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INVARIANT FACTORS ASSIGNMENT FOR A CLASS OF TIME-DELAY SYSTEMS

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It is well-known that every system with commensurable delays can be assigned a finite spectrum by feedback, provided that it is spectrally controllable. In general, the feedback involves distributed delays, and it is defined in terms of a Volterra equation. In the case of multivariable time-delay systems, one would be interested in assigning not only the location of the poles of the closed-loop system, but also their multiplicities, or, equivalently, the invariant factors of the closed-loop system. We answer this question. Our basic tool is the ring of operators that includes derivatives, localized and distributed delays. This ring is a Bezout ring. It is also an elementary divisor ring, and finally one can show that every matrix over this ring can be brought in column reduced form using right unimodular transformations. The formulation of the result we finally obtain in the case of time-delay systems differs from the well-known fundamental theorem of state feedback for finite dimensional systems, mainly because the reduced column degrees of a matrix of operators are not uniquely defined in general.

1. INTRODUCTION

Let us consider a multivariable linear system with commensurate delays, of the form

$$\dot{x}(t) = \sum_{k=0}^{p} A_k x(t-k\theta) + \sum_{k=0}^{p} B_k u(t-k\theta)\,, \quad t \geq 0\,, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the instantaneous state, $u(t) \in \mathbb{R}^m$ is the control, $0 < \theta \in \mathbb{R}$, and the $n \times n$ matrices A_k and $n \times m$ matrices B_k have real coefficients. The initial condition $x(t) = \phi(t)$, $-p\theta \leq t \leq 0$ is assumed to be known.

System 1 is called *spectrally controllable* [15] whenever

$$\operatorname{Rank}\left(sI_n - A(\mathrm{e}^{-\theta s}), -B(\mathrm{e}^{-\theta s})\right) = n\,, \quad \forall\, s \in \mathbb{C}\,, \tag{2}$$

where the matrices $A(\mathrm{e}^{-\theta s}) \in \mathbb{R}[\mathrm{e}^{-s\theta}]^{n\times n}$ and $B(\mathrm{e}^{-\theta s}) \in \mathbb{R}[\mathrm{e}^{-s\theta}]^{n\times m}$ are defined by

$$A(\mathrm{e}^{-\theta s}) = \sum_{k=0}^{p} A_k \mathrm{e}^{-ks\theta}\,,$$

and

$$B(e^{-\theta s}) = \sum_{k=0}^{p} B_k e^{-ks\theta} .$$

As it is well-known, the system (1) is stabilizable by state feedback if condition (2) holds [13]. There is some freedom in choosing the class of systems among which one researches a stabilizing feedback. In particular, it was shown that the stabilizing feedback can always be chosen in the class of finite dimensional linear systems [8], or in the class of linear systems with localized delays commensurable to θ [3, 5]. The design of a stabilizing feedback by these methods is rather involved, and one cannot fully assign the properties of the closed-loop system. In particular, the closed-loop system in general has an infinite number of poles. The larger class of feedbacks of the form

$$u(t) = \int_{t-p\theta}^{t} h(t-\tau)\, u(\tau)\, \mathrm{d}\tau + \int_{t-p\theta}^{t} f(t-\tau)x(\tau)\, \mathrm{d}\tau + \sum_{k=0}^{p} f_k x(t-k\theta) \,, \qquad (3)$$

that include distributed delays, and which was introduced by Olbrot [13, 15], permits to overcome this difficulty. It indeed leads to a simple design of a stabilizing compensator, and, if the system is spectrally controllable, the designer can assign the closed-loop system to have only a finite number of poles, and arbitrarily choose their location.

This paradigm, known as *finite spectrum assignment*, was first introduced for monovariable systems. The existence of such a simple design method is ultimately a consequence of the fact that the operator ring $\mathcal{E}$ that includes distributed delays is a Bézout ring [1, 4]. It is still valid in the case of a multivariable system (see for instance [12, 18]). In the present paper, we go further in this direction, and precise the freedom in assigning also the multiplicities of the poles of the closed-loop system. The result is expressed in terms of the degrees of the invariant factors of the closed-loop system. At the contrary of finite-dimensional systems, the possible invariant factor degrees are not constrained by some controllability indices of the pair $(A(e^{-\theta s}), B(e^{-\theta s}))$ or by some minimal column degrees of a denominator $D(s, e^{-\theta s})$ of the transfer, defined by $(sI - A(e^{-\theta s}))^{-1} B(e^{-\theta s}) = N(s, e^{-\theta s})\, D^{-1}(s, e^{-\theta s})$. However the concept of column reducedness and column degrees of the denominator is instrumental to obtain the result, and for the design methodology.

The paper is organized as follows. We first show that these two elementary concepts, invariant factors and column reduced pseudopolynomial matrices, are well-defined in the present context. Hence we show the independence between invariant factors and column degrees. There indeed exist column reduced pseudopolynomial matrices with arbitrary invariant factors and arbitrary column degrees, provided that the sum of the column degrees equals the degree of the determinant of the matrix, that is the sum of the invariant factors degrees. This leads to an algorithm for the design of a feedback assigning arbitrary invariant factors to the closed-loop system, provided that the degree condition holds. In particular, if the system is spectrally controllable, one can assign n finite poles to the closed-loop system, as in [18]. We finally point out some consequences regarding minimal realizations of the considered class of time-delay systems.

2. PSEUDOPOLYNOMIAL MATRICES

A distributed delay is an input–output relationship of the form

$$y(t) = \int_{t-\theta}^{t} h(t-\tau)\, u(\tau)\, \mathrm{d}\tau \ ,$$

where $\theta > 0$ is a real number, $y(t)$ is the output, $u(t)$ is the input, and $h(t)$ is a measurable fonction defined on $[0, \theta]$, called the kernel of the distributed delay. The Laplace transform of this distributed delay is

$$\hat{y}(s) = \mathcal{L}(h)(s)\hat{u}(s) \ ,$$

where s is the Laplace variable, $\hat{y}(s)$ and $\hat{u}(s)$ respectively denote the Laplace transforms of $y(t)$ and of $u(t)$. The Laplace transform of the kernel is defined by

$$\mathcal{L}(h)(s) = \hat{h}(s) = \int_{0}^{\theta} h(\tau)\mathrm{e}^{-s\tau}\mathrm{d}\tau \ ,$$

which is a finite integral since the domain of definition of $h(t)$ is finite. As a consequence, $\hat{h}(s)$ is analytic in the whole complex plane [11].

Let $\mathcal{G}$ be the ring of those Laplace transforms of distributed delays that are rational in the variables s and $\mathrm{e}^{-\theta s}$, and $\mathcal{E}$ be defined as the ring of polynomials in the variable s with coefficients in $\mathcal{G} + \mathbb{R}[\mathrm{e}^{-\theta f}]$.

Example 1. For instance, the distributed delay

$$y(t) = \int_{t-1}^{t} u(\tau)\, \mathrm{d}\tau \ ,$$

can be rewritten in terms of the Laplace variable s

$$\hat{y}(s) = \frac{1 - \mathrm{e}^{-s}}{s}\hat{u}(s) \ .$$

Hence $(1 - \mathrm{e}^{-s})/s$ is a typical element of $\mathcal{G}$. It is analytic in the whole complex plane, and equals 1 at $s = 0$. Further, one can see that $s + (1 - \mathrm{e}^{-s})/s$ is in $\mathcal{E}$, and that the transfer of the input–state system defined by the following Volterra integral equation

$$\dot{x}(t) = -\int_{t-1}^{t} x(\tau)\, \mathrm{d}\tau + u(t) \ ,$$

is a fraction of two elements of $\mathcal{E}$, namely

$$\frac{\hat{x}(s)}{\hat{u}(s)} = \frac{1}{s + \frac{1-\mathrm{e}^{-s}}{s}} \ .$$

The elements of $\mathcal{E}$ are called pseudopolynomials. They are fractions of the form $\alpha(s, \mathrm{e}^{-\theta s}) = \frac{n(s,\mathrm{e}^{-\theta s})}{d(s)}$, where all the zeros of $d(s) \in \mathbb{R}[s]$ are zeros of $n(s, \mathrm{e}^{-\theta s}) \in$

$\mathbb{R}[s, e^{-\theta s}]$. If $\sigma \in \mathbb{C}$ is a zero of $n(s, e^{-\theta s})$ that is not a zero of $d(s)$, it is clear that $\beta(s, e^{-\theta s}) = \frac{n(s,e^{-\theta s})}{(s-\sigma)d(s)}$ lies in $\mathcal{E}$, hence $\alpha(s, e^{-\theta s})$ can be factored as $\alpha(s, e^{-\theta s}) = (s - \sigma)\beta(s, e^{-\theta s})$. Since the quasipolynomial $n(s, e^{-\theta s})$ has in general an infinite number of zeros, it follows that $\mathcal{E}$ is not a unique factorization domain. One can further show that two elements of $\mathcal{E}$ are coprime if and only if they have neither common zero nor common factor of the form $e^{-ks\theta}$, $k \in \mathbb{N}$, and that two elements of $\mathcal{E}$ have a greatest common divisor, that is unique up to a nonzero constant.

Theorem 1. [1, 4] The ring $\mathcal{E}$ is a Bézout domain, i. e. every two elements $\alpha(s, e^{-\theta s})$, $\beta(s, e^{-\theta s})$ of $\mathcal{E}$ are coprime if and only if they satisfy a Bézout identity, $\exists \gamma(s, e^{-\theta s})$, $\delta(s, e^{-\theta s}) \in \mathcal{E}$ such that

$$\alpha(s, e^{-\theta s})\gamma(s, e^{-\theta s}) + \beta(s, e^{-\theta s})\,\delta(s, e^{-\theta s}) = 1 \,.$$

If $\alpha(s, e^{-\theta s})$ and $\beta(s, e^{-\theta s})$ are not coprime, then there exists $\gamma(s, e^{-\theta s}), \delta(s, e^{-\theta s}) \in \mathcal{E}$ such that

$$\alpha(s, e^{-\theta s})\gamma(s, e^{-\theta s}) + \beta(s, e^{-\theta s})\,\delta(s, e^{-\theta s}) = g(s, e^{-\theta s}) \,,$$

where $g(s, e^{-\theta s})$ is the greatest common divisor of $\alpha(s, e^{-\theta s})$ and $\beta(s, e^{-\theta s})$.

As a consequence, the usual definitions and characterizations of right and left coprimeness of polynomial matrices can be extended to matrices over $\mathcal{E}$.

Theorem 2. [1, 4] Let be given $A(s, e^{-\theta s}) \in \mathcal{E}^{m\times r}$, $B(s, e^{-\theta s}) \in \mathcal{E}^{n\times r}$. Then the following statements are equivalent.

(i) Every square full rank matrix $C(s, e^{-\theta s}) \in \mathcal{E}^{r\times r}$ such that $A(s, e^{-\theta s}) = A_1(s, e^{-\theta s})$ $C(s, e^{-\theta s})$ and $B(s, e^{-\theta s}) = B_1(s, e^{-\theta s})C(s, e^{-\theta s})$, for some matrices $A_1(s, e^{-\theta s}) \in \mathcal{E}^{m\times r}$ and $B_1(s, e^{-\theta s}) \in \mathcal{E}^{n\times r}$, is invertible over $\mathcal{E}$, i. e. $\det C(s, e^{-\theta s}) \in \mathbb{R} - \{0\}$.

(ii) There exists matrices $X(s, e^{-\theta s}) \in \mathcal{E}^{r\times m}$ and $Y(s, e^{-\theta s}) \in \mathcal{E}^{r\times n}$ such that $X(s, e^{-\theta s})A(s, e^{-\theta s}) + Y(s, e^{-\theta s})B(s, e^{-\theta s}) = 1$.

(iii) $r < n + m$ and there exist matrices $N(s, e^{-\theta s}) \in \mathcal{E}^{m\times(n+m-r)}$ and $D(s, e^{-\theta s}) \in \mathcal{E}^{n\times(n+m-r)}$ such that the overall matrix

$$\begin{pmatrix} A(s, e^{-\theta s}) & N(s, e^{-\theta s}) \\ B(s, e^{-\theta s}) & D(s, e^{-\theta s}) \end{pmatrix}$$

is unimodular.

One says that $A(s, e^{-\theta s})$ and $B(s, e^{-\theta s})$ are right coprime, and that their transposes $A^T(s, e^{-\theta s})$ and $B^T(s, e^{-\theta s})$ are left coprime, if one of these conditions holds.

The procedures to construct the matrices $X(s, e^{-\theta s})$, $Y(s, e^{-\theta s})$, $N(s, e^{-\theta s})$, and $D(s, e^{-\theta s})$ as in Theorem 2 are constructive [1]. Further we have the following.

Theorem 3. $\mathcal{E}$ is an elementary divisor ring, i. e., every matrix $A(s,\mathrm{e}^{-\theta s}) \in \mathcal{E}^{n\times m}$ can be factored as

$$A(s,\mathrm{e}^{-\theta s}) = U(s,\mathrm{e}^{-\theta s}) \begin{pmatrix} \Lambda(s,\mathrm{e}^{-\theta s}) & 0 \\ 0 & 0 \end{pmatrix} V(s,\mathrm{e}^{-\theta s}) ,$$

where $U(s,\mathrm{e}^{-\theta s}) \in \mathcal{E}^{n\times n}$, $V(s,\mathrm{e}^{-\theta s}) \in \mathcal{E}^{m\times m}$ are unimodular, $r = \operatorname{rank} A(s,\mathrm{e}^{-\theta s})$, $\Lambda(s,\mathrm{e}^{-\theta s}) = \operatorname{diag}\{\alpha_1(s,\mathrm{e}^{-\theta s}),\ldots,\alpha_r(s,\mathrm{e}^{-\theta s})\}$, and $\alpha_i(s,\mathrm{e}^{-\theta s})|\alpha_{i+1}(s,\mathrm{e}^{-\theta s})$, for $i = 1,\ldots,r-1$. The pseudopolynomials $\alpha_i(s,\mathrm{e}^{-\theta s})$, $i = 1,\ldots,r$, are called the invariant factors of $A(s,\mathrm{e}^{-\theta s})$.

Proof. Following [9] (see also [4] where a similar result is established), it is sufficient to show that if $\alpha(s,\mathrm{e}^{-\theta s})$, $\beta(s,\mathrm{e}^{-\theta s})$, and $\gamma(s,\mathrm{e}^{-\theta s})$ are three coprime elements of $\mathcal{E}$, then there exist $p(s,\mathrm{e}^{-\theta s})$, $q(s,\mathrm{e}^{-\theta s}) \in \mathcal{E}$ such that $p(s,\mathrm{e}^{-\theta s})\,\alpha(s,\mathrm{e}^{-\theta s})$ and $p(s,\mathrm{e}^{-\theta s})\beta(s,\mathrm{e}^{-\theta s}) + q(s,\mathrm{e}^{-\theta s})\gamma(s,\mathrm{e}^{-\theta s})$ are coprime. One can indeed show that, since $\alpha(s,\mathrm{e}^{-\theta s})$, $\beta(s,\mathrm{e}^{-\theta s})$, and $\gamma(s,\mathrm{e}^{-\theta s})$ are coprime, there exists a constant $k \in \mathbb{R}$ such that $g(s) = \gcd(\beta(s,\mathrm{e}^{-\theta s}) + k\alpha(s,\mathrm{e}^{-\theta s}),\gamma(s,\mathrm{e}^{-\theta s}))$ has only a finite number of common zeros. There exists $p(s,\mathrm{e}^{-\theta s})$, $q(s,\mathrm{e}^{-\theta s}) \in \mathcal{E}$ so that $[\beta(s,\mathrm{e}^{-\theta s}) + k\alpha(s,\mathrm{e}^{-\theta s})]p(s,\mathrm{e}^{-\theta s}) + \gamma(s,\mathrm{e}^{-\theta s})\,q(s,\mathrm{e}^{-\theta s}) = g(s)$, and another constant $k' \in \mathbb{R}$ so that $p(s,\mathrm{e}^{-\theta s}) + k'\frac{\gamma(s,\mathrm{e}^{-\theta s})}{g(s)}$ and $g(s)$ are coprime. The result follows. □

The degree of an element $\alpha(s,\mathrm{e}^{-\theta s}) = \frac{n(s,\mathrm{e}^{-\theta s})}{d(s)} \in \mathcal{E}$, where $n(s,\mathrm{e}^{-\theta s}) \in \mathbb{R}[s,\mathrm{e}^{-\theta s}]$ and $d(s) \in \mathbb{R}[s]$, is the difference $\delta = \deg_s n(s,\mathrm{e}^{-\theta s}) - \deg d(s) \in \mathbb{Z}$, and we write $\deg \alpha(s,\mathrm{e}^{-\theta s}) = \delta$. This degree function is endowed with the usual properties of a degree function, namely $\forall\,\alpha(s,\mathrm{e}^{-\theta s})$, $\beta(s,\mathrm{e}^{-\theta s}) \in \mathcal{E}$, $\deg(\alpha(s,\mathrm{e}^{-\theta s}) + \beta(s,\mathrm{e}^{-\theta s}) \leq \max(\deg\alpha(s,\mathrm{e}^{-\theta s}),\deg\beta(s,\mathrm{e}^{-\theta s}))$, and $\deg(\alpha(s,\mathrm{e}^{-\theta s})\beta(s,\mathrm{e}^{-\theta s})) = \deg\alpha(s,\mathrm{e}^{-\theta s}) + \deg\beta(s,\mathrm{e}^{-\theta s})$. $\mathcal{E}$ is not a Euclidian ring, because this degree lies in $\mathbb{Z}$ and not in $\mathbb{N}$. The degree of any element of $\mathcal{E}$ can be negative. For instance

$$\deg\frac{1-\mathrm{e}^{-s}}{s} = -1 .$$

This in particular implies that the degree of a divisor can be strictly less than the degree of some of its multiples. Consider for instance

$$1-\mathrm{e}^{-s} = s\cdot\frac{1-\mathrm{e}^{-s}}{s} .$$

One can see that if $\deg\alpha(s,\mathrm{e}^{-\theta s}) = \delta$, then

$$\alpha(s,\mathrm{e}^{-\theta s}) = \sum_{k=-\infty}^{\delta} \alpha_k(\mathrm{e}^{-\theta s})\,s^k ,$$

where the coefficients $\alpha_k(\mathrm{e}^{-\theta s}) \in \mathbb{R}[\mathrm{e}^{-\theta s}]$ are uniquely defined. $\alpha(s,\mathrm{e}^{-\theta s})$ is said to be monic if $\alpha_\delta(\mathrm{e}^{-\theta s})$ is a nonzero real constant. Notice the following.

Lemma 1. [Division Algorithm] Let $\alpha(s, \mathrm{e}^{-\theta s})$, $\beta(s, \mathrm{e}^{-\theta s}) \in \mathcal{E}$, $k \in \mathbb{Z}$, and assume that $\alpha(s, \mathrm{e}^{-\theta s})$ is monic. Then there exist pseudopolynomials $q(s, \mathrm{e}^{-\theta s})$, $r(s, \mathrm{e}^{-\theta s}) \in \mathcal{E}$ such that $\beta(s, \mathrm{e}^{-\theta s}) = q(s, \mathrm{e}^{-\theta s})\, \alpha(s, \mathrm{e}^{-\theta s}) + r(s, \mathrm{e}^{-\theta s})$ and $\deg r(s, \mathrm{e}^{-\theta s}) < k$.

This is clearly deduced from the existence of monic pseudopolynomials having negative degrees, e. g. $\frac{\theta s - 1 + \mathrm{e}^{-\theta s}}{s^2}$. We call division the operation described in Lemma 1. The following will be useful on the sequel.

Theorem 4. [1] Let $\alpha(s, \mathrm{e}^{-\theta s}) \in \mathcal{E}$. Then
(i) $\alpha(s, \mathrm{e}^{-\theta s}) \in \mathcal{G} \Longleftrightarrow \deg \alpha(s, \mathrm{e}^{-\theta s}) \leq -1$,
(ii) $\alpha(s, \mathrm{e}^{-\theta s}) \in \mathcal{G} + \mathbb{R}[\mathrm{e}^{-\theta s}] \Longleftrightarrow \deg \alpha(s, \mathrm{e}^{-\theta s}) \leq 0$.

A matrix $M(s, \mathrm{e}^{-\theta s}) \in \mathcal{E}^{n \times m}$ being given, and defining c_j as the degree of the jth column of $M(s, \mathrm{e}^{-\theta s})$, it appears that

$$M(s, \mathrm{e}^{-\theta s})\, \mathrm{diag}\, \{s^{c_1}, s^{c_2}, \ldots, s^{c_m}\} = \sum_{k=-\infty}^{0} M_k(\mathrm{e}^{-\theta s})\, s^k .$$

$M(s, \mathrm{e}^{-\theta s}) \in \mathcal{E}^{n \times m}$ is said to be *column reduced*, with column degrees $c_1, c_2, \ldots, c_m$ if the matrix $M_0(\mathrm{e}^{-\theta s})$ is of full column rank.

Theorem 5. [1] Every matrix over $\mathcal{E}$ can be brought in column reduced form through a right unimodular transformation.

The column degrees of a column reduced pseudopolynomial matrix can be negative. Hence the column degrees of a column reduced form of a pseudopolynomial matrix are not uniquely defined. Consider for instance the following unimodular matrix

$$U(s, \mathrm{e}^{-\theta s}) = \begin{pmatrix} s + \ln 2 & \frac{1-\mathrm{e}^{-s}}{s} \\ s & \frac{2-\mathrm{e}^{-s}}{s+\ln 2} \end{pmatrix} .$$

This matrix is column reduced with column degrees $c_1 = 1$, $c_2 = -1$, unimodular since its determinant equals 1, with inverse

$$U^{-1}(s) = \begin{pmatrix} \frac{2-\mathrm{e}^{-s}}{s+\ln 2} & -\frac{1-\mathrm{e}^{-s}}{s} \\ -s & s + \ln 2 \end{pmatrix}$$

hence $U(s)\, U^{-1}(s)$ is also column reduced, with column degrees $c_1 = 0$ and $c_2 = 0$.

Lemma 2. Let $k \in \mathbb{Z}$. Then there exist a unimodular matrix over $\mathcal{E}$, of the form

$$U(s, \mathrm{e}^{-\theta s}) = \begin{pmatrix} \alpha(s, \mathrm{e}^{-\theta s}) & \beta(s, \mathrm{e}^{-\theta s}) \\ \gamma(s, \mathrm{e}^{-\theta s}) & \delta(s, \mathrm{e}^{-\theta s}) \end{pmatrix} ,$$

where $\deg \alpha(s, \mathrm{e}^{-\theta s}) = k$, $\deg \delta(s, \mathrm{e}^{-\theta s}) = -k$, and the degree of $\beta(s, \mathrm{e}^{-\theta s})$ and $\gamma(s, \mathrm{e}^{-\theta s})$ is arbitrarily low.

Lemma 2 clearly follows from the example above and Lemma 1. Taking Theorem 5 into account, it leads to the following characterization of the possible column degrees of a matrix in column reduced form.

Theorem 6. Let $D(s, \mathrm{e}^{-\theta s}) \in \mathcal{E}^{m \times m}$, $c_1, c_2, \ldots, c_m \in \mathbb{Z}$, and assume further that $\det D(s, \mathrm{e}^{-\theta s}) \neq 0$. Then there exists a matrix $U(s, \mathrm{e}^{-\theta s})$, unimodular over $\mathcal{E}$, such that $D(s, \mathrm{e}^{-\theta s})\, U(s, \mathrm{e}^{-\theta s})$ is column reduced with column degrees $c_1, c_2, \ldots, c_m$ if and only if

$$\sum_{i=1}^{m} c_i = \deg \det D(s, \mathrm{e}^{-\theta s}) .$$

3. APPLICATION TO FINITE POLE PLACEMENT

The matrices $sI_n - A(\mathrm{e}^{-\theta s})$ and $B(\mathrm{e}^{-\theta s})$ that appear in equation (2) are clearly polynomial in s, $\mathrm{e}^{-\theta s}$. They have a fortiori their entries in $\mathcal{E}$. Applying Laplace transform, equation (3) becomes

$$\hat{u}(s) = H(s, \mathrm{e}^{-\theta s})\hat{u}(s) + F(s, \mathrm{e}^{-\theta s})\hat{x}(s) ,$$

where

$$H(s, \mathrm{e}^{-\theta s}) = \int_0^{p\theta} h(\tau)\mathrm{e}^{-\theta s}\, \mathrm{d}\tau ,$$

$$F(s, \mathrm{e}^{-\theta s}) = \int_0^{p\theta} f(\tau)\mathrm{e}^{-s\tau}\, \mathrm{d}\tau + \sum_{k=0}^{p} f_k \mathrm{e}^{-ks\theta} .$$

If the entries of $H(s, \mathrm{e}^{-\theta s})$ and $F(s, \mathrm{e}^{-\theta s})$ are rational in the variable s, $\mathrm{e}^{-\theta s}$, then they also lie, respectively, in $\mathcal{G}$ and in $\mathcal{G} + \mathbb{R}[\mathrm{e}^{-\theta f}]$. Thus the closed-loop system reads

$$M(s, \mathrm{e}^{-\theta s}) \begin{pmatrix} \hat{x}(s) \\ \hat{u}(s) \end{pmatrix} = \begin{pmatrix} \hat{\phi}(s) \\ 0 \end{pmatrix} ,$$

where $\hat{\phi}(s)$ depends from the initial condition of the system, and

$$M(s, \mathrm{e}^{-\theta s}) = \begin{pmatrix} sI_n - A(\mathrm{e}^{-\theta s}) & -B(\mathrm{e}^{-\theta s}) \\ -F(s, \mathrm{e}^{-\theta s}) & I_m - H(s, \mathrm{e}^{-\theta s}) \end{pmatrix}$$

is a matrix with entries in $\mathcal{E}$. Notice that the entries of $H(s, \mathrm{e}^{-\theta s})$ lie in $\mathcal{G}$, and that of $F(s, \mathrm{e}^{-\theta s})$ lie in $\mathcal{G} + \mathbb{R}[\mathrm{e}^{-\theta f}]$ if, and only if, the entries of the kernels $f(\tau)$ and $g(\tau)$ are linear combinations of exponentials, of the form $\mathrm{e}^{\alpha\tau}, \alpha \in \mathbb{R}$, $\mathrm{e}^{\mathrm{Re}(\beta)\tau} \cos(\mathrm{Im}(\beta)\tau)$, or $\mathrm{e}^{\mathrm{Re}(\beta)\tau} \sin(\mathrm{Im}(\beta)\tau)$, defined on the compact support $[k_1\theta, k_2\theta]$, $k_1 < k_2 \in \mathbb{N}$, and of their derivatives [2, 7].

The design of a stabilizing feedback hence comes down to choosing $F(s, \mathrm{e}^{-\theta s})$ and $H(s, \mathrm{e}^{-\theta s})$ over $\mathcal{E}$ so that the zeros of $M(s, \mathrm{e}^{-\theta s})$ lie in the left half complex plane. The finite spectrum assignment is obtained when $M(s, \mathrm{e}^{-\theta s})$ has only a finite number of zeros, say $q \in \mathbb{N}$. In that case, the determinant of M is a polynomial in s of degree q. Such an assignment is possible for every given self-conjugated set of q zeros if and only if the pair (A, B) is spectrally controllable [13, 15, 18]. The design procedure [12] is almost similar to the classical design method for a linear time-invariant system [6, 10], thanks to the properties of the operator ring $\mathcal{E}$. The first step of this procedure consists of finding matrices $X(s,\mathrm{e}^{-\theta s})$, $Y(s,\mathrm{e}^{-\theta s})$, $N(s,\mathrm{e}^{-\theta s})$, and $D(s,\mathrm{e}^{-\theta s})$ over $\mathcal{E}$, such that

$$U(s,\mathrm{e}^{-\theta s}) = \begin{pmatrix} X(s,\mathrm{e}^{-\theta s}) & N(s,\mathrm{e}^{-\theta s}) \\ Y(s,\mathrm{e}^{-\theta s}) & D(s,\mathrm{e}^{-\theta s}) \end{pmatrix}$$

is unimodular, i. e. it possesses an inverse over $\mathcal{E}$, and

$$\begin{pmatrix} sI_n - A(\mathrm{e}^{-\theta s}) & -B(\mathrm{e}^{-\theta s}) \end{pmatrix} U(s,\mathrm{e}^{-\theta s}) = \begin{pmatrix} I_n & 0 \end{pmatrix}.$$

Hence it is easy to see that the zeros of $M(s,\mathrm{e}^{-\theta s})$ are those of

$$D_{HF}(s,\mathrm{e}^{-\theta s}) = (I_n - H(s,\mathrm{e}^{-\theta s}))\, D(s,\mathrm{e}^{-\theta s}) - F(s,\mathrm{e}^{-\theta s})N(s,\mathrm{e}^{-\theta s}) \,, \tag{4}$$

for every choice of $H(s,\mathrm{e}^{-\theta s})$ and $F(s,\mathrm{e}^{-\theta s})$. Further we can notice that $N(s,\mathrm{e}^{-\theta s})$ and $D(s,\mathrm{e}^{-\theta s})$ are right coprime over $\mathcal{E}$, and assume that $D(s,\mathrm{e}^{-\theta s})$ is column reduced. The second step consists of choosing a matrix $D_{HF}(s,\mathrm{e}^{-\theta s})$ that is column reduced, with the same column degrees as $D(s,\mathrm{e}^{-\theta s})$. Hence there exists a solution $H(s,\mathrm{e}^{-\theta s})$, $F(s,\mathrm{e}^{-\theta s})$ to equation (4). By division, we can assume that $\deg F(s,\mathrm{e}^{-\theta s}) \leq 0$, hence it appears that indeed $H(s,\mathrm{e}^{-\theta s})$ lies in $\mathcal{G}^{m\times m}$, and that $F(s,\mathrm{e}^{-\theta s})$ is a matrix over $\mathcal{G} + \mathbb{R}[\mathrm{e}^{-\theta f}]$. Using Theorem 5, it appears that the freedom in choosing the invariant factors of the closed–loop system, that are those of $D_{HF}(s,\mathrm{e}^{-\theta s})$ is only limited by n.

Theorem 7. Let the system (1) be given and assume that it is spectrally controllable, and $\psi_1(s,\mathrm{e}^{-\theta s}), \psi_2(s,\mathrm{e}^{-\theta s}), \ldots, \psi_m(s,\mathrm{e}^{-\theta s}) \in \mathcal{E}$. Then there exists a feedback (3) such that $\psi_1(s,\mathrm{e}^{-\theta s}), \psi_2(s,\mathrm{e}^{-\theta s}), \ldots, \psi_m(s,\mathrm{e}^{-\theta s}) \in \mathcal{E}$ are the invariant factors of the closed-loop system if and only if the pseudopolynomials $\psi_i(s,\mathrm{e}^{-\theta s})$ are monic, $\psi_i(s,\mathrm{e}^{-\theta s})$ divides without remainder $\psi_{i+1}(s,\mathrm{e}^{-\theta s})$, $i = 1,2,\ldots,m-1$, and

$$\sum_{i=1}^{m} \deg \psi_i(s,\mathrm{e}^{-\theta s}) = n \,.$$

Corollary 1. [Finite pole placement] Under the same condition (2), one can choose a feedback (3) such that $\psi_1(s), \psi_2(s), \ldots, \psi_m(s) \in \mathbb{R}[s]$ are the invariant factors of the closed–loop system if and only if the polynomials $\psi_i(s)$ are monic, $\psi_i(s)$ divides without remainder $\psi_{i+1}(s)$, $i = 1,2,\ldots,m-1$, and

$$\sum_{i=1}^{m} \deg \psi_i(s) = n \,,$$

where here the degree is understood as that of a polynomial in s.

Example 2. For instance, consider the system

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u(t-1) \,.$$

Hence the matrices $N(s, \mathrm{e}^{-\theta s})$ and $D(s, \mathrm{e}^{-\theta s})$ as above can be taken as

$$N(s, \mathrm{e}^{-s}) = \begin{pmatrix} \mathrm{e}^{-s} & 0 \\ 0 & \mathrm{e}^{-s} \end{pmatrix},$$

and

$$D(s, \mathrm{e}^{-s}) = \begin{pmatrix} s & -1 \\ -1 & s \end{pmatrix}.$$

One can then verify that (4) is satisfied taking

$$H(s, \mathrm{e}^{-s}) = \begin{pmatrix} -\alpha(s, \mathrm{e}^{-s}) & -\beta(s, \mathrm{e}^{-s}) \\ -\beta(s, \mathrm{e}^{-s}) & -\alpha(s, \mathrm{e}^{-s}) \end{pmatrix},$$

and

$$F(s, \mathrm{e}^{-s}) = \begin{pmatrix} -\gamma(s, \mathrm{e}^{-s}) & 0 \\ 0 & \gamma(s, \mathrm{e}^{-s}) \end{pmatrix},$$

where

$$\alpha(s, \mathrm{e}^{-s}) = \frac{1 + s\left(1 - 2\frac{\mathrm{e}-\mathrm{e}^{-s}}{\mathrm{e}-\mathrm{e}^{-1}}\mathrm{e}^{-s+1}\right)}{s^2 - 1},$$

$$\beta(s, \mathrm{e}^{-s}) = \frac{s + 1 - 2\frac{\mathrm{e}-\mathrm{e}^{-s}}{\mathrm{e}-\mathrm{e}^{-1}}\mathrm{e}^{-s+1}}{s^2 - 1},$$

and

$$\gamma(s, \mathrm{e}^{-s}) = 2\mathrm{e}\frac{\mathrm{e} - \mathrm{e}^{-s}}{\mathrm{e} - \mathrm{e}^{-1}}.$$

4. CONCLUSION

The freedom in assigning the closed–loop invariant factors of a system with commensurable time–delay system has been described, provided that it is spectrally controllable. An algorithm has been proposed, for the synthesis of the assigning control law, which in general is expressed in terms of a Volterra integral equation that involves both localized and distributed delays. This provides a slight improvement in the method proposed in [18] where the closed–loop system is nilpotent, i. e. has a unique nonunit invariant factor, which may in application ameliorate the transient behavior of the system.

There are many more applications to the control of systems with commensurable delays, of the theory of pseudopolynomial matrices. For instance, using standard techniques (see for instance [6]), one can see that if a transfer matrix is factored as

$$T(s, \mathrm{e}^{-\theta s}) = N(s, \mathrm{e}^{-\theta s})\, D^{-1}(s, \mathrm{e}^{-\theta s})$$

where $N(s, \mathrm{e}^{-\theta s})$ and $D(s, \mathrm{e}^{-\theta s})$ are right coprime, $D(s, \mathrm{e}^{-\theta s})$ is column reduced, with column degrees $c_1 \geq c_2 \geq \ldots \geq c_m \geq 0$, then it has a minimal realization. The following is hence a clear consequence of Theorem 5.

Theorem 8. Let $T(s, e^{-\theta s})$ be the transfer matrix of a time–delay system. Then $T(s, e^{-\theta s})$ admits a minimal realization, i. e. spectrally controllable (in the sense of (2)) and spectrally observable, if and only if every right coprime factorization of $T(s, e^{-\theta s})$ over $\mathcal{E}$, in the form $T(s, e^{-\theta s}) = N(s, e^{-\theta s})\, D^{-1}(s, e^{-\theta s})$, is so that $\deg\det D(s, e^{-\theta s}) \geq 0$.

For instance, the transfer function

$$T(s, e^{-\theta s}) = \frac{1 + e^{-2s}}{s + \frac{\pi}{2} e^{-s}},$$

does not admit any minimal realization since a coprime factorization is provided by $T(s, e^{-\theta s}) = N(s, e^{-\theta s})\, D^{-1}(s, e^{-\theta s})$, where

$$N(s, e^{-\theta s}) = \frac{1 + e^{-2s}}{s^2 + \frac{\pi^2}{4}},$$

and

$$D(s, e^{-\theta s}) = \frac{s + \frac{\pi}{2} e^{-s}}{s^2 + \frac{\pi^2}{4}}.$$

One hence checks that $\deg\det D(s, e^{-\theta s}) = -1$. There are still some difficulties when applying this theory. First, the calculations are quite involved, and the absence of any specialized toolbox is a serious drawback. Second, the robustness of the control law is not ensured. In particular, some experimentations have shown that a numerical implementation of Volterra integral equations may result in an unstable closed–loop system [17]. Further studies are needed for practical implementations.

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