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CONVERGENCE THEOREMS FOR MEASURES WITH VALUES IN RIESZ SPACES

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In some recent papers, results of uniform additivity have been obtained for convergent sequences of measures with values in l-groups. Here a survey of these results and some of their applications are presented, together with a convergence theorem involving Lebesgue decompositions.

1. INTRODUCTION

This short note is an overview of some recent results, obtained by the Author in some joint papers with A. Boccuto, concerning convergence theorems for sequences of measures, of the type of Vitali–Hahn–Saks.

From these results, other relevant theorems are deduced, such as Schur-type theorems, Dieudonné-type theorems, and also some theorems concerning Lebesgue-type decompositions for convergent sequences of measures.

We recall the so-called Vitali–Hahn–Saks (V-H-S) theorem (see [12]):

Given a sequence of σ-additive measures, defined on some σ-algebra $\mathcal{B}$ of subsets of some abstract set X, from pointwise convergence of these measures on all elements of $\mathcal{B}$ it follows that they are *uniformly* σ-additive, and the limit function is still σ-additive on $\mathcal{B}$.

This theorem has been generalized in many directions since then. We shall only mention [1, 2, 6, 7, 8, 9]. However, in the framework of Riesz-space-valued measures the results were not sufficiently general, mainly because in such spaces in general there is no topology inducing the usual (O)-convergence.

In some recent papers, ([3, 4, 5]), a new instrument has been introduced, which allows to obtain sufficiently general convergence theorems for Riesz-space valued measures.

The basic tool is an equivalent formulation of order convergence, named (D)-convergence, which is then used to formulate a suitable condition of convergence for sequences of *functions* taking values in a Riesz space: such condition (which we call (RD)-convergence) is *formally* stronger than *pointwise* order convergence, and *substantially* weaker than *uniform* order convergence.

Here, we shall outline the main results obtained in [3, 4, 5], and then we prove a convergence theorem involving the Lebesgue decompositions of a sequence of measures (μ_n), (with respect to a scalar non-negative measure λ), assuming that the measures (μ_n) are (RD)-convergent.

This result is simply meant as an application of the Vitali–Hahn–Saks theorem, without aiming at full generality.

2. PRELIMINARIES

We begin recalling the following:

Definitions 2.1. A Riesz space R is said to be *Dedekind complete* if every nonempty subset of R, bounded from above, has supremum in R.

A sequence $(r_n)_n$ in R is said to be *order-convergent* (or *(o)-convergent*) to r if there exists a sequence $(p_n)_n$ in R such that $p_n \downarrow 0$ and $|r_n - r| \leq p_n$, $\forall n \in \mathbb{N}$: this will be written $(o)\lim_n r_n = r$. Order convergence can be formulated simply as *coincidence* of $\liminf r_n$ and $\limsup r_n$, as soon as (r_n) is bounded (see also [13]).

A bounded double sequence $(a_{i,l})_{i,l}$ in R is called *(D)-sequence* or *regulator* if for all $i \in \mathbb{N}$ we have $a_{i,l} \downarrow 0$ as $l \to +\infty$.

We say that $b \in R$, $b \geq 0$, *dominates* a sequence $(r_n)_n$ of elements of R if there exists $n_0 \in \mathbb{N}$ such that $|r_n| \leq b$ for $n \geq n_0$. Moreover, given a regulator $(a_{i,l})_{i,l}$, we call *bound* of $(a_{i,l})_{i,l}$ every element b of the type $b = \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$, for some $\varphi \in \mathbb{N}^{\mathbb{N}}$.

A sequence $(r_n)_n$ in R is said to be *(D)-convergent* to $r \in R$ (and we write $(D)\lim_n r_n = r$) if there exists a regulator $(a_{i,l})_{i,l}$ whose every bound dominates the sequence $(r_n - r)_n$.

In general, order-convergence implies (D)-convergence, while the converse is false, unless R is *weakly σ-distributive*, according with the following definition.

A Riesz space R is said to be *weakly σ-distributive* if for every (D)-sequence $(a_{i,l})_{i,l}$ we have:

$$\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \right) = 0.$$

From now on R will denote a weakly σ-distributive and Dedekind complete Riesz space: therefore *order convergence and (D)-convergence shall be considered as equivalent.*

This is not a sharp requirement: one can easily see that weak σ-distributivity is a necessary and sufficient condition for *uniqueness* of the (D)-limit.

The main motivation for working with (D)-convergence, rather than (O)-convergence, in a weakly σ-distributive Riesz space, is contained in the following concept.

Definition 2.2. If E is any nonempty set, we say that a sequence $(f_n)_n$ of elements of R^E *(RD)-converges to* $f \in R^E$ if there exists a regulator whose every bound dominates every sequence of the type $(f_n(x) - f(x))_n$, with $x \in E$. Analogously, we say that $(f_n)_n$ *(UD)-converges to* f if there exists a regulator whose every bound dominates the sequence $\left(\bigvee_{x\in E} |f_n(x) - f(x)|\right)_n$.

We remark here that (RD)-convergence is somewhat stronger than pointwise (D)-convergence (which in our context is equivalent to pointwise (O)-convergence), while of course (UD)-convergence corresponds to *uniform* convergence. However (RD)-convergence can be shown to be equivalent to pointwise (O)-convergence, as soon as the latter is topological.

The next Lemma shows a further feature of such a kind of convergence, see [14].

Lemma 2.3. Let $(a^k_{i,j})$ be any countable family of regulators. Then for each fixed element $u \in R$ there exists a regulator $(a_{i,j})$ such that, for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ one has

$$u \wedge \sum_{k=1}^{\infty} \left(\bigvee_{i=1}^{\infty} a^k_{i,\varphi(i+k)} \right) \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}.$$

We now introduce the following:

Definitions 2.4. Let Ω be any infinite set, $\mathcal{A} \subset \mathcal{P}(\Omega)$ be an algebra, and $\mathcal{E} \subset \mathcal{A}$ be any non-empty sub-family of $\mathcal{A}$. Given a finitely additive bounded measure (or, in short, *mean*) $m : \mathcal{A} \to R$, we define the *$\mathcal{E}$-semivariation* $v_{\mathcal{E}}(m) : \mathcal{A} \to R$ by:

$$v_{\mathcal{E}}(m)(A) \;=\; \sup_{B \in \mathcal{E}, B \subset A} |m(B)|, \ \forall A \in \mathcal{A}.$$

When $\mathcal{E} = \mathcal{A}$, we get the *semivariation* of m :

$$v(m) := v_{\mathcal{A}}(m)$$

A mean $m : \mathcal{A} \to R$ is said to be *σ-additive* (or, in short, *measure*) if there exists a (D)-sequence $(u_{i,l})_{i,l}$ such that, $\forall \varphi \in \mathbb{N}^{\mathbb{N}}$ and for every decreasing sequence $(H_s)_s$ in $\mathcal{A}$, $H_s \downarrow \emptyset$, there exists $\overline{s}$:

$$v_{\mathcal{A}}(m)(H_{\overline{s}}) \leq \bigvee_{i=1}^{\infty} u_{i,\varphi(i)}.$$

If a sequence of measures $m_j : \mathcal{A} \to R$, $j \in \mathbb{N}$, is given, *uniform σ-additivity* is defined as above, but with $\overline{s}$ *independent* of j (see also [3]).
A finitely additive measure $m : \mathcal{A} \to R$ is said to be *(s)-bounded in $\mathcal{E}$* or simply *$\mathcal{E}$-(s)-bounded,* if there exists a (D)-sequence $(w_{i,l})_{i,l}$ such that, $\forall \varphi \in \mathbb{N}^{\mathbb{N}}$ and for every disjoint sequence $(H_s)_s$ in $\mathcal{E}$ there exists $\overline{s}$: $\forall s \geq \overline{s}$,

$$v_{\mathcal{E}}(m)(H_s) \leq \bigvee_{i=1}^{\infty} w_{i,\varphi(i)}.$$

If $\mathcal{E}$ is as above, we say that the maps $m_j : \mathcal{A} \to R$, $j \in \mathbb{N}$, are *$\mathcal{E}$-uniformly (s)-bounded* if the above condition holds, but with $\overline{s}$ independent of j (see also [3]). When $\mathcal{E} = \mathcal{A}$ we simply speak of *(s)-boundedness or uniform (s)-boundedness.*

Given a sequence of means $(m_j)_{j\in\mathbb{N}\cup\{0\}}$; $m_j : \mathcal{A} \to R$, we say that the m_j's (RD)-*converge to* m_0 *in* $\mathcal{E}$ if the sequence of functions $(m_j : \mathcal{E} \to R)_j$ (RD)-converges to m_0.

Let now Ω, R and $\mathcal{A}$ be as above. From now on, we assume that $\mathcal{F}$, $\mathcal{G} \subset \mathcal{A}$ are two fixed lattices, such that the complement (with respect to Ω) of every element of $\mathcal{F}$ belongs to $\mathcal{G}$.

Definitions 2.5. A mean $m : \mathcal{A} \to R$ is said to be *regular* if there exists a (D)-sequence $(\gamma_{i,l})_{i,l}$ in R such that for each $A \in \mathcal{A}$ and $W \in \mathcal{F}$ there exists sequences $(F_n)_n$, $(F'_n)_n$ in $\mathcal{F}$, $(G_n)_n$, $(G'_n)_n$ in $\mathcal{G}$, such that

$$F_n \subset F_{n+1} \subset A \subset G_{n+1} \subset G_n \quad \forall n, \tag{1}$$

$$W \subset F'_{n+1} \subset G'_n \subset F'_n \quad \forall n, \tag{2}$$

and the sequences $(v_{\mathcal{A}}(m)(G_n \setminus F_n))_n$ and $(v_{\mathcal{A}}(m)(G'_n \setminus W))_n$ (D)-converge to 0 with respect to $(\gamma_{i,l})_{i,l}$.

The means $m_j : \mathcal{A} \to R$, $j \in \mathbb{N}$, are said to be *uniformly regular* if there exists a (D)-sequence $(\gamma_{i,l})_{i,l}$ in R such that $\forall A \in \mathcal{A}$ and $\forall W \in \mathcal{F}$ there exist sequences $(F_n)_n, (G_n)_n, (F'_n)_n, (G'_n)_n$ satisfying (1) and (2), and such that the sequences $(\psi_n)_n$, $(\omega_n)_n$ of elements of $R^{\mathbb{N}}$, defined by setting

$$\psi_n(j) = v_{\mathcal{A}}(m_j)(G_n \setminus F_n), \tag{3}$$

$$\omega_n(j) = v_{\mathcal{A}}(m_j)(G'_n \setminus W) \quad n, j \in \mathbb{N},$$

(UD)-converge to 0 with respect to $(\gamma_{i,l})_{i,l}$.

We now introduce the concept of absolute continuity in our setting.

Definition 2.6. Let m be any R-valued finitely additive measure on $\mathcal{A}$. Given any other finitely additive measure $\nu : \mathcal{A} \to \mathbb{R}_0^+$, we say that m is *absolutely continuous* with respect to ν (and write $m \ll \nu$) if there exists a (D)-sequence $(a_{i,l})_{i,l}$ such that, whenever $(H_k)_k$ is a sequence from $\mathcal{A}$ satisfying $\lim_k \nu(H_k) = 0$, for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ an integer $\overline{k}$ can be found, such that $|m(H_k)| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$, for all $k \geq \overline{k}$.
In case ν is fixed, and $(m_j)_j$ is a sequence of finitely additive measures on $\mathcal{A}$, *uniform absolute continuity* of the m_j's with respect to ν can be defined in a similar way, but clearly the integer $\overline{k}$ must be independent of j.

One can easily see that, in case m and ν are σ-additive and non-negative, this definition of absolute continuity is equivalent to the so-called *(0-0)* one:

$$\mu \ll \nu \text{ if and only if } \nu(A) = 0 \text{ implies } m(A) = 0.$$

The following theorem will be needed in the sequel. (See [3], Theorem 4.8.)

Theorem 2.7. Let $(m_n)_n$ be any sequence of uniformly bounded, uniformly s-bounded R-valued finitely additive measures on an algebra Σ. If the measures m_n are absolutely continuous with respect to the same finitely additive measure $\nu : \Sigma \to \mathbb{R}_0^+$, then they are uniformly absolutely continuous.

We shall also need the notion of *singularity.*

Definition 2.8. Assume that m and ν are as in Definition 2.6. We say that m and ν are *singular* if there exist a regulator $(a_{i,l})_{i,l}$ and a sequence $(A_k)_k$ from $\mathcal{A}$ such that $\lim_k \nu(A_k) = 0$ and such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ an integer k_0 can be found, satisfying $v(m)(A_k^c) \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$, for all $k \geq k_0$.

When this is the case, we write $m \perp \nu$ (or also $\nu \perp m$).

A concept of *uniform singularity* for a sequence of measures m_j can be introduced, by means of the same formulation as in Definition 2.8, but requiring that the integer k_0 does not depend on $j \in \mathbb{N}$.

Proposition 2.9. Assume that m and ν are σ-additive and non-negative. Then m and ν are singular if and only if there exists a set $A \in \mathcal{A}$ such that $m(A) = 0 = \nu(A^c)$.

Proof. Of course, just the "only if" part needs proving. So, assume $m \perp \nu$, and let $(A_k)_k$ and $(a_{i,l})_{i,l}$ be the sequence and the regulator related to singularity according with Definition 2.8. Without loss of generality, we can assume that $\sum \nu(A_k) < \infty$. Thus, if A denotes the set $A := \limsup A_k$, we have $\nu(A) = 0$. On the other hand, $m(A^c) = (O) - \lim m(B_k^c)$, where $B_k = \cup_{j=k}^{\infty} A_j$, because of σ-additivity of m. As $m(B_k^c) \leq m(A_k^c)$ for each integer k, we obtain $m(A^c) = 0$. □

3. THE VITALI–HAHN–SAKS THEOREM

In this section, we shall report the main results of [3], in the formulation we need later.

We first deal with the σ-additive case.

Theorem 3.1. Let $(m_n)_n$ be any sequence of uniformly bounded, σ-additive measures, defined on the σ-algebra $\mathcal{A}$ and taking values in R. Let $\mathcal{G}$ be any lattice in $\mathcal{A}$, and assume that $\mathcal{G}$ is closed under countable disjoint unions.

If the measures m_n are (RD)-convergent in $\mathcal{G}$, then they are uniformly $\mathcal{G}$-s-bounded.

Of course, if $\mathcal{G}$ coincides with $\mathcal{A}$, from (RD)-convergence in $\mathcal{A}$ it follows uniform s-boundedness.

A typical consequence of the Vitali–Hahn–Saks theorem is the so-called Schur theorem.

Corollary 3.2. (Schur Theorem.) Let $(m_k)_k$ be a sequence of σ-additive measures, defined on $\mathcal{P}(\mathbb{N})$ and taking values in R.

If the measures are (RD)-convergent to some measure m, then they are *(UD)*-convergent (see Definition 2.2).

In finitely additive setting, we need a further assumption, in order to obtain a result of the type of Vitali–Hahn–Saks. Moreover, for the sake of simplicity, we shall assume (RD)-convergence in the whole σ-algebra $\mathcal{A}$.

Theorem 3.3. Let $(m_n)_n$ be any sequence of uniformly bounded, finitely additive measures, defined on a σ-algebra $\mathcal{A}$. If the measures m_n are all absolutely continuous with respect to a finitely additive measure $\nu : \mathcal{A} \to \mathbb{R}_0^+$ and if they are (RD)-convergent to some limit m_0, then the measures m_n are uniformly s-bounded and uniformly absolutely continuous with respect to ν.

4. THE DIEUDONNÉ THEOREM

In this section we list some formulations of the Dieudonné-type theorems proved in [5].

We assume, as usual, that $\mathcal{A}$ is a σ-algebra of subsets of a set Ω, and that $\mathcal{F}$, $\mathcal{G}$ are two sublattices of $\mathcal{A}$, such that the complement of every element $F \in \mathcal{F}$ belongs to $\mathcal{G}$.

The following Lemma is crucial: it states that uniform s-boundedness on $\mathcal{G}$ implies, for a sequence of regular means, uniform s-boundedness in $\mathcal{A}$. The proof is in [5].

Lemma 4.1. Under the same hypotheses and notations as above, let $(m_j : \mathcal{A} \to R)_j$ be a sequence of uniformly bounded, regular and $\mathcal{G}$-uniformly (s)-bounded means. Then the m_j's are $\mathcal{A}$-uniformly (s)-bounded, and uniformly regular.

Theorem 4.2. (Dieudonné) Let Ω, R, $\mathcal{G}$, $\mathcal{F}$ be as above, and assume that $\mathcal{G}$ is stable under countable disjoint unions. Suppose that $(m_j : \mathcal{A} \to R)_j$ is a sequence of uniformly bounded regular σ-additive measures such that there exists

$$m_0 = (RD) \lim_j m_j \quad \text{in } \mathcal{G}.$$

Then we have:

i) The measures m_j, $j \in \mathbb{N}$, are $\mathcal{A}$-uniformly (s)-bounded and uniformly regular.

ii) There exists in R the limit $m_0 = (RD) \lim_j m_j$ in $\mathcal{A}$.

iii) The m_j's are uniformly σ-additive.

iv) m_0 is regular and σ-additive.

The proof is an easy consequence of Theorem 3.1 and of the previous Lemma 4.1.

Under suitable additional conditions, we can also state a finitely additive version of Dieudonné's theorem.

Theorem 4.3. Let Ω, R, $\mathcal{A}$, $\mathcal{G}$, $\mathcal{F}$ be as usual, and assume that $\mathcal{G}$ is stable under countable disjoint unions. Suppose that $(m_j : \mathcal{A} \to R)_j$ is a sequence of uniformly bounded regular finitely additive measures, absolutely continuous with respect to a real-valued, nonnegative, finitely additive measure ν on $\mathcal{A}$. Assume that there exists

$$m_0 = (RD) \lim_j m_j \quad \text{in } \mathcal{G}.$$

Then we have:

i) The means m_j, $j \in \mathbb{N}$, are $\mathcal{A}$-uniformly (s)-bounded, uniformly regular and uniformly absolutely continuous with respect to ν.

ii) There exists in R the limit $m_0 = (RD) \lim_j m_j$ in $\mathcal{A}$.

iii) m_0 is (s)-bounded, regular and absolutely continuous with respect to ν.

5. CONVERGENCE OF LEBESGUE DECOMPOSITIONS

In this section, under a further condition on the Riesz space R, we shall see that, assuming (RD)-convergence of a sequence (m_n) of measures, it is possible to deduce (RD)-convergence of their absolutely continuous and singular parts, with respect to a given scalar non-negative measure ν.

We first introduce a definition.

Definition 5.1. We say that a complete Riesz space is *super-Dedekind complete* if, for every subset $A \subset R$, bounded from above, there exists a countable subset $A_0 \subset A$, such that $\sup A = \sup A_0$.

It is well-known that, under this assumption, a Lebesgue decomposition holds, for R-valued measures. (See [10, 15]). So, from now on, R will be assumed to be super-Dedekind complete. However, we shall need a particular formulation, so we prefer to state it explicitly.

Theorem 5.2. Let $m : \mathcal{A} \to R$ be any non-negative σ-additive measure, defined on the σ-algebra $\mathcal{A}$, and let $\nu : \mathcal{A} \to \mathbb{R}_0^+$ be any σ-additive measure. Then there exists a set $V \in \mathcal{A}$ such that the measure $m|_V \perp \nu$ and $m|_{V^c} \ll \nu$.

Proof. Define $\mathcal{H} := \{H \in \mathcal{A} : m(H) \neq 0, \nu(H) = 0\}$, and put $h := \sup_{H \in \mathcal{H}} m(H)$. As R is super-Dedekind complete, there exists a sequence $(H_n)_n$ in $\mathcal{H}$ such that $h = \sup_{n \in \mathbb{N}} m(H_n)$. Without loss of generality, we can assume $(H_n)_n$ to be increasing. Thus, the required set is $V := \cup_{n \in \mathbb{N}} H_n$. Indeed, it is clear that $m|_V$ is singular with respect to ν, and we can easily see that $m|_{V^c} \ll \nu$, because for any set K disjoint from V, $\nu(K) = 0$ and $m(K) > 0$ would contradict maximality of V in $\mathcal{H}$.□

The measures $m|_{V^c}$ and $m|_V$ are called respectively the *absolutely continuous part* and the *singular part* of m with respect to ν.

Now, assume that $(m_j)_j$ is any sequence of σ-additive measures on the σ-algebra $\mathcal{A}$, and taking values in the positive cone of R. For every σ-additive measure $\nu : \mathcal{A} \to \mathbb{R}_0^+$ it is possible to find a *unique* set $V \in \mathcal{A}$ such that

$$m_j|_V \perp \nu, \qquad m_j|_{V^c} \ll \nu :$$

indeed, denoting by V_j the set corresponding to the measure m_j according with Theorem 5.2, it is enough to set $V := \cup_{j\in\mathbb{N}} V_j$.

Remark 5.3. It is obvious that, in the situation here described, the measures $m_j|_V$ are *uniformly* singular with respect to ν. However, in general, the measures $m_j|_{V^c}$ are *not* uniformly absolutely continuous, even when $R = \mathbb{R}$: it is enough to choose ν as the usual Lebesgue measure on the unit interval, and $m_j := j\nu|_{[0,\frac{1}{j}]}$. As $m_j \ll \nu$ for all j, clearly $V = \emptyset$; however, the measures m_j are not uniformly absolutely continuous.

Of course, uniform absolute continuity of the absolutely continuous parts, $m_j|_{V^c}$, is ensured as soon as the sequence $(m_j)_j$ is uniformly s-bounded. The following theorem deals with such situation.

Theorem 5.4. Assume that $(m_j)_j$ is a sequence of σ- additive measures, defined on the same σ-algebra $\mathcal{A}$ and taking values in the positive cone of R. Let $\nu : \mathcal{A} \to \mathbb{R}_0^+$ be any σ-additive measure.

If the sequence $(m_j)_j$ is (RD)-convergent to some measure m, then the absolutely continuous and singular parts of m_j respectively converge to the absolutely continuous and singular parts of m.

P r o o f. Convergence, and non-negativity, imply uniform boundedness of the measures m_j. Thanks to Theorem 3.1, the measures m_j are uniformly s-bounded, hence also the measures $m_j|_V$ and $m_j|_{V^c}$ are. This also implies that the absolutely continuous parts, $m_j|_{V^c}$, are uniformly absolutely continuous. As to convergence, it's clear from the hypotheses that $(RD)\lim_j m_j|_{V^c} = m|_{V^c}$ and $(RD)\lim_j m_j|_V = m|_V$.

From the *(0-0)* definition of absolute continuity,it is clear that $m|_{V^c} \ll \nu$, while $\nu(V) = 0$ immediately implies that $m|_V \perp \nu$. □

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