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T–LAW OF LARGE NUMBERS FOR FUZZY NUMBERS[1]

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The notions of a t-norm and of a fuzzy number are recalled. The law of large numbers for fuzzy numbers is defined. The fuzzy numbers, for which the law of large numbers holds, are investigated. The case when the law of large numbers is violated is studied.

1. DEFINITIONS

1.1. Triangular norms, T-addition of fuzzy quantities

Since the law of large numbers for fuzzy numbers is based on the T-sum of fuzzy quantities, we recall some important results concerning t-norms.

Definition 1. A *triangular norm* (shortly *t-norm*) T is a non-decreasing, associative and commutative $[0,1]^2 \to [0,1]$ mapping that satisfies the boundary condition $\forall\, x \in [0,1]$, $T(x,1) = x$.

Definition 2. A continuous t-norm T is called Archimedean if

$$\forall\, x \in]0,1[, \quad T(x,x) < x.$$

Note some well-known t-norms. The strongest t-norm is minimum $T_M(x,y) = \min(x,y)$. On the other hand, the weakest t-norm is drastic product T_D defined as follows

$$T_D(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1, \\ 0 & \text{elsewhere.} \end{cases}$$

Continuous Archimedean t-norms can be divided into two classes: strict and nilpotent. An example of a strict (i. e. continuous, strictly increasing on $]0,1]^2$) t-norm is the algebraic product T_p, and an example of a nilpotent t-norm (Archimedean, continuous, not strict) is the Lukasiewicz t-norm $T_L(x,y) = \max(0, x+y-1)$.

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We recall that fuzzy quantity A is a fuzzy subset of the real line. The addition of fuzzy quantities is based on a given t-norm T, following Zadeh's extension principle, by

$$A \oplus_T B(z) = \sup_{x+y=z} (T(A(x), B(y))), \quad z \in R$$

where A, B are given fuzzy quantities. If T is an Archimedean continuous t-norm with additive generator f then the addition of fuzzy quantities can be expressed as follows

$$A \oplus_T B(z) = f^{(-1)} \left(\inf_{x+y=z} (f \circ A(x) + f \circ B(y)) \right), \quad z \in R,$$

where $F^{(-1)}$ is the pseudoinverse of f, defined by

$$f^{(-1)}(x) = f^{-1} (\min(f(0), x)), \quad x \in [0, +\infty[.$$

Definition 3. Let $A_1, \ldots, A_n$ be fuzzy quantities, $n \in N$. Then their T-arithmetic mean $M_{n,T}$ is defined as follows

$$M_{n,T}(x) = \frac{1}{n} (A_1 \oplus_T \cdots \oplus_T A_n)(x) := (A_1 \oplus_T \cdots \oplus_T A_n)(nx).$$

1.2. Fuzzy numbers

Definition 4. A fuzzy quantity A is called a *fuzzy number* if it is convex fuzzy subsets of R with one modal value only, i. e.

— $\forall x < y < z \in R$, $A(y) \geq \min(A(x), A(z))$,

— $\forall!\, a \in R$, $A(a) = 1$.

Any continuous fuzzy number can be written as a triple $A = (a, F, G)$ where F and G are continuous decreasing functions from $[0, \infty[$ to $[0, 1]$, such that $F(x) = G(x) = 1$ if and only if

$$x = 0 \quad \text{and} \quad A = \begin{cases} F(x-a) & \text{if} \quad x \leq a \\ G(x-a) & \text{if} \quad x > a. \end{cases}$$

The functions F and G will be called *the shape functions.*

A special case of a continuous fuzzy number is a LR-fuzzy number, i. e., continuous fuzzy number with bounded support.

Definition 5. A fuzzy quantity A is a so called *LR-fuzzy number* $A = (a, \alpha, \beta)_{LR}$ if the corresponding membership function satisfies for all $x \in R$

$$A(x) = \begin{cases} L\left(\frac{a-x}{\alpha}\right), & \text{for } a - \alpha \leq x \leq a, \\ R\left(\frac{x-a}{\beta}\right), & \text{for } a \leq x \leq a + \beta, \\ 0 & \text{else,} \end{cases}$$

where a is the peak (modal value) of A, $\alpha > 0$ and $\beta > 0$ is *the left* and *the right spread*, respectively, and L and R are decreasing continuous functions from $[0,1]$ to $[0,1]$ such that $L(x) = R(x) = 1$ iff $x = 0$ and $L(x) = R(x) = 0$ iff $x = 1$.

Note that, if shape functions L and R are linear, i.e., $L(x) = R(x) = 1 - x$, we will speak about triangular, or linearly fuzzy number $A = (A, \alpha, \beta)$.

1.3. The law of large numbers for fuzzy numbers

Definition 6. [2] Let A be a given fuzzy number. Let D be a subset of R and $\overline{D}$ be the complement of D. The grade of *possibility* of statement "D contains the value of A" is defined by

$$\pi(D) = \sup_{x \in D} A(x).$$

The grade of necessity of the statement "D contains the value of A" is defined by

$$\mathcal{N}(D) = 1 - \pi(\overline{D}).$$

Fullér [2] introduced the law of large numbers for special LR-fuzzy numbers and t-norms bounded by the Hamacher product t-norm.

Theorem 1. (The law of large numbers, [2]) Let $A_n = (a_n, \alpha, \alpha)$, $n \in N$ be symmetric triangular fuzzy numbers and let $T \leq T_{H_0}$, where $T_{H_0}(x,y) = \frac{xy}{x+y-xy}$ (whenever $(x,y) \neq (0,0)$) is the Hamacher product. If $a = \lim_{n\to\infty} \frac{a_1+\cdots+a_n}{n}$ exists, then for any $\varepsilon > 0$

$$\begin{aligned} &\lim_{n\to\infty} \mathcal{N}\left(a^{(n)} - \varepsilon \leq M_{n,T} \leq a^{(n)} + \varepsilon\right) = 1, \\ &\mathcal{N}\left(\lim_{n\to\infty} M_{n,T} = a\right) = 1, \end{aligned} \tag{1}$$

where $a^{(n)} = \frac{a_1+\cdots a_n}{n}$.

However, note that Fullér's law of large numbers is not defined correctly. He requires (in his proof) that conditions (1) are equivalent to condition $\lim_{n\to\infty} M_{n,T}(z) = \chi_a(z)$, what is not true. See the following example.

Example 1. Let A_i be crisp fuzzy numbers $A_n = \chi_{a_n}$, $a_n = \left(\frac{1}{2}\right)^n$, $n \in N$. The limit of the arithmetic means of peaks a_i is equal to zero, $\lim_{n\to\infty} a^{(n)} = 0$ and the T-arithmetic mean of fuzzy numbers A_i, $i = 1, \ldots, n$, is for arbitrary t-norm T defined by $M_{n,T}(z) = \chi_{a(n)}(z)$. However $\lim_{n\to\infty} M_{n,T}(z) = 0 \neq \chi_0(z)$ whenever $z = 0$.

Hence, we have to define the law of large numbers for fuzzy numbers as follows:

$$\lim_{n\to\infty} \frac{1}{n}\left(A_1 \oplus_T \cdots \oplus_T A_n\right)(z) = \begin{cases} 1 & \text{if } z = a, \\ 0 & \text{otherwise}, \end{cases} \tag{2}$$

where A_n, $n \in N$ are fuzzy numbers with peaks a_i and $a = \lim_{n\to\infty} \frac{a_1+\cdots+a_n}{n}$.

In the same paper Fullér showed that for T_M ($T_M(x,y) = \min(x,y)$) the law of large numbers is violated. Note that the law of large numbers (2) with respect to T_M (and then for all t-norms) holds only for every special fuzzy LR-fuzzy numbers $A_n = (a, \alpha_n, \beta_n)_{LR}$ such that $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \alpha_i = 0$ and the same for the right spreads.

On the other hand the class of sequences of fuzzy numbers such that the law of large numbers holds with respect to the weakest t-norm T_D is very large. It contains, e.g., all sequences $\{A_n = (a, \alpha_n, \beta_n)_{L_n R_n},\ n \in N\}$ of LR-fuzzy numbers with bounded spreads α_n, $\beta_n \le C$, as well as all sequences $\{A_n = (a, F_n, G_n),\ n \in N\}$ of fuzzy numbers with bounded shapes ($F_n, G_n \le F$, $n \in N$, where F is shape function) and unbounded support, etc.

2. T-LAW OF LARGE NUMBERS FOR FUZZY NUMBERS

Recently, Hong [4] has shown that to each shape function $L : [0,1] \to [0,1]$ there exists a concave shape function L^* such that $L^* \ge L$.

Lemma 1. For any additive generator f of a strict t-norm and for any shape function F there exist a shape function F_f such that $F_f \ge F$ and $f \circ F_f$ is convex.

Proof. Let f be a generator of a strict t-norm. Then $f \circ F$ is increasing continuous function. By the same arguments as used by Hong in [4], there exists a convex function $F^* \le F \circ F$. Put $F_f = f^{-1} \circ F^*$. Then $F_f \ge f^{-1}(f \circ F) = F$ and $f \circ F_f = F^*$ is convex. □

Lemma 2. For any nilpotent t-norm T_1 there exists a strict t-norm T_2 such that $T_1 \le T_2$.

The proof is obvious. If we realize that $T_L \le T_p$, it is enough to apply Mesiar's transformation principle [11]. □

Theorem 2. (T-law of large numbers) Let $A_n = (a, \alpha_n, \beta_n)_{LR}$, $n \in N$ be LR-fuzzy numbers with bounded spread sequences, i.e., $\alpha_n \le c$, $\beta_n \le c$, $n \in N$, for some $c > 0$. Then the law of large numbers holds with respect to any continuous Archimedean t-norm T, i.e.

$$\lim_{n\to\infty} \frac{1}{n} (A_1 \oplus_T \cdots \oplus_T a_n)(z) = \begin{cases} 1 & \text{if } z = a, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let T be a strict t-norm and let $L_f \ge L$, $R_f \le R$ be shapes such that the composites $f \circ L_f$ and $f \circ R_f$ are convex (existence of L_f, R_f is ensured by Lemma 1). Then $A_n \le B_n = (a, c, c)_{L_f R_f}$, $n \in N$. It is obvious that

$$\frac{1}{n} (A_1 \oplus_T \cdots \oplus_T A_n)(a) = 1 \quad \text{for all } n \in N.$$

Further, for $z \neq a$, say for $z \in]a, a+c]$, we have by [10, 11]

$$\begin{aligned} & \frac{1}{n}(A_1 \oplus_T \cdots \oplus_T A_n)(z) \leq \frac{1}{n}(B_1 \oplus_T \cdots \oplus_T B_n)(z) \\ = \; & f^{-1}\left(n \cdot f \circ R_f\left(\frac{z-a}{c}\right)\right) \to 0. \end{aligned}$$

For $z > a + c$, $\frac{1}{n}(A_1 \oplus_T \cdots \oplus_T A_n)(z) = 0$. The case when $z < a$ is similar.

We have proved that the law of large numbers holds for arbitrary strict t-norm. As far as it holds for any weaker t-norm, by Lemma 2 it holds for any nilpotent t-norm, too. □

Theorem 2 can be easily generalized to the case of LR-fuzzy numbers $A_n = (a, \alpha_n, \beta_n)_{L_n R_n}$ such that $\alpha_n \leq c$, $\beta_n \leq c$, $L_n \leq L_f$, $R_n \leq R_f$, $n \in N$.

The requirements of Theorem 2 are sufficient conditions only. In the following example the spreads of incoming fuzzy numbers are unbounded.

Example 2. Let $A_n = (0, \sqrt{n}, \sqrt{n})$, $n \in N$ be linear fuzzy numbers. Consider the Lukasiewicz t-norm T_L. Then

$$M_{n,T_L} = \frac{1}{n}(A_1 \oplus_{T_L} \cdots \oplus_{T_L} A_n) = \frac{1}{n}\left(0, \sqrt{n}, \sqrt{n}\right) = \frac{1}{n} A_n = \left(0, \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\right)$$

and

$$\lim_{n \to \infty} M_{n,T_L} = \chi_0.$$

In the next example the shape functions are not bounded by any shape function.

Example 3. Let $A_n = (0,1,1)_{L_n L_n}$, $n \in N$ be LR-fuzzy numbers with the shape functions $L_n(x) = 1 - x^{\sqrt{\ln(n+3)}}$. The limit function

$$L(x) = \lim_{n \to \infty} L_n = \left\langle \begin{array}{ll} 1 & x \in [0, 1[\\ 0 & x = 1 \end{array} \right.$$

is not a shape function. Consider the Lukasiewicz t-norm T_L. Then

$$\frac{1}{n}(0,0,0) \leq \frac{1}{n}(A_1 \oplus_{T_L} \cdots \oplus_{T_L} A_n) \leq \frac{1}{n}(A_n \oplus_{T_L} \cdots \oplus_{T_L} A_n).$$

Hence for $x \neq 0$

$$\frac{1}{n}(A_n \oplus_{T_L} \cdots \oplus_{T_L} A_n)(x) = \max\left(0, 1 - n|x|^{\sqrt{\ln(n+3)}}\right),$$

then

$$\lim_{n \to \infty} M_{n,T_L} = \chi_0,$$

i.e., the law of large numbers holds.

In the following example the spreads and the shape functions are unbounded.

Example 4. Let $A_n = (0, \ln(n+3), \ln(n+3))_{L_n L_n}$, $n \in N$ be LR-fuzzy numbers with shape functions L_n as in previous Example 3. $L_n(x) = 1 - x^{\sqrt{\ln(n+3)}}$. Consider the Lukasiewicz t-norm T_L. Then using the same way as in the previous example we can show

$$\lim_{n\to\infty} M_{n,T_L} = \chi_0,$$

i. e., the law of large numbers holds.

For continuous fuzzy numbers $A_n = (a, F, G)$, $n \in N$, with unbounded supports the law of large numbers holds, too.

Theorem 3. Let $A_n = (a, F, G)$, $n \in N$ be fuzzy numbers with unbounded support, i. e., $\operatorname{supp} A_n = (-\infty, \infty)$. Then the law of large numbers (2) holds with respect to any continuous Archimedean t-norm T.

Proof. Let T be a strict t-norm and let $F_f \geq F$, $G_f \geq G$ be shapes such that the composites $f \circ F_f$ and $f \circ G_f$ are convex. Then $A_n \leq B_n = (a, F_f, G_f)$, $n \in N$.

By [12] for all $z \in R$ we have

$$\begin{aligned} & \frac{1}{n}(A_1 \oplus_T \cdots \oplus_T A_n)(z) \leq \frac{1}{n}(B_1 \oplus_T \cdots \oplus_T B_n)(z) \\ = & \left(a,\, F_f^{(n)}(nz),\, G_f^{(n)}(nz)\right) \to (a, 0, 0), \end{aligned}$$

where

$$F_f^{(n)}(x) = \left(n \cdot f \circ F_f\left(\frac{x}{n}\right)\right)$$

for all $x \geq 0$, similarly for $G_f^{(n)}$.

By the same arguments as in the previous proof, the law of large numbers holds for any nilpotent t-norm, too. □

The requirement of equal peaks is not a necessary condition, too. See the following example.

Example 5. Consider the Gaussian fuzzy numbers

$$G(\mu_n, \sigma_n^2) \sim A_n(x) = e^{-\frac{(x-\mu_n)^2}{\sigma_n^2}}, \quad n \in N,$$

we can be write as triples $\left(\mu_n,\, F^{(\sigma_n^2)},\, F^{(\sigma_n^2)})\right)$, where $F(x) = e^{-x^2}$, and the product t-norm T_p. Then by [10]

$$\begin{aligned} M_{n,T_p} &= \frac{1}{n}\left(A_1 \oplus_{T_p} \cdots \oplus_{T_p} A_n\right) \\ &= \left(\frac{\mu_1 + \cdots + \mu_n}{n},\, F^{(\sigma_1^2 + \cdots \sigma_n^2)}(nx),\, F^{(\sigma_1^2 + \cdots \sigma_n^2)}(nx)\right), \end{aligned}$$

where

$$F^{(\sigma_1^2+\cdots\sigma_n^2)}(nx) = e^{-\frac{n^2x^2}{\sigma_1^2+\cdots+\sigma_n^2}}.$$

If exists

$$\mu = \lim_{n\to\infty} \frac{\mu_1 + \cdots + \mu_n}{n} \quad \text{and} \quad \lim_{n\to\infty} \frac{n^2}{\sigma_1^2 + \cdots + \sigma_n^2} = \infty$$

then the law of large numbers holds. As an example, take $\mu_n = \frac{1}{2n}$, $n \in N$ and $\sigma_1^2 = \sigma_2^2 = \ldots$.

Theorem 4. Let $A_n = (a_n, F, G)$, $n \in N$, and let an Archimedean continuous t-norm T with additive generator f, such that $f \circ F$ and $f \circ G$ are convex, be given. Let

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} a_i = a \in R;$$

$$b = \lim_{n\to\infty} \left(\sum_{i=1}^{n} a_i - na \right) = \lim_{n\to\infty} \sum_{i=1}^{n} (a_i - a), \quad |b| \in]0, \infty[;$$

$$c = \begin{cases} (f \circ F)'(0) & \text{if} \quad b > 0 \\ (f \circ G)'(0) & \text{if} \quad b < 0 \end{cases} = 0.$$

Then the law of large numbers (2) holds.

P r o o f. Suppose that $b > 0$. Then there is some $n_0 \in N$ such that for all $n \geq n_0$, $\sum_{i=1}^{n} a_i > na$. Then by [12]

$$\begin{aligned} &\frac{1}{n} (A_1 \oplus_T \cdots \oplus_T A_n)(a) = (A_1 \oplus_T \cdots \oplus_T A_n)(na) \\ = \quad & f^{(-1)} \left(n \cdot f \circ F \left(\frac{\sum_{i=1}^{n} a_i - na}{n} \right) \right). \end{aligned}$$

Now

$$\lim_{n\to\infty} \frac{f \circ F \left(\left(\sum_{i=1}^{n} a_i - na\right) \big/ n \right)}{\frac{1}{n}} = (f \circ F)'(0) \cdot b = cb = 0$$

and consequently

$$\lim_{n\to\infty} f^{(-1)} \left(n \cdot f \circ L \left(\left(\sum_{i=1}^{n} a_i - na \right) \big/ n \right) \right) = f^{(-1)}(0) = 1.$$

For $z < a$ we get

$$\begin{aligned} &\frac{1}{n} (A_1 \oplus_T \cdots \oplus_T A_n)(z) = (A_1 \oplus_T \cdots \oplus_T A_n)(nz) \\ = \quad & f^{(-1)} \left(n \cdot f \circ F \left(\frac{1}{n} \sum_{i=1}^{n} a_i - z \right) \right) \to 0 \end{aligned}$$

and similarly for $z > a$. □

However, we cannot drop the requirement of the Archimedean property for T.

Theorem 5. Let $A_n = (a, \alpha, \beta)_{LR}$, $n \in N$ and let T be a continuous t-norm. We denote

$$A(z) = \sup\left(c \in [0,1]; c \le A_1(z),\, T(c,c) = c\right).$$

Then

$$\lim_{n\to\infty} \frac{1}{n} \left(A_1 \oplus_T \cdots \oplus_T A_n\right)(z) = A(z).$$

Proof. Following [1], if $T(c,c) = c$ and $A_1(z) = c$, then $\frac{1}{n}(A_1 \oplus_T \cdots \oplus_T A_n)(z) = c = A(z)$ for all $n \in N$. If $A_1(z)$ is not an idempotent element of T, then it is contained in some interval $]\alpha_k, \beta_k[$ from the ordinal sum decomposition of T. Then by [1] and Theorem 2, $\lim_{n\to\infty} \frac{1}{n}(A_1 \oplus_T \cdots \oplus_T A_n)(z) = \alpha_k = A(z)$. □

Corollary 1. Let $A_n = (a, \alpha, \beta)_{LR}$, $n \in N$ and let T be a non-Archimedean continuous t-norm. Then the law of large numbers does not hold.

3. SUFFICIENT CONDITIONS FOR THE T-LAW OF LARGE NUMBERS IN SPECIFIC CASES

Let $A_n = (a, \alpha_n, \beta_n)_{L_n R_n}$. Let T be an Archimedean t-norm with an additive generator f. Then the validity of the T-law of large numbers for $(A_n)_{n\in N}$ is equivalent to the validity of the condition:

$$\lim_{n\to\infty} n\, f^*\left(A^{(n)}\right) \ge f(0) \quad \text{for all } x \ne 0, \tag{3}$$

where f^* is some convex lower bound of an additive generator f of T, and $A^{(n)}$ is the lowest concave upper bound of centralized fuzzy numbers $A_1, \dots, A_n$ we are dealing with, i.e.,

$$A^{(n)}(x) \ge A_i(x - a_i), \quad \forall x \in R,\ \forall i \in \{1, \dots, n\}.$$

Proposition 1. The sequence of linear fuzzy numbers $(A_n)_{n\in N}$, $A_n = (a, \alpha_n, \beta_n)$, $n \in N$, obeys the T_p-law of large numbers whenever $\lim_{n\to\infty} \frac{n}{c_n} = \infty$, where $c_n = \max(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n)$.

Proof. It is immediate that

$$A^{(n)} = \left(0, \underbrace{\max(\alpha_1, \dots, \alpha_n)}_{a_n}, \underbrace{\max(\beta_1, \dots, \beta_n)}_{b_n}\right).$$

Recall that $f(x) = -\log x$ is a (convex) additive generator of t-norm T_p. Then the condition (3) is fulfilled whenever

$$\infty = \lim_{n\to\infty} -n \log\left(\max\left(0, 1 - \frac{x}{a_n}\right)\right) = \lim_{n\to\infty} -n \log\left(\max\left(0, 1 - \frac{x}{b_n}\right)\right)$$

for any $x > 0$, i.e.,

$$\lim_{n\to\infty} -n\log\left(\max\left(0,\, 1-\frac{x}{c_n}\right)\right) = +\infty.$$

If $\{c_n\}_{n\in N}$ is bounded, then the latest equality is obvious (and then $\lim_{n\to\infty}\frac{n}{c_n} = \infty$). For $c_n \to \infty$, it is

$$\lim_{n\to\infty} -n\log\left(\max\left(0,\, 1-\frac{x}{c_n}\right)\right) = x\lim_{n\to\infty}\frac{n}{c_n} = +\infty$$

(for all $x > 0$) iff

$$\lim_{n\to\infty}\frac{n}{c_n} = \infty. \qquad \square$$

Remark. Note that $\lim\limits_{n\to\infty}\frac{n}{c_n} = \infty$ if and only if $\lim\limits_{n\to\infty}\frac{n}{\alpha_n} = \lim\limits_{n\to\infty}\frac{n}{\beta_n} = \infty$.

We can generalize Proposition 1 in several directions. Firstly, let f^* be a convex lower bound of an additive generator f of a t-norm T (if f is itself convex, we can take $f^* = f$). Then the conclusions of Proposition 1 with respect to the T-law of large numbers are valid whenever $(f^*)'(1^-) < 0$. Recall that in the case of $T = T_p$, $(-\log 1^-)' = -1$. For the Hamacher product T_{H_0}, $f(x) = \frac{1}{x} - 1$ and $f'(1^-) = -1$.

For the Yager t-norm T_p^Y, $p \in]0,\infty[$, $f_p(x) = (1-x)^p$. In the case $p \leq 1$, f_p are concave and the corresponding f^* is, e.g., $f^* = 1 - x$ with $(f^*)'(1^-) = -1$. In all these cases Proposition 1 can be applied. However, for $p > 1$, $f_p'(1^-) = 0$, and Proposition 1 cannot be applied.

Proposition 2. The sequence of linear (triangular) fuzzy numbers $(A_n)_{n\in N}$, $A_n = (a, \alpha_n, \beta_n)$, $n \in N$ obeys the T_p^Y-law of large numbers for a given $p > 1$ whenever $\lim_{n\to\infty}\frac{n}{c_n^p} = \infty$, where c_n is defined as in Proposition 1.

Proof. It is enough to deal with unbounded sequence $\{c_n\}_{n\in N}$. Then

$$\lim_{n\to\infty} n\left(1-\left(\max\left(0,\, 1-\frac{x}{c_n}\right)\right)\right)^p = x\lim_{n\to\infty}\frac{n}{c_n^p}, \quad x > 0$$

and

$$x\lim_{n\to\infty}\frac{n}{c_n^p} \geq 1 = f_p(0) \quad \text{for all } x > 0$$

is equivalent to

$$\lim_{n\to\infty}\frac{n}{c_n^p} = \infty. \qquad \square$$

Note that Proposition 2 can be derived directly from results of Kolesárová [5, 6]. Indeed, following [5, 6], it is

$$\frac{1}{n}\left(A_1 \oplus_{T_p^Y} \cdots \oplus_{T_p^Y} A_n\right) = \left(a,\, \frac{1}{n}\left(\sum_{i=1}^n \alpha_i^q\right) 1/q,\, \frac{1}{n}\left(\sum_{i=1}^n \beta_i^q\right) 1/q\right),$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Then the T_p^Y-law of large numbers is equivalent to the equalities

$$\lim_{n\to\infty} \frac{1}{n} \left(\sum_{i=1}^{n} \alpha_i^q \right)^{1/q} = \lim_{n\to\infty} \frac{1}{n} \left(\sum_{i=1}^{n} \beta_i^q \right)^{1/q} = 0. \tag{4}$$

It is a matter of simple calculations to show that equalities (4) are fulfilled whenever

$$\lim_{n\to\infty} \frac{\alpha_n^p}{n} = \lim_{n\to\infty} \frac{\beta_n^p}{n} = 0,$$

or equivalently

$$\lim_{n\to\infty} \frac{n}{c_n^p} = \infty.$$

On the other hand, we can generalize also the shapes L and R. If L^* and R^* are the corresponding concave upper bounds, the non-zero value of derivatives $(f^* \circ L^*(0^+))'$ and $(f^* \circ R^*(0^+))'$ allows to apply Proposition 1.

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