_b^2 \frac{1}{m}(X_0^2 - X_n^2)$ and $\boldsymbol{A}_m = E\boldsymbol{C}_m$. Since $A_m \overset{m\to\infty}{\longrightarrow} 0$, $C_m \underset{\text{a.s.}}{\overset{m\to\infty}{\longrightarrow}} 0$ due to Borel–Cantelli lemma and since $B_m^{[i]} \underset{\text{a.s.}}{\overset{m\to\infty}{\longrightarrow}} 0$ due to SLLN–MD, it can be derived that for

the common limits $\Pi_2^{[i]}, \tau_2^{[i]}$ of the solutions $p^{[i]}, t^{[i]}$ the following relations hold:

$$\begin{aligned} \Pi_2^{[k]} &= \frac{1}{1-\sigma_b^{2k}} \sum_{i=0}^{k-1} \sigma_b^{2i} \sigma_Y^{2[k-i]}, \\ \Pi_2^{[l]} &= \sigma_b^{2l} \Pi_2^{[k]} + \sum_{i=0}^{l-1} \sigma_b^{2i} \sigma_Y^{2[l-i]} \quad \text{for } l = 1, \dots, k-1. \end{aligned} \tag{10}$$

Moreover, since $\Pi_2 = \frac{1}{1-\sigma_b^2} \frac{1}{k} \sum_{i=1}^{k} \sigma_Y^{2[i]}$ (see [5]), one can easily check that $\Pi_2 = \frac{1}{k} \sum_{i=1}^{k} \Pi_2^{[i]}$ really holds.

Remark 3.4. It can be easily derived that the limit $\overline{\sigma}^2$ occurred in Theorem 3.2 is of the form

$$\overline{\sigma}^2 := \lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} \sigma_t^2 E X_{t-1}^2 = \frac{1}{k} \sum_{i=1}^{k} \sigma_Y^{2[i]} \Pi_2^{[i-1]}.$$

In the following let us define $\lambda_n := \frac{1}{n} \sum_{i=1}^{k} \sum_{t\in I_i} \left(X_{t-1}^2 - \overline{X^2}^{[i-1]} \right)^2$, $\rho_t := u_t^2 - \sigma_t^2 - \sigma_B^2 X_{t-1}^2$, $\sigma_B^{2*} := \lambda_n^{-1} \frac{1}{n} \sum_{i=1}^{k} \sum_{t\in I_i} u_t^2 \left(X_{t-1}^2 - \overline{X^2}^{[i-1]} \right)$ and $\sigma_Y^{2[i]*} := \overline{u^2}^{[i]} - \overline{X^2}^{[i-1]} \sigma_B^{2*}$.
So that the estimators (8) and hence (9) be well-defined, we have to moreover assume one technical assumption:

A4: Y_t cannot take only two values for each t.

Lemma 3.4. Assumption A4 ensures that λ_n is strictly positive almost surely for large n.

Proof. Since $\overline{X^2}^{[i]} \xrightarrow[\text{a.s.}]{m\to\infty} \Pi_2^{[i]}$, we can concentrate on the expression $\frac{1}{n} \sum_{i=1}^{k} \sum_{t\in I_i} \left(X_{t-1}^2 - \Pi_2^{[i-1]} \right)^2$. Suppose in contradiction that $X_t^2 - \Pi_2^{[i]} = 0$ a.s. for all $t \in I_i$ and for all i. Then X_t can reach only two values $V_1^{[i]} = \sqrt{\Pi_2^{[i]}}$ and $V_2^{[i]} = -\sqrt{\Pi_2^{[i]}}$. In this case we get either $Y_t = V_1^{[i]} - b_t X_{t-1}$ or $Y_t = V_2^{[i]} - b_t X_{t-1}$ for $t \in I_i$. Since Y_t is independent of b_t and X_{t-1}, it implies that Y_t can take also only two values that is the contradiction. □

It is easy to derive that $\lambda_n = \frac{1}{n} \sum_{t=1}^{n} X_{t-1}^4 - \frac{1}{k} \sum_{i=1}^{k} \left(\overline{X^2}^{[i-1]} \right)^2$. Since $\Pi_2^{[0]} = \Pi_2^{[k]}$, we can define $\Lambda \overset{\text{a.s.}}{:=} \lim_{n\to\infty} \lambda_n$, where

$$\Lambda = \Pi_4 - \frac{1}{k} \sum_{i=1}^{k} \left(\Pi_2^{[i]} \right)^2. \tag{11}$$

Remark 3.5. Limit Π_4 is given by $\Pi_4 = \frac{1}{1-\mu_{b,4}}(6\sigma_b^2\overline{\sigma}^2 + \mu_{Y,4})$ (see [5]).

Opposed to the stationary case, in this general non-stationary model Lemma 3.4 does not imply that $\Lambda > 0$. Hence, we have to require this property as an additional assumption:

A5: $\Lambda > 0$ holds.

Lemma 3.5. Under A0–A5, $\hat{\sigma}_B^2 - \sigma_B^{2*} \xrightarrow[\text{a.s.}]{n\to\infty} 0$ and $\hat{\sigma}_Y^{2[i]} - \sigma_Y^{2[i]*} \xrightarrow[\text{a.s.}]{n\to\infty} 0$ for $i = 1,\dots,k$ hold.

Proof. Firstly, since $\hat{u}_t^2 = (X_t - \hat{\beta}X_{t-1})^2 = u_t^2 - 2(\hat{\beta}-\beta)X_{t-1}u_t + (\hat{\beta}-\beta)^2X_{t-1}^2$, we get

$$\begin{aligned}\hat{\sigma}_B^2 - \sigma_B^{2*} &= \lambda_n^{-1}\frac{1}{n}\sum_{i=1}^{k}\sum_{t\in I_i}(\hat{u}_t^2 - u_t^2)\left(X_{t-1}^2 - \overline{X^2}^{[i-1]}\right) \\ &= (\hat{\beta}-\beta)^2 - \lambda_n^{-1}\frac{1}{k}\sum_{i=1}^{k}(\hat{\beta}-\beta)\Big[2\frac{1}{m}\sum_{t\in I_i}X_{t-1}^3u_t - 2\overline{X^2}^{[i-1]}\frac{1}{m}\sum_{t\in I_i}X_{t-1}u_t\Big],\end{aligned} \tag{12}$$

where $\frac{1}{n}\sum_{i=1}^{k}\sum_{t\in I_i}X_{t-1}^2\left(X_{t-1}^2 - \overline{X^2}^{[i-1]}\right) = \lambda_n$ was used. Because $\{X_{t-1}^3u_t\}$ and $\{X_{t-1}u_t\}$ are $L_{1+\delta}$-uniformly bounded $\mathcal{F}_t$-m.d.s., Lemma 3.2 together with SLLN-MD and Lemma 3.3 imply that the expression in brackets converges almost surely to 0. The fact that $\hat{\beta} \xrightarrow[\text{a.s.}]{n\to\infty} \beta$ (see Theorem 3.1) concludes the proof of the first part. Further, for $i = 1,\dots,k$ we have

$$\begin{aligned}\hat{\sigma}_Y^{2[i]} - \sigma_Y^{2[i]*} &= \frac{1}{m}\sum_{t\in I_i}(\hat{u}_t^2 - u_t^2) - (\hat{\sigma}_B^2 - \sigma_B^{2*})\overline{X^2}^{[i-1]} \\ &= (\hat{\beta}-\beta)\Big[-2\frac{1}{m}\sum_{t\in I_i}X_{t-1}u_t + (\hat{\beta}-\beta)\overline{X^2}^{[i-1]}\Big] - (\hat{\sigma}_B^2 - \sigma_B^{2*})\overline{X^2}^{[i-1]}.\end{aligned} \tag{13}$$

The same arguments as before together with the first statement of this lemma imply that (13) converges a.s. to 0 that finishes the proof. □

Theorems

Theorem 3.3. Under A0–A5, $\hat{\sigma}_B^2 \xrightarrow[\text{a.s.}]{n\to\infty} \sigma_B^2$ holds.

Proof. Because of Lemma 3.5, it is sufficient to show that

$$\sigma_B^{2*} - \sigma_B^2 \xrightarrow[\text{a.s.}]{n\to\infty} 0. \tag{14}$$

Using the previous notation we can write

$$\sigma_B^{2*} = \lambda_n^{-1}\frac{1}{n}\sum_{i=1}^{k}\sum_{t\in I_i}\left(\rho_t + \sigma_B^2X_{t-1}^2 + \sigma_Y^{2[i]}\right)\left(X_{t-1}^2 - \overline{X^2}^{[i-1]}\right).$$

Since $\sum_{i=1}^{k} \sigma_Y^{2[i]} \sum_{t\in I_i} \left(X_{t-1}^2 - \overline{X^2}^{[i-1]}\right) = 0$, we get

$$\sigma_B^{2*} - \sigma_B^2 = \lambda_n^{-1} \frac{1}{n} \sum_{i=1}^{k} \sum_{t\in I_i} \left(X_{t-1}^2 - \overline{X^2}^{[i-1]}\right) \rho_t. \tag{15}$$

Further, let us define

$$\sigma_B^{2**} := \lambda_n^{-1} \frac{1}{n} \sum_{i=1}^{k} \sum_{t\in I_i} \left(X_{t-1}^2 - \Pi_2^{[i-1]}\right) \rho_t + \sigma_B^2. \tag{16}$$

In the following let us show that $\sigma_B^{2*} - \sigma_B^{2**} \xrightarrow[\text{a. s.}]{n\to\infty} 0$. Combining (15) an (16) we get

$$\sigma_B^{2*} - \sigma_B^{2**} = \lambda_n^{-1} \frac{1}{k} \sum_{i=1}^{k} \left(\Pi_2^{[i-1]} - \overline{X^2}^{[i-1]}\right) \frac{1}{m} \sum_{t\in I_i} \rho_t. \tag{17}$$

Since $E(\rho_t|\mathcal{F}_{t-1}) = E(u_t^2 - \sigma_t^2 - \sigma_B^2 X_{t-1}^2|\mathcal{F}_{t-1}) = 0$, $\{\rho_t\}$ is an $\mathcal{F}_t$-m.d.s. that is moreover $L_{1+\delta}$-uniformly bounded. Hence, Lemma 3.2 and SLLN–MD imply that $\frac{1}{m} \sum_{t\in I_i} \rho_t \xrightarrow[\text{a. s.}]{n\to\infty} 0$. Lemma 3.3 then gives that $\sigma_B^{2*} - \sigma_B^{2**} \xrightarrow[\text{a. s.}]{n\to\infty} 0$.
Finally, since $\left\{\left(X_{t-1}^2 - \Pi_2^{[i-1]}\right)\rho_t\right\}$ for $t \in I_i$ and $\forall i$ remain to be $L_{1+\delta}$-bounded martingale differences, convergence of $\sigma_B^{2**} - \sigma_B^2 \xrightarrow[\text{a. s.}]{n\to\infty} 0$ in (16) is again a consequence of SLLN–MD and hence the proof is finished. □

Theorem 3.4. Under A0 – A5, $\hat{\sigma}_Y^{2[i]} \xrightarrow[\text{a. s.}]{n\to\infty} \sigma_Y^{2[i]}$ holds for $i = 1, \dots, k$.

Proof. Due to Lemma 3.5, it remains to show that $\sigma_Y^{2[i]*} - \sigma_Y^{2[i]} \xrightarrow[\text{a. s.}]{n\to\infty} 0$ holds for each i.
We can write

$$\begin{aligned}\sigma_Y^{2[i]*} - \sigma_Y^{2[i]} &= \frac{1}{m} \sum_{t\in I_i} (u_t^2 - \sigma_B^{2*} X_{t-1}^2) - \sigma_Y^{2[i]} = \frac{1}{m} \sum_{t\in I_i} (u_t^2 - \sigma_Y^{2[i]} - \sigma_B^2 X_{t-1}^2) \\ &- (\sigma_B^{2*} - \sigma_B^2)\overline{X^2}^{[i-1]} = \frac{1}{m} \sum_{t\in I_i} \rho_t - (\sigma_B^{2*} - \sigma_B^2)\overline{X^2}^{[i-1]}.\end{aligned} \tag{18}$$

Hence, the desired result directly follows from (14), Lemma 3.3 and SLLN–MD. □

3.3. Asymptotic normality

In case of asymptotic normality requirements for higher moments are needed. In contrast to the previous paragraph let us strengthen Assumptions A1 and A2 into the form:

A1': $E|X_0|^{8+\delta} < \infty$ and $\tau_t := E|Y_t|^{8+\delta} \le K < \infty\, \forall t$ and for some $\delta > 0$,

A2': $\tau_b := E|b_t|^{8+\delta} < 1$ for some $\delta > 0$.

Similarly as in case of EY_t^2 and EY_t^4, some restrictions for higher moments of the process $\{Y_t\}$ must be considered. One possibility under which all following proofs can be done analogously is to assume that $EY_t^j \overset{t\to\infty}{\longrightarrow} \mu_{Y,j}$ for $j = 3, \dots, 8$. But this structure reduces the idea of seasonal behaviour of the error term. To sustain a seasonal variation of higher moments of $\{Y_t\}$ analogously as in (3) we will assume the following restrictions:

A3': $EY_t^j = \mu_{Y,j}^{[i]}$ for $t = I_i,\ j = 3, \dots, 8.$

Remark 3.6. Assumption A3' is trivially fulfilled for example if Y_t are identically distributed within each I_i.

Auxiliary lemmas

Lemma 3.6. Assumptions A0, A1' and A2' imply that $E|X_t|^{8+\delta} \le C < \infty \quad \forall t.$

Proof. Analogously as for Lemma 3.1. □

Lemma 3.7. Under Assumptions A0, A1', A2' and A3', limits $\Pi_j^{[i]}, \tau_j^{[i]}$ for $j = 3, \dots, 8$ exist for all $i = 1, \dots, k$. Moreover, $\tau_j^{[i]} = \Pi_j^{[i]}$ holds a.s.

Proof. We can use the analogous procedure as in Lemma 3.3 applied to $EX_t^j = E(b_t X_{t-1} + Y_t)^j$ in case of $\Pi_j^{[i]}$ and to $X_t^j = (b_t X_{t-1} + Y_t)^j$ in case of $\tau_j^{[i]}$, respectively. □

Remark 3.7. Existence of $\Pi_j^{[i]}$ for $j = 2, 4, 6, 8$ only is essential for crucial Theorems 3.5 and 3.6 hold. For Theorems 4.3 and 4.4 limits $\Pi_j^{[i]}$ moreover for $j = 3, 5$ have to exist. However, in both cases the restriction A3' for $j = 7$ is redundant and hence can be omitted.

Lemma 3.8. Under Assumptions A0, A1', A2', A3', A4 and A5, $\sqrt{n}(\hat{\sigma}_B^2 - \sigma_B^{2*}) \overset{n\to\infty}{\longrightarrow} 0$ in probab. and $\sqrt{n}\left(\hat{\sigma}_Y^{2[i]} - \sigma_Y^{2[i]*}\right) \overset{n\to\infty}{\longrightarrow} 0$ in probab. for $i = 1, \dots, k$ hold.

Proof. The first statement directly yields from (12) multiplied by $\sqrt{n}$, since $\sqrt{n}(\hat{\beta} - \beta)$ converges in distribution (see Theorem 3.2) while $(\hat{\beta} - \beta)$ and the expression in brackets converges almost surely to 0.
The second property is analogously seen when multiplying (13) by $\sqrt{n}$. Lemma 3.3 and the first part of this lemma have to be moreover used in this case. □

Theorems

Theorem 3.5. Under Assumptions A0, A1', A2', A3', A4 and A5, the asymptotic distribution of $\sqrt{n}(\hat{\sigma}_B^2 - \sigma_B^2)$ is $N(0, \Lambda^{-2}\hat{\Sigma}_B)$, where Λ is defined by (11) and

$$\hat{\Sigma}_B := \frac{1}{k}\sum_{i=1}^{k} \lim_{m\to\infty} \frac{1}{m} \sum_{t\in I_i} E\Big[\Big(X_{t-1}^2 - \Pi_2^{[i-1]}\Big)^2 \rho_t^2\Big].$$

P r o o f. Due to Lemma 3.8, it remains to examine the expression $\sqrt{n}(\sigma_B^{2*} - \sigma_B^2) = \sqrt{n}(\sigma_B^{2*} - \sigma_B^{2**}) + \sqrt{n}(\sigma_B^{2**} - \sigma_B^2)$.
Firstly, let us show that $\sqrt{n}(\sigma_B^{2*} - \sigma_B^{2**}) \overset{n\to\infty}{\longrightarrow} 0$ in probab.
Since $\sqrt{n}(\sigma_B^{2*} - \sigma_B^{2**}) = \lambda_n^{-1}\frac{1}{\sqrt{k}}\sum_{i=1}^k \Big(\Pi_2^{[i-1]} - \overline{X^2}^{[i-1]}\Big)\frac{1}{\sqrt{m}}\sum_{t\in I_i}\rho_t$ holds due to (17) and since $\overline{X^2}^{[i]} \underset{\text{a. s.}}{\overset{m\to\infty}{\longrightarrow}} \Pi_2^{[i]}$, it is sufficient to show that $\frac{1}{\sqrt{m}}\sum_{t\in I_i}\rho_t$ is $O_p(1)$. It follows directly from the Chebyshev's inequality, for all $\varepsilon > 0$ there exists $K_\varepsilon > 0$ such that

$$P\Big(\Big|\frac{1}{\sqrt{m}}\sum_{t\in I_i}\rho_t\Big| \geq K_\varepsilon\Big) \leq \frac{1}{K_\varepsilon^2}\frac{1}{m}\sum_{t\in I_i} E\rho_t^2 \leq \frac{C}{K_\varepsilon^2} < \varepsilon,$$

since $E\Big(\sum_{t\in I_i}\rho_t\Big)^2 = \sum_{t\in I_i} E\rho_t^2$ and $E\rho_t^2 \leq C < \infty$ for all t.
Secondly, for $t \in I_i$ and $i = 1, \dots, k$ let us define $Z_t^{[i]} := \Big(X_{t-1}^2 - \Pi_2^{[i-1]}\Big)\rho_t$. Then using (16) we get $\sqrt{n}(\sigma_B^{2**} - \sigma_B^2) = \lambda_n^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^k\sum_{t\in I_i} Z_t^{[i]}$. Further, put $s_n^2 := \sum_{i=1}^k\sum_{t\in I_i} E\Big(Z_t^{[i]}\Big)^2$. Then $\frac{1}{n}s_n^2 \overset{n\to\infty}{\longrightarrow} \hat{\Sigma}_B$. Derivation of $\hat{\Sigma}_B$ is quite technical and time consuming and it is presented in Appendix A.1, its explicit form is given by (25).
Consequently, it remains to show that $\frac{1}{s_n}\sum_{i=1}^k\sum_{t\in I_i} Z_t^{[i]}$ has the asymptotic distribution $N(0,1)$. Since $Z_t^{[i]}$ are martingale differences, CLT-MD can be applied. Hence, it remains to verify assumptions of this theorem which are of the form:

i) $\frac{1}{s_n^2}\sum_{i=1}^k\sum_{t\in I_i} E\Big[\Big(Z_t^{[i]}\Big)^2\Big|\mathcal{F}_{t-1}\Big] \overset{n\to\infty}{\longrightarrow} 1$ in probab.,

ii) $\frac{1}{s_n^2}\sum_{i=1}^k\sum_{t\in I_i} E\Big[\Big(Z_t^{[i]}\Big)^2 I_{[|Z_t^{[i]}|\geq\varepsilon s_n]}\Big] \overset{n\to\infty}{\longrightarrow} 0$ for all $\varepsilon > 0$.

The first condition can be checked by explicit expansion of all terms that is done in Appendix A.1. Since $\{M_{1,t}^{[i]}\}$ defined in appendix by (24) is an $\mathcal{F}_t$-m.d.s. that satisfies SLLN–MD, $\frac{1}{n}\sum_{i=1}^k\sum_{t\in I_i} M_{1,t}^{[i]} \overset{n\to\infty}{\longrightarrow} 0$ in probab. Taking conditional and unconditional expectations of remaining terms in (23), we can show, using Lemmas 3.3, 3.7 and SLLN–MD, desired convergence of all remaining terms.
The second condition directly yields from the fact that $E\Big|Z_t^{[i]}\Big|^{2+\delta'} \leq C < \infty\ \forall t$ for $\delta' = \frac{\delta}{4}$ and from convergence $\frac{1}{n}s_n^2 \overset{n\to\infty}{\longrightarrow} \hat{\Sigma}_B$. □

Theorem 3.6. Under Assumptions A0, A1', A2', A3', A4 and A5, the asymptotic distribution of $\sqrt{n}(\hat{\sigma}_Y^{2[i]} - \sigma_Y^{2[i]})$ for $i = 1, \dots, k$ is $N\left(0, \hat{\Sigma}_Y^{[i]}\right)$, where

$$\hat{\Sigma}_Y^{[i]} := k\hat{\Sigma}_1^{[i]} - 2\Pi_2^{[i-1]}\Lambda^{-1}\hat{\Sigma}_2^{[i]} + \left(\Pi_2^{[i-1]}\right)^2\Lambda^{-2}\hat{\Sigma}_B,$$

$$\hat{\Sigma}_1^{[i]} := \lim_{m\to\infty}\frac{1}{m}\sum_{t\in I_i} E\rho_t^2,$$

$$\hat{\Sigma}_2^{[i]} := \lim_{m\to\infty}\frac{1}{m}\sum_{t\in I_i} E\left[\left(X_{t-1}^2 - \Pi_2^{[i-1]}\right)\rho_t^2\right].$$

P r o o f. The asymptotic distribution of $\sqrt{n}(\hat{\sigma}_Y^{2[i]} - \sigma_Y^{2[i]})$ is, due to Lemma 3.8, the same as that of $\sqrt{n}\left(\sigma_Y^{2[i]*} - \sigma_Y^{2[i]}\right) = \sqrt{\frac{k}{m}}\sum_{t\in I_i}\rho_t - \sqrt{n}(\sigma_B^{2*} - \sigma_B^2)\overline{X^2}^{[i-1]}$, which is seen from (18).
Further, since $\sqrt{n}(\sigma_B^{2*} - \sigma_B^{2**})\overline{X^2}^{[i-1]} \xrightarrow{n\to\infty} 0$ in probab. and $\sqrt{n}(\sigma_B^{2*} - \sigma_B^2)\left(\overline{X^2}^{[i-1]} - \Pi_2^{[i-1]}\right) \xrightarrow{n\to\infty} 0$ in probab., we can concentrate on the expression

$$\sqrt{\frac{k}{m}}\sum_{t\in I_i}\rho_t - \sqrt{n}(\sigma_B^{2**} - \sigma_B^2)\Pi_2^{[i-1]} = \sqrt{\frac{k}{m}}\sum_{t\in I_i}\rho_t - \left[\lambda_n^{-1}\frac{1}{\sqrt{n}}\sum_{j=1}^{k}\sum_{t\in I_j} Z_t^{[j]}\right]\Pi_2^{[i-1]}.$$

Since $(\lambda_n^{-1} - \Lambda^{-1})\frac{1}{\sqrt{n}}\sum_{j=1}^{k}\sum_{t\in I_j} Z_t^{[j]} \xrightarrow{n\to\infty} 0$ in probab., we can equivalently examine the asymptotic distribution of

$$\sqrt{\frac{k}{m}}\sum_{t\in I_i}\rho_t - \left[\Lambda^{-1}\frac{1}{\sqrt{n}}\sum_{j=1}^{k}\sum_{t\in I_j} Z_t^{[j]}\right]\Pi_2^{[i-1]} = \frac{1}{\sqrt{n}}\sum_{j=1}^{k}\sum_{t\in I_j}\Omega_t^{[i,j]},$$

where $\Omega_t^{[i,j]} := \left[k\delta_{i,t} - \Lambda^{-1}\left(X_{t-1}^2 - \Pi_2^{[j-1]}\right)\Pi_2^{[i-1]}\right]\rho_t$ and $\delta_{i,t} = 1$ for $t \in I_i$ and 0 elsewhere.
It is easy to check that $\Omega_t^{[i,j]}$ are martingale differences for each i and hence the standard application of CLT-MD as in the previous case can be used. Let us therefore define $s_n^{2[i]} := \sum_{j=1}^{k}\sum_{t\in I_j} E\left(\Omega_t^{[i,j]}\right)^2$. Then,

$$\begin{aligned} s_n^{2[i]} = {} & \sum_{j=1}^{k}\sum_{t\in I_j} E\Big[k^2\rho_t^2\delta_{i,t} - 2k\Pi_2^{[i-1]}\Lambda^{-1}\left(X_{t-1}^2 - \Pi_2^{[j-1]}\right)\rho_t^2\delta_{i,t} \\ & + \left(\Pi_2^{[i-1]}\right)^2\Lambda^{-2}\left(X_{t-1}^2 - \Pi_2^{[j-1]}\right)^2\rho_t^2\Big] \\ = {} & k^2\sum_{t\in I_i} E\rho_t^2 - 2k\Pi_2^{[i-1]}\Lambda^{-1}\sum_{t\in I_i} E\left(Z_t^{[i]}\rho_t\right) + \left(\Pi_2^{[i-1]}\right)^2\Lambda^{-2}\sum_{j=1}^{k}\sum_{t\in I_j} E\left(Z_t^{[j]}\right)^2 \end{aligned}$$

and hence $\frac{1}{n}s_n^{2[i]} \xrightarrow{n\to\infty} \hat{\Sigma}_Y^{[i]}$. The appropriate limits are derived in Appendix A.2 and their final forms are given by (28) and (31).
Finally, we have to prove that $\frac{1}{s_n^{[i]}}\sum_{j=1}^{k}\sum_{t\in I_j}\Omega_t^{[i,j]}$ has the asymptotic distribution $N(0,1)$ for all i. The corresponding conditions of CLT–MD are of the following form:

i) $\frac{1}{s_n^{2[i]}}\sum_{j=1}^{k}\sum_{t\in I_j} E\Big[\Big(\Omega_t^{[i,j]}\Big)^2\Big|\mathcal{F}_{t-1}\Big] \xrightarrow{n\to\infty} 1$ in probab.,

ii) $\frac{1}{s_n^{2[i]}}\sum_{j=1}^{k}\sum_{t\in I_j} E\Big[\Big(\Omega_t^{[i,j]}\Big)^2 I_{[|\Omega_t^{[i,j]}|\geq\varepsilon s_n^{[i]}]}\Big] \xrightarrow{n\to\infty} 0$ for all $\varepsilon > 0$.

Their verification can be done using expressions (26), (27), (29) and (30) from Appendix A.2 analogously as in the previous proof. □

Remark 3.8. It is worth mentioning that while $\hat{\Sigma}_B$ depends on k only through an average of some terms, $\hat{\Sigma}_Y^{[i]}$ includes linear term $k\hat{\Sigma}_1^{[i]}$ that increases asymptotic variance of each estimator $\hat{\sigma}_Y^{2[i]}$ with increasing number of seasonal periods (and hence unknown seasonal coefficients). Further, the relation between $\hat{\Sigma}_Y^{[i]}$ and $\hat{\Sigma}_B$ is seen.

4. ALTERNATIVE APPROACH

4.1. Estimation procedure

In contrast to the procedure described in Section 3, this approach primarily gives estimators of σ_b^2 and σ_t^2 instead of σ_B^2 and σ_t^2. In its first stage it does not require the OLS estimator $\hat{\beta}$. The idea is however very similar to the previous one.

Since the standard procedure is based on relation (5) for unknown residuals u_t, we have decided to use similar relation for the observed process itself. One can see that

$$E(X_t^2|\mathcal{F}_{t-1}) = \sigma_t^2 + \sigma_b^2 X_{t-1}^2 \quad \text{a.s.}$$

and hence analogously as in the previous case the estimators of unknown parameters can be obtained by minimizing $\sum_{t=1}^{n}(X_t^2 - \sigma_t^2 - \sigma_b^2 X_{t-1}^2)^2$ or equivalently as OLS estimators in the regression model $X_t^2 = \sigma_t^2 + \sigma_b^2 X_{t-1}^2 + \eta_t$, where η_t are $\mathcal{F}_t$-m.d.s.

Using the same arguments we can derive that they are given by the following formulas:

$$\tilde{\sigma}_b^2 = \frac{\sum_{i=1}^{k}\sum_{t\in I_i} X_t^2\Big(X_{t-1}^2 - \overline{X^2}^{[i-1]}\Big)}{\sum_{i=1}^{k}\sum_{t\in I_i}\Big(X_{t-1}^2 - \overline{X^2}^{[i-1]}\Big)^2}, \tag{19}$$

$$\tilde{\sigma}_Y^{2[i]} = \overline{X^2}^{[i]} - \overline{X^2}^{[i-1]}\tilde{\sigma}_b^2, \quad i = 1,\dots,k. \tag{20}$$

Consequently we can define the estimator of σ_B^2 as $\tilde{\sigma}_B^2 = \tilde{\sigma}_b^2 - \hat{\beta}^2$.

4.2. Asymptotic properties

These estimators are also strongly consistent and asymptotically normal. The proofs of these properties are even easier than those of Theorems 3.5 and 3.6 since Lemmas 3.5 and 3.8 are not needed. The main steps of the proofs are however the same and hence we can directly formulate the following theorems.

Theorem 4.1. Under A0 – A5, $\tilde{\sigma}_b^2 \overset{n\to\infty}{\underset{\text{a.s.}}{\longrightarrow}} \sigma_b^2$ holds.

Proof. In this case we can directly analyze the difference $\tilde{\sigma}_b^2 - \sigma_b^2$. Expression (19) can be reformulated in the way
$\tilde{\sigma}_b^2 = \sigma_b^2 + \lambda_n^{-1}\frac{1}{n}\sum_{i=1}^k \sum_{t\in I_i} (X_t^2 - \sigma_b^2 X_{t-1}^2)\left(X_{t-1}^2 - \overline{X^2}^{[i-1]}\right)$. Extending the previous expression by the term $\sum_{i=1}^k \sigma_Y^{2[i]} \sum_{t\in I_i}\left(X_{t-1}^2 - \overline{X^2}^{[i-1]}\right) = 0$, we can further write

$$\tilde{\sigma}_b^2 - \sigma_b^2 = \lambda_n^{-1}\frac{1}{n}\sum_{i=1}^k \sum_{t\in I_i}\left(X_{t-1}^2 - \overline{X^2}^{[i-1]}\right)\eta_t. \tag{21}$$

Analogously as in (16) let us define

$$\sigma_b^{2**} := \lambda_n^{-1}\frac{1}{n}\sum_{i=1}^k \sum_{t\in I_i}\left(X_{t-1}^2 - \Pi_2^{[i-1]}\right)\eta_t + \sigma_b^2. \tag{22}$$

Since $\{\eta_t\}$ is also an $\mathcal{F}_t$-m.d.s., we can proceed in the same way as in the proof of Theorem 3.3. □

Remark 4.1. Strong consistency of $\hat{\beta}$ directly implies that $\tilde{\sigma}_B^2 \overset{n\to\infty}{\underset{\text{a.s.}}{\longrightarrow}} \sigma_B^2$ holds.

Theorem 4.2. Under A0 – A5, $\tilde{\sigma}_Y^{2[i]} \overset{n\to\infty}{\underset{\text{a.s.}}{\longrightarrow}} \sigma_Y^{2[i]}$ holds for $i = 1, \dots, k$.

Proof. Firstly, let us define $\sigma_Y^{2[i]\#} := \overline{X^2}^{[i]} - \overline{X^2}^{[i-1]}\sigma_b^2$. Then $\tilde{\sigma}_Y^{2[i]} - \sigma_Y^{2[i]\#} \overset{n\to\infty}{\underset{\text{a.s.}}{\longrightarrow}} 0$, due to Theorem 4.1 and Lemma 3.3.
Further, $\sigma_Y^{2[i]\#} - \sigma_Y^{2[i]} = \frac{1}{m}\sum_{t\in I_i}\eta_t \overset{m\to\infty}{\underset{\text{a.s.}}{\longrightarrow}} 0$ holds, since $\{\eta_t\}$ satisfy SLLN–MD and hence the proof is finished. □

Theorem 4.3. Under Assumptions A0, A1', A2', A3', A4 and A5, the asymptotic distribution of $\sqrt{n}(\tilde{\sigma}_b^2 - \sigma_b^2)$ is $N(0, \Lambda^{-2}\tilde{\Sigma}_b)$, where

$$\tilde{\Sigma}_b := \frac{1}{k}\sum_{i=1}^k \lim_{m\to\infty}\frac{1}{m}\sum_{t\in I_i} E\left[\left(X_{t-1}^2 - \Pi_2^{[i-1]}\right)^2 \eta_t^2\right].$$

Proof. Combining (21), (22) and Lemma 3.3 we get $\sqrt{n}(\tilde{\sigma}_b^2 - \sigma_b^{2**}) \overset{n\to\infty}{\longrightarrow} 0$ in probab., since it can be shown that $\frac{1}{\sqrt{m}}\sum_{t\in I_i}\eta_t$ is $O_p(1)$. To find the asymptotic distribution of $\sqrt{n}(\sigma_b^{2**} - \sigma_b^2)$ we can proceed analogously as in the proof of Theorem 3.5 where ρ_t is interchanged with η_t. All conditions can be verified similarly. Derivation of $\tilde{\Sigma}_b$ is presented in Appendix A.3, its explicit form is given by (32). □

Theorem 4.4. Under Assumptions A0, A1', A2', A3', A4 and A5, the asymptotic distribution of $\sqrt{n}(\tilde{\sigma}_Y^{2[i]} - \sigma_Y^{2[i]})$ for $i = 1, \dots, k$ is $N(0, \tilde{\Sigma}_Y^{[i]})$, where

$$\tilde{\Sigma}_Y^{[i]} := k\tilde{\Sigma}_1^{[i]} - 2\Pi_2^{[i-1]}\Lambda^{-1}\tilde{\Sigma}_2^{[i]} + \left(\Pi_2^{[i-1]}\right)^2\Lambda^{-2}\tilde{\Sigma}_b,$$

$$\tilde{\Sigma}_1^{[i]} := \lim_{m\to\infty} \frac{1}{m}\sum_{t\in I_i} E\eta_t^2,$$

$$\tilde{\Sigma}_2^{[i]} := \lim_{m\to\infty} \frac{1}{m}\sum_{t\in I_i} E\Big[\left(X_{t-1}^2 - \Pi_2^{[i-1]}\right)\eta_t^2\Big].$$

Proof. As in the previous case, basic steps of the proof will be similar to those in the proof of Theorem 3.6. Using analogous arguments one can verify that the asymptotic distribution of $\sqrt{n}(\tilde{\sigma}_Y^{2[i]} - \sigma_Y^{2[i]}) = \sqrt{\frac{k}{m}}\sum_{t\in I_i}\eta_t - \sqrt{n}(\tilde{\sigma}_b^2 - \sigma_b^2)\overline{X^2}^{[i-1]}$ is the same as that of $\sqrt{\frac{k}{m}}\sum_{t\in I_i}\eta_t - \sqrt{n}(\sigma_b^{2**} - \sigma_b^2)\Pi_2^{[i-1]}$ and consequently as that of $\frac{1}{\sqrt{n}}\sum_{j=1}^k\sum_{t\in I_j}\Psi_t^{[i,j]}$, where $\Psi_t^{[i,j]} := \left[k\delta_{i,t} - \Lambda^{-1}\left(X_{t-1}^2 - \Pi_2^{[j-1]}\right)\Pi_2^{[i-1]}\right]\eta_t$. Derivation of the asymptotic variance $\tilde{\Sigma}_Y^{[i]}$ and checking of conditions of CLT–MD are then made analogously as in the proof of the mentioned theorem. Again, the exact form of $\tilde{\Sigma}_Y^{[i]}$ is derived in Appendix A.4, formulas for $\tilde{\Sigma}_1^{[i]}$ and $\tilde{\Sigma}_2^{[i]}$ are given by (33) and (34). □

5. COMPARISON OF BOTH APPROACHES

5.1. Theoretical comments

Basic difference between two presented methods is the fact that in the former one the parameter β has to be firstly estimated to obtain residuals $\hat{u}_t$. Then the remaining parameters σ_B^2 and σ_t^2 are estimated. On the other hand, in the latter method estimates of σ_b^2 and σ_t^2 are directly computed.

In the alternative method described in Section 4 one avoids estimation of residuals in the first stage that may incorporate inaccuracy before remaining parameters are estimated. On the other hand, the fact that both parameters β and σ_B^2 are estimated together in the alternative method may be also its disadvantage, since impact of each parameter can not be well separated. It can consequently lead to inaccurate estimates of the whole σ_b^2.

Theoretical comparison of both approaches is however hardly to be done, even in case of estimates of σ_t^2. It arises from the fact that asymptotic variances of $\hat{\sigma}_Y^{2[i]}$ and $\tilde{\sigma}_Y^{2[i]}$ depend on asymptotic variances $\hat{\Sigma}_B$ and $\tilde{\Sigma}_b$, respectively, that are incomparable.

5.2. Numerical comparison

We made several simulations to find out in which cases the standard method is preferable to the alternative one and vice versa. Selected results of our simulation study are presented in the sequel.

We simulated 21 types of RCA(1) processes satisfying model (1), where b_t and Y_t were supposed to be normally distributed. In each case different parameters of distribution of b_t were considered. Their values are summarized in the table. In spite of the fact that this paper concerns with generally heteroscedastic RCA(1) models, the homoscedastic processes Y_t with $\sigma_t^2 = \sigma_Y^2 = 1$ were used for this presentation. The reason is that comparison of both approaches can be well demonstrated in homoscedastic case. We additionally made analogous simulations for $\sigma_Y^2 = 5$ and for seasonal heteroscedasticity with $k = 2$ and $k = 4$. The conclusions are however very similar to those presented here.

All estimates were based on 100 independent realizations with 1000 observations in each case. Since estimates $\hat{\sigma}_B^2$ and $\tilde{\sigma}_B^2$ differ from $\hat{\sigma}_b^2$ and $\tilde{\sigma}_b^2$ only of $\hat{\beta}^2$, it is sufficient to compare only one of these pairs. We chose to present here estimates of σ_b^2 and σ_Y^2 that are summarized in the table.

Table. Estimates of σ_b^2 and σ_Y^2 in a homoscedastic RCA(1) model.

	Parameters of distribution of b_t			Estimates			
	β	σ_B^2	σ_b^2	$\hat{\sigma}_b^2$	$\tilde{\sigma}_b^2$	$\hat{\sigma}_Y^2$	$\tilde{\sigma}_Y^2$
A	0	0.1	0.1	0.0911	**0.0922**	1.0088	**1.0086**
	0.2	0.06	0.1	0.0902	**0.0914**	1.0103	**1.0097**
B	0	0.2	0.2	0.1903	**0.1927**	1.0122	**1.0109**
	0.3	0.11	0.2	0.2701	0.2608	0.9126	0.9256
C	0	0.26	0.26	0.2458	**0.2499**	1.0199	**1.0166**
	0.4	0.1	0.26	0.2483	0.2467	1.0093	1.0123
D	0.1	0.35	0.36	0.3144	**0.3157**	1.0587	1.0593
	0.5	0.11	0.36	0.3500	**0.3554**	1.0141	**1.0061**
	0	0.5	0.5	0.3947	**0.4034**	1.1936	**1.1822**
	0.2	0.46	0.5	0.4071	0.4031	1.1395	1.1557
E	0.3	0.41	0.5	0.4301	0.4150	1.1202	1.1563
	0.4	0.34	0.5	0.4551	0.4294	1.0946	1.1548
	0.5	0.25	0.5	0.4661	0.4429	1.0620	1.1062
	0	0.64	0.64	0.4442	**0.4656**	1.4430	**1.4023**
F	0.1	0.63	0.64	0.4469	**0.4591**	1.4628	**1.4463**
	0.2	0.6	0.64	0.4614	0.4588	1.4305	1.4481
	0.6	0.28	0.64	0.5919	0.5314	1.1199	1.2876
	0	0.74	0.74	0.4833	**0.5075**	1.7607	**1.6942**
G	0.2	0.7	0.74	0.4946	**0.5084**	1.8308	**1.7993**
	0.3	0.65	0.74	0.5201	0.4883	1.6550	1.7692
	0.7	0.25	0.74	0.6806	0.5765	1.1144	1.4931

In order to make comparison of both methods, all processes were separated into 7 groups so as to have the same second moment σ_b^2 within each group. Boldfaced values in the table are those where the alternative approach gives estimates with smaller estimated bias than the standard method. One can see that priority of one of the method does not depend on the value of the second moment σ_b^2 but one the value β alone. Our simulations show that the alternative method is preferable for processes where the true parameter β is close to 0 regardless of the value σ_b^2 ranking within $(0,1)$. It holds both for estimates of σ_b^2 and σ_Y^2.

These empirical results may correspond with the fact that under the null hypothesis $H_0 : \beta = 0$, the RCA(1) process is second order equivalent to the special case of the ARCH(1) process in the sense that both processes have the same conditional expectation and variance. In a homoscedastic case it was proved in [12], generalization for heteroscedastic processes is given in [6]. In the latter paper there is shown that estimation procedure for ARCH processes is the same as the alternative method presented here for RCA processes.

Finally, from the table one can deduce some common features of both methods. It is seen that both methods overestimate parameters σ_Y^2 and underestimate σ_b^2. The higher the value of σ_b^2, the greater the over- and underestimation. Comparing estimated variance of presented estimates (that are not given here), one can see that there is no significant and systematic superiority of one of the method. When parameter σ_b^2 is greater than 0.8 and it is tending to 1, processes start to be very unstable and both methods give inaccurate estimates with extremely high estimated variances.

APPENDIX

A.1 Derivation of $\hat{\Sigma}_B$

The expression $\left(Z_t^{[i]}\right)^2 = \left(X_{t-1}^2 - \Pi_2^{[i-1]}\right)^2 \rho_t^2 = \left(X_{t-1}^2 - \Pi_2^{[i-1]}\right)^2 (u_t^2 - \sigma_t^2 - \sigma_B^2 X_{t-1}^2)^2$, where $u_t = B_t X_{t-1} + Y_t$, can be expanded, for $t \in I_i$, to the following form:

$$\begin{aligned}\left(Z_t^{[i]}\right)^2 &= X_{t-1}^8 (B_t^2 - \sigma_B^2)^2 + 2X_{t-1}^6 \left[2B_t^2 Y_t^2 - \Pi_2^{[i-1]}(B_t^2 - \sigma_B^2)^2\right] \\ &+ X_{t-1}^4 \left[Y_t^4 + \left(\sigma_Y^{2[i]}\right)^2 - 2Y_t^2 \sigma_Y^{2[i]} + \left(\Pi_2^{[i-1]}\right)^2 (B_t^2 - \sigma_B^2)^2 - 8\Pi_2^{[i-1]} B_t^2 Y_t^2\right] \\ &+ 2X_{t-1}^2 \Pi_2^{[i-1]} \left[2Y_t^2 \sigma_Y^{2[i]} - Y_t^4 - \left(\sigma_Y^{2[i]}\right)^2 + 2\Pi_2^{[i-1]} B_t^2 Y_t^2\right] \\ &+ \left(\Pi_2^{[i-1]}\right)^2 \left[Y_t^4 + \left(\sigma_Y^{2[i]}\right)^2 - 2Y_t^2 \sigma_Y^{2[i]}\right] + M_{1,t}^{[i]},\end{aligned} \tag{23}$$

where

$$\begin{aligned}M_{1,t}^{[i]} &= 4X_{t-1}^7 (B_t^2 - \sigma_B^2) B_t Y_t + 2X_{t-1}^6 (B_t^2 - \sigma_B^2)\left[Y_t^2 - \sigma_Y^{2[i]}\right] \\ &+ 4X_{t-1}^5 B_t Y_t \left[Y_t^2 - \sigma_Y^{2[i]} - 2\Pi_2^{[i-1]}(B_t^2 - \sigma_B^2)\right] - 4X_{t-1}^4 (B_t^2 - \sigma_B^2)\Pi_2^{[i-1]}\left[Y_t^2 - \sigma_Y^{2[i]}\right] \\ &+ 4X_{t-1}^3 B_t Y_t \Pi_2^{[i-1]}\left[2\sigma_Y^{2[i]} - 2Y_t^2 + \Pi_2^{[i-1]}(B_t^2 - \sigma_B^2)\right] \\ &+ 2X_{t-1}^2 (B_t^2 - \sigma_B^2)\left(\Pi_2^{[i-1]}\right)^2 \left[Y_t^2 - \sigma_Y^{2[i]}\right] + 4X_{t-1} B_t Y_t \left(\Pi_2^{[i-1]}\right)^2 \left[Y_t^2 - \sigma_Y^{2[i]}\right].\end{aligned} \tag{24}$$

One can easily check that $EM_{1,t}^{[i]} = 0$, hence after some algebra in (23) we get for $\hat{\Sigma}_B := \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{k}\sum_{t\in I_i} E\left(Z_t^{[i]}\right)^2$ the following equality:

$$\begin{aligned}\hat{\Sigma}_B = \Pi_8^{[i-1]}\mathrm{var}(B_t^2) + \frac{1}{k}\sum_{i=1}^{k}\Bigg\{ 2\Pi_6^{[i-1]}\Big[2\sigma_B^2\sigma_Y^{2[i]} - \Pi_2^{[i-1]}\mathrm{var}(B_t^2)\Big] \\ + \Pi_4^{[i-1]}\Big[\mathrm{var}\left(Y^{2[i]}\right) + \left(\Pi_2^{[i-1]}\right)^2\mathrm{var}(B_t^2) - 8\Pi_2^{[i-1]}\sigma_B^2\sigma_Y^{2[i]}\Big] \\ + \left(\Pi_2^{[i-1]}\right)^2\Big[4\Pi_2^{[i-1]}\sigma_B^2\sigma_Y^{2[i]} - \mathrm{var}\left(Y^{2[i]}\right)\Big]\Bigg\},\end{aligned} \tag{25}$$

where $var\left(Y^{2[i]}\right) = \mu_{Y,4}^{[i]} - \left(\sigma_Y^{2[i]}\right)^2$.

A.2 Derivation of $\hat{\Sigma}_Y^{[i]}$

Firstly, let us derive $\hat{\Sigma}_1^{[i]} := \lim_{m\to\infty}\frac{1}{m}\sum_{t\in I_i} E\rho_t^2$. It is easy to show that for $t \in I_i$

$$\rho_t^2 = X_{t-1}^4(B_t^2-\sigma_B^2)^2 + 4X_{t-1}^2B_t^2Y_t^2 + Y_t^4 - 2Y_t^2\sigma_Y^{2[i]} + \left(\sigma_Y^{2[i]}\right)^2 + M_{2,t}^{[i]}, \tag{26}$$

where

$$\begin{aligned} M_{2,t}^{[i]} &= 4X_{t-1}^3(B_t^2-\sigma_B^2)B_tY_t + 2X_{t-1}^2(B_t^2-\sigma_B^2)\Big[Y_t^2 - \sigma_Y^{2[i]}\Big] \\ &\quad + 4X_{t-1}B_tY_t\Big[Y_t^2 - \sigma_Y^{2[i]}\Big].\end{aligned} \tag{27}$$

Again, $EM_{2,t}^{[i]} = 0$ holds and hence

$$\hat{\Sigma}_1^{[i]} = \Pi_4^{[i-1]}\mathrm{var}(B_t^2) + 4\Pi_2^{[i-1]}\sigma_B^2\sigma_Y^{2[i]} + \mathrm{var}\left(Y^{2[i]}\right). \tag{28}$$

Further, let us expand $Z_t^{[i]}\rho_t$ for $t \in I_i$:

$$\begin{aligned} Z_t^{[i]}\rho_t = X_{t-1}^6(B_t^2-\sigma_B^2)^2 + X_{t-1}^4\Big[4B_t^2Y_t^2 - \Pi_2^{[i-1]}(B_t^2-\sigma_B^2)^2\Big] \\ + X_{t-1}^2\Big[Y_t^4 - 2Y_t^2\sigma_Y^{2[i]} + \left(\sigma_Y^{2[i]}\right)^2 - 4\Pi_2^{[i-1]}B_t^2Y_t^2\Big] \\ + \Pi_2^{[i-1]}\Big[2Y_t^2\sigma_Y^{2[i]} - Y_t^4 - \left(\sigma_Y^{2[i]}\right)^2\Big] + M_{3,t}^{[i]},\end{aligned} \tag{29}$$

where

$$\begin{aligned} M_{3,t}^{[i]} = 4X_{t-1}^5(B_t^2-\sigma_B^2)B_tY_t + 2X_{t-1}^4(B_t^2-\sigma_B^2)\Big[Y_t^2 - \sigma_Y^{2[i]}\Big] \\ + 4X_{t-1}^3B_tY_t\Big[Y_t^2 - \sigma_Y^{2[i]} - \Pi_2^{[i-1]}(B_t^2-\sigma_B^2)\Big] \\ - 2X_{t-1}^2(B_t^2-\sigma_B^2)\Pi_2^{[i-1]}\Big[Y_t^2 - \sigma_Y^{2[i]}\Big] - 4X_{t-1}B_tY_t\Pi_2^{[i-1]}\Big[Y_t^2 - \sigma_Y^{2[i]}\Big]\end{aligned} \tag{30}$$

and $EM_{3,t}^{[i]} = 0$. Derivation of $\hat{\Sigma}_2^{[i]} = \lim_{m\to\infty} \frac{1}{m} \sum_{t\in I_i} E\left(Z_t^{[i]}\rho_t\right)$ is now straightforward:

$$\hat{\Sigma}_2^{[i]} = \Pi_6^{[i-1]} \mathrm{var}(B_t^2) + \Pi_4^{[i-1]} \left[4\sigma_B^2\sigma_Y^{2[i]} - \Pi_2^{[i-1]} \mathrm{var}(B_t^2)\right] - 4\left(\Pi_2^{[i-1]}\right)^2 \sigma_B^2\sigma_Y^{2[i]}. \tag{31}$$

A.3 Derivation of $\tilde{\Sigma}_b$

One can observe that the only difference between $\hat{\Sigma}_B$ and $\tilde{\Sigma}_b$ is in using ρ_t or η_t, respectively. Let us compare both terms:

$$\rho_t = u_t^2 - \sigma_B^2 X_{t-1}^2 - \sigma_t^2 = (B_t^2 - \sigma_B^2)X_{t-1}^2 + 2X_{t-1}B_tY_t + Y^2 - \sigma_t^2,$$
$$\eta_t = X_t^2 - \sigma_b^2 X_{t-1}^2 - \sigma_t^2 = (b_t^2 - \sigma_b^2)X_{t-1}^2 + 2X_{t-1}b_tY_t + Y^2 - \sigma_t^2.$$

Since the difference is only in using b_t and σ_b^2 instead of B_t and σ_B^2, we can formally interchange these terms in (23) and (24) and get the limit $\tilde{\Sigma}_b$ analogously as in (25). The only one difference is that two terms $4X_{t-1}^5 b_t Y_t^3$ and $-8X_{t-1}^3\Pi_2^{[i-1]} b_t Y_t^3$ from (24) do not already have zero expectation and have to be taken under consideration. Hence, the final form of $\tilde{\Sigma}_b$ is

$$\begin{aligned}\tilde{\Sigma}_b &= \Pi_8^{[i-1]} \mathrm{var}(b_t^2) + \frac{1}{k}\sum_{i=1}^{k}\Bigg\{ 2\Pi_6^{[i-1]}\left[2\sigma_b^2\sigma_Y^{2[i]} - \Pi_2^{[i-1]}\mathrm{var}(b_t^2)\right] + 4\Pi_5^{[i-1]}\beta\mu_{Y,3}^{[i]} \\ &+ \Pi_4^{[i-1]}\left[\mathrm{var}\left(Y^{2[i]}\right) + \left(\Pi_2^{[i-1]}\right)^2 \mathrm{var}(b_t^2) - 8\Pi_2^{[i-1]}\sigma_b^2\sigma_Y^{2[i]}\right] - 8\Pi_3^{[i-1]}\Pi_2^{[i-1]}\beta\mu_{Y,3}^{[i]} \\ &+ \left(\Pi_2^{[i-1]}\right)^2\left[4\Pi_2^{[i-1]}\sigma_b^2\sigma_Y^{2[i]} - \mathrm{var}\left(Y^{2[i]}\right)\right]\Bigg\}. \end{aligned} \tag{32}$$

A.4 Derivation of $\tilde{\Sigma}_Y^{[i]}$

Analogously as in Appendix A.3 we get the desired limits by interchanging b_t and σ_b^2 with B_t and σ_B^2 in (26), (27), (29) and (30). After this operation, the only term with non-zero expectation is $4X_{t-1}^3 b_t Y_t^3$ from (30). By evaluating the appropriate limits we get

$$\tilde{\Sigma}_1^{[i]} = \Pi_4^{[i-1]}\mathrm{var}(b_t^2) + 4\Pi_2^{[i-1]}\sigma_b^2\sigma_Y^{2[i]} + \mathrm{var}\left(Y^{2[i]}\right), \tag{33}$$

$$\begin{aligned}\tilde{\Sigma}_2^{[i]} &= \Pi_6^{[i-1]}\mathrm{var}(b_t^2) + \Pi_4^{[i-1]}\left[4\sigma_b^2\sigma_Y^{2[i]} - \Pi_2^{[i-1]}\mathrm{var}(b_t^2)\right] \\ &+ 4\Pi_3^{[i-1]}\beta\mu_{Y,3}^{[i]} - 4\left(\Pi_2^{[i-1]}\right)^2\sigma_b^2\sigma_Y^{2[i]}. \end{aligned} \tag{34}$$

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