Vojtech Veselý

# Static output feedback controller design

*Kybernetika*, Vol. 37 (2001), No. 2, [205]--221

Persistent URL: http://dml.cz/dmlcz/135401

KYBERNETIKA — VOLUME 37 (2001), NUMBER 2, PAGES 205-221

# STATIC OUTPUT FEEDBACK CONTROLLER DESIGN

VOJTECH VESELÝ

In this paper new necessary and sufficient conditions for static output feedback stabilizability for continuous and discrete time linear time invariant systems have been proposed. These conditions form the basis for the procedure of static output feedback controller design proposed in this paper. The proposed LMI based algorithms are computationally simple and tightly connected with the Lyapunov stability theory and LQ optimal state feedback design. The structure of the output feedback gain matrix, including a decentralized one, can be prescribed by the designer. In this way the decentralized output feedback controller can be designed.

## 1. INTRODUCTION

One of the most often mentioned open problems in control theory is the output feedback stabilization problem [3]. Simply stated, the problem is as follows: Given a dynamic system, find a static output feedback so that the closed loop system has some desirable characteristic, or determine that such a feedback does not exist. Various approaches have been used to study two aspects of the stabilization problem, namely conditions under which the linear system described in the state-space can be stabilized via an output feedback and the respective procedure to obtain a stabilizing control law. A body of literature deals with the output stabilization problem for the linear time invariant systems. Various approaches and results are surveyed in [8, 11, 16] and in references therein. In the above papers the authors basically conclude that the problem of static output feedback is still open despite the availability of many approaches and numerical algorithms. This statement is justified by the fact no testable necessary and sufficient conditions exist to test the stabilizability of a given system using a static output feedback, and that numerical algorithms cannot be shown to be convergent in general [16].

A necessary and sufficient condition for output feedback stabilizability of a linear continuous time invariant system is given in [11], and of a linear discrete time invariant system in [14]. The results given in the above two papers are not constructive and do not solve the computational aspects of the problem. Nevertheless, the relationship of the above results with the linear quadratic regulator is helpful, for continuous time systems [5] and for discrete time systems [14] it has inspired to propose an algorithm which iterates the algebraic Riccati equation until the constraints are satisfied.

It has recently been shown that an extremely wide array of output feedback controller design problems may be reduced to the problem of finding a feasible point under a Biaffine Matrix Inequality (BMI) constraint. The BMI has been introduced by [15] and [7] as a geometric reformulation of many problems in the output feedback controller design and robust control. However it is known that BMI problems are *NP*-hard [17]. The main result of [17] shows that it is rather unlikely to find an algorithm for solving general BMI problems and it is also shown that simultaneous stabilization of $N$ plants with a static output feedback is an *NP*-hard problem. The BMI feasibility problem is discussed in [7] and the paper presents a branch and bound global optimization algorithm which finds an $\varepsilon$-global minimum in a finite number of iterations. In this paper the BMI problem of the output feedback controller design is reduced to an LMIs problem.

The theory of linear matrix inequalities (LMIs) [2] has been used for output feedback controller design in [1, 9, 13, 18]. Most of the above works present an iterative algorithm in which a set of equations, or set of LMI problems are repeated until certain convergence criteria are met. In [18] a necessary and sufficient condition for simultaneous stabilizability via static output feedback is obtained and an iterative LMI algorithm is proposed to obtain the output feedback gain. In [9] necessary and sufficient conditions for the existence of an $H_\infty$ controller of any order are given in terms of three LMIs. The authors in [10] study the conditions under which the designing output feedback controllers can be divided into two stages and a dynamic output feedback can be obtained. In [1] the authors have proposed an LMI based algorithm which does not require iteration of LMI problems. The goal is to eliminate the need for iteration by an appropriate choice of the initializing state feedback matrix. The V-K iteration algorithm proposed in [4] is based on an alternative solution of two convex LMI optimization problems obtained by fixing the Lyapunov matrix or the gain controller matrix. This algorithm is guaranteed to converge but not necessarily to the global optimum of the problem depending on the starting conditions.

In the present paper new necessary and sufficient conditions for output feedback stabilizability of linear continuous and discrete time systems are the basis for the proposed static output feedback design procedure. For iterative and non iterative LMI based algorithms and for the structurally constrained state feedback method the structure of the static output feedback gain matrix can be prescribed by the designer including the decentralized case. The design procedure of the output feedback controller design for the continuous time version has been completed up to the LMI based algorithms. The discrete time approach to this problem is outlined up to the classical iterative algorithm, the remaining part being still under research.

The paper is organized as follows. In Section 2, problem formulation and some preliminary results are presented. The main results are given in Section 3. In Section 4, the obtained theoretical results are applied to some examples. The notation is standard, and will be defined as the need arises. Much of the notation and terminology follows references [8, 11] and [14].

## 2. PRELIMINARIES AND PROBLEM FORMULATION

We shall consider the following linear time invariant continuous and discrete time systems

CTS

$$\dot{x} = Ax(t) + Bu(t), \quad y(t) = Cx(t), \quad x(0) = x_0 \tag{1}$$

DTS

$$x(t+1) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \quad x(0) = x_0 \tag{2}$$

and the static output feedback

$$u(t) = FCx(t) \tag{3}$$

or structurally constrained state feedback [5]

$$u(t) = (K + L)\, x(t) \tag{4}$$

where $x(t) \in R^n$ is the plant state; $u(t) \in R^m$ is the control input; $y(t) \in R^l$ is the output vector of the system; $A$, $B$, $C$, and $F$, $K$, $L$ are constant matrices of appropriate dimensions. The closed loop system can be described by

CTS

$$\dot{x} = (A + B(K + L))\, x(t) =: (A + BFC)\, x(t) \tag{5}$$

DTS

$$x(t+1) = (A + B(K + L))\, x(t) =: (A + BFC)\, x(t) \tag{6}$$

As it is well known [8], the fixed order dynamic output feedback of order less or equal to $n$ is a special case of the static output feedback problem.

The problem studied in this paper can be formulated as follows: For a continuous and discrete time linear system described by (1) or (2) design a static output feedback controller with the gain matrix $F$ and control algorithm (3) or (4) so that the closed loop system (5) or (6) is stable. The following performance indices are associated with the systems (1) and (2):

CTS

$$J_c = \int_0^{\infty} (x(t)^T Q x(t) + u(t)^T R u(t))\, \mathrm{d}t \tag{7}$$

DTS

$$J_d = \sum_{t=0}^{\infty} [x(t)^T Q x(t) + u(t)^T R u(t)] \tag{8}$$

where $Q = Q^T \geq 0$ and $R = R^T > 0$ are matrices of compatible dimensions.

**Definition.** Consider the system (1) or (2). If there exists a control law $u^*$ and a positive scalar $J^*$ such that the closed loop system is stable and the closed loop value cost function (7) or (8) satisfies $J_c \leq J^*$ ($J_d \leq J^*$), then $J^*$ is said to be the guaranteed cost and $u^*$ is said to be the guaranteed cost control law for system (1) or (2).

Let us recall several commonly used notions for continuous time systems. The matrix $D \in R^{n\times n}$ is called stable if all its eigenvalues lie in the left half complex plane, i.e. if $\mathrm{Re}\{\lambda_i(D)\} < 0$ for $i = 1, 2, \dots, n$. System (1) with a stable matrix $A$ is called stable. System (1) is called output feedback stabilizable if there exists a real output feedback gain matrix $F$ such that $A + BFC$ is a stable matrix. The pair $(A, C)$ is called detectable if there exists a real matrix $X$ such that $A + XC$ is stable. The adequate notions can be recalled for discrete time systems.

The following results are analogous to the corresponding discrete time cases (cf. [12]).

**Lemma 1.** The matrix $A$ is stable iff there exist $P > 0$, $Q > 0$ satisfying the following Lyapunov matrix equation

$$A^T P + PA + Q = 0. \tag{9}$$

## 3. STATIC OUTPUT FEEDBACK CONTROLLER DESIGN

In the next developments (subsection 3.1) we employ Lemma 2.

**Lemma 2.** Let two matrices $C \in R^{l\times n}$ and $K \in R^{m\times n}, l \leq n, m \leq n$ be given. If for the prescribed gain matrix structure $F$ it holds $FC \neq 0$, then there exists no unique solution with respect to two matrices $F \in R^{m\times l}$ and $L \in R^{m\times n}$ such that the following equality holds

$$FC = K + L. \tag{10}$$

Proof. This is a standard fact. □

Owing to Lemma 2 one can recalculate the obtained results from a static structurally constrained state feedback to a static output feedback. Note that the structure of gain matrix $F$ can be determined by the designer.

### 3.1. Structurally constrained state feedback design

Consider the system described by (1) with control algorithm (3). The following lemma is well-known.

**Lemma 3.** Let the linear time invariant system be given. Then the following statements are equivalent.

1. The system (1) is stabilizable via the state feedback

$$u(t) = Kx(t). \tag{11}$$

2. There exist matrices $Q > 0$ and $R > 0$ of compatible dimensions such that the following ARE
$$PA + A^T P - PBR^{-1}B^T P + Q = 0 \tag{12}$$
has a unique solution $P = P^T > 0$.

3. There exist matrices $P > 0$ and $R > 0$ of compatible dimensions satisfying the following ARI
$$PA + A^T P - PBR^{-1}B^T P < 0. \tag{13}$$

In fact, if $(A, B)$ is stabilizable via state feedback (11), then, for any $Q > 0, R > 0$, the ARE (12) and ARI (13) must have a unique solution $P > 0$.

In what follows the idea of the static structurally constrained state optimal control problem [5] is used with the new necessary and sufficient conditions for static output feedback stabilizability of linear systems proposed in this paper. The main results for a continuous time system are given in the following theorem.

**Theorem 1.** Let the linear time invariant continuous system (1) be given. Consider the static structurally constrained feedback (3). Then the following statements are equivalent.

1. The system is static structurally constrained feedback stabilizable.

2. There exists a symmetric and positive definite matrix $P$ and matrices $K$ and $L$ satisfying the following matrix inequality
$$(A + B(K + L))^T P + P(A + B(K + L)) < 0. \tag{14}$$

3. There exist positive definite matrices $P$ and $R$, and matrices $K$ and $L$ satisfying the following matrix inequalities
$$\begin{aligned} &A_L^T P + PA_L - K^T RK - PBR^{-1}B^T P < 0 \\ &(RK + B^T P)\phi^{-1}(RK + B^T P)^T - R < 0 \end{aligned} \tag{15}$$
where $A_L = A + BL$ and $\phi = -(A_L^T P + PA_L - PBR^{-1}B^T P - K^T RK)$.

P r o o f. The proof that the first and second statement are equivalent is given in Lemma 1. To prove the third statement recall that, for any matrices $E_{11}, E_{12}$ and $E_{22}$ where $E_{11}$ and $E_{22}$ are symmetric, the following are equivalent:

$$\begin{bmatrix} E_{11} & E_{12} \\ E_{12}^T & E_{22} \end{bmatrix} > 0$$

$$\begin{aligned} &E_{22} > 0, \quad E_{11} - E_{12}E_{22}^{-1}E_{12}^T > 0 \\ &E_{11} > 0, \quad E_{22} - E_{12}^T E_{11}^{-1} E_{12} > 0. \end{aligned} \tag{16}$$

Using the Schur complement formula (16) the inequality (14) which can be rewritten as

$$-[(A_L + BK)^T P + P(A_L + BK)] > 0 \tag{17}$$

is for $R > 0$ equivalent to the following inequality

$$\begin{bmatrix} R & RK + B^T P \\ (RK + B^T P)^T & \phi \end{bmatrix} > 0.$$

From (16) it is straightforward to show the equivalence of (15) and above inequality, which proves the equivalence of the third and the second statements. □

**Corollary 1.** Find $(P, K)$ solving the following two matrix equations

$$A^T P + PA - PB(I + R^{-1})B^T P + Q_c = 0 \tag{18}$$

where $Q_c = (L + B^T P)^T (L + B^T P) + Q - K^T RK - L^T L + (K + L)^T R(K + L)$

$$K = -R^{-1} B^T P \tag{19}$$

then the following inequality holds

$$J_c = \int_0^\infty x(t)^T [Q + (K + L)^T R(K + L)]\, x(t)\, \mathrm{d}t \le x_0^T P x_0 = J^*$$

necessary and sufficient conditions for system stabilizability(15) hold.

Equations (18) and (19) inspired the following algorithm for the calculation of the gain matrix $F$.

**Algorithm A.**

*Step 1.* Set $i = 1,\ L_0 = 0,\ P_0 = I,\ R_c = (I + R^{-1})^{-1}$ where $I$ is the identity matrix of corresponding dimension.

*Step 2.* Compute $Q_{ci}$.

*Step 3.* Calculate $P_i = P_i^T > 0$

$$A^T P_i + P_i A + Q_{ci} - P_i B R_c^{-1} B^T P_i = 0. \tag{20}$$

*Step 4.* Compute the gain matrix

$$K_i = -R^{-1} B^T P_i. \tag{21}$$

*Step 5.* For given matrices $K_i, C$, using Lemma 2 compute the matrices $F_i$ and $L_i$ in the same way for all $i$.

*Step 6.* Calculate $er = \|L_i - L_{i-1}\|$ if $er \le error$ stop, else set $i = i + 1$ and go to Step 2.

If the sequence $L_0, L_1, \ldots$ converges, say to $L$, gain matrix $K$ is given by (19). It is easy to show that Algorithm A can be rewritten to LMI based algorithm. Note that for the stabilization problem the matrix $Q_{ci}$ is

$$Q_{ci} = Q_{ci}(Algorithm\ A) - (K+L)^T R(K+L) - Q.$$

Although the convergence of the *Algorithm A* has not been formally proven, it has converged for the most tests performed in connection with this research. Typically, the number of iterations for convergence varies from $15-50$ depending on the values of matrices $Q$ and $R$.

In this section we will present a new procedure to design a static structurally constrained state feedback for discrete time system (2). The following theorem will be employed in the further development.

**Theorem 2.** Let the discrete linear time invariant system be described by (2). Then the following statements are equivalent.

1. The system (2) is static structurally constrained state feedback stabilizable.
2. There exists a symmetric and positive definite matrix $P$ and matrices $K$ and $L$ satisfying the following matrix inequality.

$$[A + B(K+L)]^T P[A + B(K+L)] - P < 0. \tag{22}$$

3. There exist symmetric and positive definite matrices $P$ and $R$ and matrices $K$ and $L$ satisfying the following matrix inequalities

$$-\phi_d < 0, \quad G\phi_d^{-1}G^T - I < 0 \tag{23}$$

where

$$\phi_d = -(A^T PA - P - A^T PB(B^T PB + R)^{-1}B^T PA + Q_{db} + Q_{da})$$

$$G = \frac{1}{\sqrt{2}}(B^T PB + R)^{-\frac{1}{2}}B^T PA + \sqrt{2}(B^T PB + R)^{\frac{1}{2}}K$$

$$Q_{daa} = \frac{1}{\sqrt{2}}(B^T PB + R)^{-1}B^T PA + \sqrt{2}L$$

$$Q_{da} = Q_{daa}^T(B^T PB + R)Q_{daa} + (K+L)^T B^T PB(K+L)$$

$$Q_{db} = -2K^T(B^T PB + R)K - 2L^T(B^T PB + R)L.$$

Proof. The proof of equivalence of the first and second statements is evident from a similar lemma for discrete time systems as Lemma 1. The proof of equivalence of the second and third statements is similar to that of Theorem 1. According to (16) the matrix inequality

$$\begin{bmatrix} I & G \\ G^T & \phi_d \end{bmatrix} > 0$$

is for $R > 0$ equivalent to

$$\phi_d > 0, \quad I - G\phi_d^{-1}G^T > 0$$

or

$$I > 0, \quad \phi_d - G^T G > 0.$$

Obviously the former is equivalent to (23), the latter can be after some manipulation rearranged into the form (22). Thus (22) is equivalent to (23) (or statement 2 is equivalent to 3) which completes the proof. □

**Corollary 2.** Let us find $(P, K)$ solving the following two matrix equations

$$A^T PA - P - A^T PB(B^T PB + R)^{-1} B^T PA + Q_d = 0 \tag{24}$$

$$K = -\frac{1}{2}(B^T PB + R)^{-1} B^T PA \tag{25}$$

where

$$Q_d = Q + Q_{da}$$

then the necessary and sufficient conditions (23) hold.

An algorithm to handle the LQR problem with a static structurally constrained feedback can be stated as follows.

**Algorithm B.**

*Step 1.* Set $i = 1, L_0 = 0, K_0 = 0, P_0 = I$.

*Step 2.* Compute

$$Q_{di} = Q + Q_{dai}.$$

*Step 3.* Calculate $P_i = P_i^T > 0$ from the following ARE

$$A^T P_i A - P_i - A^T P_i B(B^T P_i B + R)^{-1} B^T P_i A + Q_{di} = 0.$$

*Step 4.* Compute the gain matrix $K_i$

$$K_i = -\frac{1}{2}(B^T P_i B + R)^{-1} B^T P_i A.$$

*Step 5.* For given matrices $K_i, C$, using Lemma 2 calculate the matrices $F_i$ and $L_i$.

*Step 6.* Calculate

$$er = ||L_i - L_{i-1}||$$

if $er \leq error$ stop, else set $i = i + 1$ and go to Step 2.

If the sequence $L_0, L_1, \ldots$ converges, say to $L$, the gain matrix $K$ is given by (25).

### 3.2. Static output feedback design

In this subsection we will present new procedures to design a static output feedback controller for a continuous and discrete time system. The main results for a continuous time system with a static output feedback are summarized in the following theorem.

**Theorem 3.** Let the linear continuous time system (1) be given. Then, the following statements are equivalent.

1. System (1) is static output feedback stabilizable.
2. There exists a symmetric and positive definite matrix $P$ and a matrix $F$ satisfying the following matrix inequality

$$(A + BFC)^T P + P(A + BFC) < 0. \tag{26}$$

3. There exist positive definite matrices $P$ and $R$ and a matrix $F$ satisfying the following matrix inequalities

$$\phi_u = -(A^T P + PA - C^T F^T RFC - PBR^{-1}B^T P) > 0 \tag{27}$$

$$(RFC + B^T P)\phi_u^{-1}(RFC + B^T P)^T - R < 0. \tag{28}$$

Proof. The proof of this theorem goes in the same way as for Theorem 1. □

**Theorem 4.** Let the system (1) be given. Then the following statements are equivalent.

- The system (1) is static output feedback stabilizable with guaranteed cost

$$\int_0^\infty (x(t)^T Qx(t) + u(t)^T Ru(t))\,\mathrm{d}t \le x_0^T Px_0 = J^* \tag{29}$$

and $P > 0$.

- There exist matrices $P > 0, R > 0, Q > 0$ and a matrix $F$ such that the following inequality holds

$$(A + BFC)^T P + P(A + BFC) + Q + C^T F^T RFC \le 0. \tag{30}$$

- There exist matrices $P > 0, R > 0, Q > 0$ and a matrix $F$ such that the following inequalities hold

$$A^T P + PA - PBR^{-1}B^T P + Q \le 0 \tag{31}$$

$$(B^T P + RFC)\phi_u^{-1}(B^T P + RFC)^T - R \le 0 \tag{32}$$

where

$$\phi_u = -(A^T P + PA - PBR^{-1}B^T P + Q).$$

Proof. Let the control algorithm with output feedback be given as

$$u(t) = Fy(t) = FCx(t),$$

then for the closed loop system one obtains

$$\dot{x} = (A + BFC)\, x(t).$$

For $V = x(t)^T Px(t)$, the time derivative of $V$ along system (1) is

$$\frac{\mathrm{d}V}{\mathrm{d}t} = x^T[(A + BFC)^T P + P(A + BFC)]x.$$

If inequality (30) holds, then there exist matrices $P > 0, R > 0, Q > 0$ and $F$ such that

$$\frac{\mathrm{d}V}{\mathrm{d}t} \leq -x(t)^T (Q + C^T F^T RFC)\, x(t) < 0.$$

Therefore the closed loop system is asymptotically stable. Furthermore, by integrating both sides of the inequality from 0 to $T$ and using the initial condition $x_0$, we obtain

$$V(0) - V(T) \geq \int_0^T x(t)^T (Q + C^T F^T RFC)\, x(t)\, \mathrm{d}t.$$

As the closed loop system is asymptotically stable if $T \to \infty$, then

$$x(T)^T Px(T) \to 0.$$

Hence, we get

$$\int_0^\infty x(t)^T (Q + C^T F^T RFC)\, x(t)\, \mathrm{d}t \leq x_0^T P x_0 \tag{33}$$

and the control algorithm $u = Fy$ is a guaranteed cost control law and

$$J^* = x_0^T P x_0$$

is a guaranteed cost function for uncertain closed loop system. The equivalence of the second and third statements is proved in Theorem 3. □

Define $S = P^{-1}$ and using the Schur complement formula the inequality (31) is equivalent to the following linear matrix inequalities

$$\begin{bmatrix} SA^T + AS - BR^{-1}B^T & S\sqrt{Q} \\ \sqrt{Q}S & -I \end{bmatrix} < 0$$

$$\gamma I < S \tag{34}$$

where $\gamma \geq 0$ is some non-negative constant. When one knows $P = S^{-1}$, inequality (32) can be rewritten as follows

$$\begin{bmatrix} -R & B^T P + RFC \\ (B^T P + RFC)^T & -\phi_u \end{bmatrix} < 0. \tag{35}$$

The algorithm for static output feedback stabilization of system(1) with a guaranteed cost (33) and non-iterative LMI approach is given as follows.

**Algorithm C.**

*Step 1.* Using LMI based algorithm calculate $S$ from inequalities (34), $P = S^{-1}$.

*Step 2.* Via LMI based algorithm calculate $F$ from inequalities (35).

*Step 3.* If the solution (34) is not feasible, system (1) is not stabilizable, and if (35) is not feasible and the closed loop system(5) is not stable, change $Q$ and $R$.

If the solutions (34) and (35) are feasible with respect to $S$ and $F$, then system (1) is quadratically stable with a guaranteed cost control algorithm

$$u(t) = Fy(t)$$

and

$$J^* = x_0^T P x_0$$

is a guaranteed cost for uncertain closed loop system. Theorem 3 implies the following corollary.

**Corollary 3.** Let the system (1) be given. Then the following statements are equivalent:

- The system (1) is static output feedback stabilizable.
- There exist positive definite matrices $Q > 0, R > 0, P > 0$ and a matrix $F$ satisfying the following matrix inequalities.

$$A^T P + PA - PBR^{-1}B^T P - C^T F^T RFC + Q < 0 \tag{36}$$

$$(B^T P + RFC)\Phi_s^{-1}(B^T P + RFC)^T - R < 0 \tag{37}$$

where

$$\Phi_s = -(A^T P + PA - PBR^{-1}B^T P - C^T F^T RFC + Q).$$

¿From Corollary 3 one obtains the following design procedure for static output feedback stabilization of system (1) based on V-K LMI iterative algorithm.

**Algorithm D.**

*Step 1.* Set $j = 1, F_0 = 0$.

*Step 2.* Using the LMI based algorithm calculate $S_j = P_j^{-1} > 0$ from inequality (36).

*Step 3.* Using LMI based algorithm calculate the gain matrix $F_j$ from inequality (37).

*Step 4.* Calculate $er = ||F_j - F_{j-1}||$, if $er \leq error$ stop else $j = j + 1$ and go to Step 2.

*Step 5.* If there is no solution, change $Q$ and $R$.

The philosophy of Algorithm D is very closely related to the *V-K iteration* algorithm proposed in [4]. The V-K iteration algorithm is based on an alternative solution of the two convex LMI optimization problems, obtained by either fixing the matrix $P$ or the gain $F$. This algorithm is *guaranteed to converge*, but not necessarily to the global optimum of the problem depending on the starting condition of matrix $F$. If Algorithm D is feasible, for the closed loop system the following cost is guaranteed

$$\int_0^\infty x(t)^T Qx(t)\,\mathrm{d}t \le x_0^T P x_0.$$

The main results for discrete time system (2) and output feedback are summarized in the following theorem.

**Theorem 5.** Let the linear discrete time system (2) be given. Then the following statements are equivalent.

1. The system (2) is static output feedback stabilizable.
2. There exist a symmetric and positive definite matrix $P$ and a gain matrix $F$ satisfying the following inequality

$$(A+BFC)^T P(A+BFC) - P < 0. \tag{38}$$

3. There exist matrices $P = P^T > 0,\ R = R^T > 0$ and a gain matrix $F$ satisfying the following inequalities

$$-\Phi_d < 0, \quad G_d\Phi_d^{-1}G_d^T - I < 0 \tag{39}$$

where

$$\Phi_d = -(A^TPA - P - A^TPB(B^TPB+R)^{-1}B^TPA - C^TF^TRFC) \tag{40}$$

$$G_d = (B^TPB+R)^{-\frac{1}{2}}B^TPA + (B^TPB+R)^{\frac{1}{2}}FC. \tag{41}$$

Proof. The proof of this theorem goes the same way as for Theorem 2. □

**Corollary 4.** Approximate solution of (39) can be given as a solution of the following two matrix equality.

$$A^TPA - P - A^TPB(B^TPB+R)^{-1}B^TPA + Q = 0 \tag{42}$$

$$F = -(B^TPB+R)^{-1}B^TPAC^T(CC^T)^{-1}. \tag{43}$$

Equations (42), (43) directly determine the Lyapunov matrix $P = P^T > 0$ which satisfies the condition of (39). The gain matrix $F$ is only an approximate solution of the second inequality of (39). Hence, the sufficient and necessary conditions (39) for (43) may not be fulfilled in some cases.

An algorithm to handle the static output feedback can be stated as follows.

**Algorithm E.**

*Step 1.* Set $i = 1,\ F_0 = 0$.

*Step 2.* Compute

$$Q_{di} = Q - C^T F_{i-1}^T R F_{i-1} C.$$

*Step 3.* Solve ARE

$$A^T P_i A - P_i - A^T P_i B(B^T P_i B + R)^{-1} B^T P_i A + Q_{di}$$

for $P_i = P_i^T > 0$.

*Step 4.* Compute

$$F_i = -(B^T P_i B + R)^{-1} B^T P_i A C^T (CC^T)^{-1}.$$

*Step 5.* Calculate

$$er = ||P_i - P_{i-1}||$$

if $er \leq error$ stop else increase $i$ by one and go to Step 2.

*Step 6.* Check the stability of closed loop system $A + BFC$ or the inequality

$$B^T PA(I - C^T(CC^T)^{-1})Q_d^{-1}(I - C^T(CC^T)^{-1}C)^T APB - (B^T PB + R) < 0$$

if it holds stop else change $Q, R$ and go to Step 2.

## 4. EXAMPLES

As a first concrete example we have taken the problem of the design of a PI controller to control a small DC motor rotation. A continuous model of the DC motor is given by (1) where

$$A = \begin{bmatrix} -4.701 & 1 & 0 \\ -8.2986 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -.0721 \\ 15.0218 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The results of gain matrix calculation employing the derived algorithms are summarized as follows.
Algorithm A. Structurally constrained state feedback. Let the matrix $F$ be described as

$$F = [f_1 \quad f_2]$$

and

$$K = [K_{11} \quad K_{12} \quad K_{13}], \quad L = [L_{11} \quad L_{12} \quad L_{13}].$$

Due to (10) for entries of matrices $F$ and $L$ we obtain the following equations

$$f_1 C_{11} = K_{13} + L_{13}, \quad f_2 C_{23} = K_{13} + L_{13}, \quad 0 = K_{12} + L_{12}.$$

The solution can be

$$K_{12} = -L_{12}, \quad L_{11} = L_{13} = 0, \quad f_1 = \frac{K_{11}}{C_{11}}, \quad f_2 = \frac{K_{13}}{C_{23}}$$

For $Q = diag\{2 \quad 1 \quad 2\}$ and $R = 1$ the gain matrix $F$ and eigenvalues of the closed loop $(CL)$ system are

$$F = [.1763 \quad 1.4142]; \quad eig\, CL = \{-3.3446; -.6718 \pm j2.429\}.$$

The above DC motor continuous model has been recalculated with sample period $T = 0.1s$ to a discrete model. Results of the design of a PS controller are summarized as follows. For $Q = diag\{.2 \quad .2 \quad .2\}$ and $R = 10$ the gain matrix $F$ and eigenvalues of the closed loop system for Algorithm B are

$$F = [-.0312 \quad -.0675] \quad eig\, CL = \{.9857; .7886 \pm j.1083\}.$$

The number of iterations = 21 and error of calculation = $2.0955.10^{-7}$
From eqs. (42) and (43) one obtains the gain matrix

$$F = [-.0847 \quad -.1313] \quad eig\, CL = \{.9628 \quad .801 \pm j.0518\}.$$

Algorithm E gives the following results

$$F = [-.089 \quad -.0963] \quad eig\, CL = \{.9748; .7957 \pm j.0623\}.$$

The number of iterations = 67 and error of calculation = $2.4117.10^{-7}$.

As a second example we have taken the problem of the design of a decentralized PI controller for the system given by (1) where

$$A = \begin{bmatrix} 0 & -.2878 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -.7662 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.485 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1.788 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -.6638 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1.2642 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -.1363 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -.5725 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$B^T = \begin{bmatrix} .2794 & -.0582 & 0 & 0 & -.3277 & .0084 & 0 & 0 & 0 & 0 \\ 0 & 0 & .5725 & -.0655 & 0 & 0 & .2085 & -.0371 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The structure of the output feedback gain matrix $F$ is given by the designer as follows

$$F = \begin{bmatrix} f11 & 0 & f13 & 0 \\ 0 & f22 & 0 & f24 \end{bmatrix}.$$

The results of calculation for Algorithm A are as follows. For $Q = diag\{1\}, R = diag\{100\}$ the gain matrix

$$F = \begin{bmatrix} .02812 & 0 & .080178 & 0 \\ 0 & .01058 & 0 & .076501 \end{bmatrix}$$

and

$$eig\, CL = \{-.12624 \pm j1.8422; -.12819; -.2866, \ldots, -.89345 \pm j.827\}.$$

The results of computation for LMI based Algorithm A are as follows. For $Q = diag\{7\}, R = diag\{50\}$ and $\gamma = .03$ the gain matrix

$$F = \begin{bmatrix} .2623 & 0 & .0345 & 0 \\ 0 & .0294 & 0 & .0112 \end{bmatrix}.$$

The closed loop eigenvalues are

$$eig\, CL = \{-.0233 \pm j.0049; -2769 \pm j.2368; \ldots; -.8953 \pm j.8261\}.$$

Guaranteed cost

$$\int_0^\infty x(t)^T (Q + (K+L)^T R(K+L))\, x(t)\, \mathrm{d}t \leq 61.7315 \|x_0\|^2.$$

For Algorithm C and $Q = diag\{1\}, R = diag\{50\}$ and $\gamma = .03$ one obtains the gain matrix

$$F = \begin{bmatrix} 0.0268 & 0 & .1013 & 0 \\ 0 & .0515 & 0 & 0.0831 \end{bmatrix}.$$

The eigenvalues of the closed loop system are

$$eig\, CL = \{-.1364 \pm j.1968; -.216 \pm j.0279; \ldots; -.8933 \pm j.8263\}.$$

The guaranteed cost is

$$\int_0^\infty x(t)^T (Q + C^T F^R F C)\, x(t)\, \mathrm{d}t \leq 47.7658 \|x_0\|^2.$$

Algorithm D for the same $Q, R$ and $\gamma$ give the following results

$$F = \begin{bmatrix} .0184 & 0 & .083 & 0 \\ 0 & .0419 & 0 & .072 \end{bmatrix}$$

The eigenvalues of the closed loop system are

$$eig\, CL = \{-.1632 \pm j.1791; -.1734 \pm j.0217; \ldots; -.8935 \pm .8263\}.$$

The guaranteed cost is

$$\int_0^\infty x(t)^t Q x(t)\, \mathrm{d}t \leq 45.5703 \|x_0\|^2.$$

## 5. CONCLUSION

The main aim of this paper is to propose new methods for solving the problem of the controller design via static output feedback for linear continuous and discrete time systems. In this paper new necessary and sufficient conditions have been proposed for static output feedback stabilizability for continuous and discrete time systems. These conditions are tightly connected with the Lyapunov function and LQ optimal state feedback control gain matrix design. For the proposed Algorithm A and non-iterative LMI Algorithm C and V-K iterative LMI based Algorithm D, the gain matrix structure of static output feedback $F$ could be prescribed by the designer including the decentralized case. The proposed algorithms are computationally simple.

## ACKNOWLEDGEMENT

This research was supported by the Scientific Grant Agency of the Slovak Republic through Grant 1/7608/20.
The author is grateful to the referee for the careful reading of the manuscript and many helpful suggestions which improved the paper.

(Received December 15, 1999.)

## REFERENCES

[1] I. E. Benton and D. Smith: A non iterative LMI based algorithm for robust static output feedback stabilization. Internat. J. Control *72* (1999), 14, 1322–1330.
[2] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnam: Linear matrix inequalities in system and control theory. SIAM *15* (1994), Philadelphia.
[3] V. Blondel, M. Gevers, and A. Lindquist: Survey on the state of systems and control. European J. Control *1* (1995), 5, 2–23.
[4] L. El Ghaoui and V. Balakrishnan: Synthesis of fixed structure controllers via numerical optimization. In: Proc. of the 33rd Conference on Decision and Control, Lake Buena Vista 1994, pp. 2678–2683.
[5] J. C. Geromel and P. L. D. Peres: Decentralized load-frequency control. Proc. IEE–D *132* (1985), 225–230.
[6] K. C. Goh, M. G. Safonov, and J. H. Ly: Robust synthesis via bilinear matrix inequalities. Internat. J. Robust and Nonlinear Control *6* (1996), 1079–1095.
[7] K. G. Goh, M. G. Safonov, and G. P. Papavassilopoulos: Global optimization for the biaffine matrix inequality problem. J. Global Optimization *7* (1995), 365–380.
[8] T. Iwasaki, R. L. Skelton, and J. G. Geromel: Linear quadratic suboptimal control with static output feedback. Systems Control Lett. *23* (1994), 421–430.
[9] T. Iwasaki and R. E. Skelton: All controllers for the general $H_\infty$ control problem: LMI existence conditions and state space formulas. Automatica *30* (1994), 8, 1307–1314.
[10] I. E. Kose and F. Jabbari: Robust control of linear systems with real parametric uncertainty. Automatica *35* (1999), 679–687.
[11] V. Kučera and C. E. deSouza: A necessary and sufficient condition for output feedback stabilizability. Automatica *31* (1995), 9, 1357–1359.
[12] P. Lancaster: Theory of Matrices. Academic Press, New York – London 1969.
[13] Li Yu and Jian Chu: An LMI approach to guaranteed cost control of linear uncertain time-delay systems. Automatica *35* (1999), 1155–1159.

[14] D. Rosinová, V. Veselý, and V. Kučera: A necessary and sufficient condition for output feedback stabilizability of linear discrete-time systems. In: IFAC Conference Control System Design, Bratislava 2000, pp. 164–167.
[15] M. G. Safonov, K. G. Goh, and J. Ly: Control system synthesis via bilinear matrix inequalities. In: Proc. American Control Conference, Baltimore 1994, IEEE Press, New York, pp. 1–5.
[16] V. L. Syrmos, C. T. Abdalah, P. Dorato, and K. Grigoriadis: Static output feedback - A survey. Automatica *33* (1997), 2, 125–137.
[17] O. Toker and H. Özbay: On the NP-hardness of solving bilinear matrix inequalities and simultaneous stabilization with static output feedback. In: Proc. American Control Conference 1995, pp. 2525–2526.
[18] Yong-Yan Gao and You-Xian Sun: Static output feedback simultaneous stabilization: LMI approach. Internat. J. Control *70* (1998), 5, 803–814.

*Prof. Ing. Vojtech Veselý, DrSc., Department of Automatic Control Systems, Faculty of Electrical Engineering and Information Technology, Slovak University of Technology, 812 19 Bratislava. Slovak Republic.*
*e-mail: vesely@kasr.elf.stuba.sk*