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ℓ^1-OPTIMAL CONTROL FOR MULTIRATE SYSTEMS UNDER FULL STATE FEEDBACK[1,2]

JOHANNES AUBRECHT AND PETROS G. VOULGARIS

This paper considers the minimization of the ℓ^∞-induced norm of the closed loop in linear multirate systems when full state information is available for feedback. A state-space approach is taken and concepts of viability theory and controlled invariance are utilized. The essential idea is to construct a set such that the state may be confined to that set and that such a confinement guarantees that the output satisfies the desired output norm conditions. Once such a set is computed, it is shown that a memoryless nonlinear controller results, which achieves near-optimal performance. The construction involves the solution of several finite linear programs and generalizes to the multirate case earlier work on linear time-invariant (LTI) systems.

1. INTRODUCTION

Multirate sampled data systems arise in many applications in which it is desirable to use multiple sampling rates for controlling a continuous-time system. The impetus to use multiple sampling rates could result from, for instance, differing bandwidths of input signals or differing limitations of the physical sensors and actuators used to implement a control algorithm. In addition, if the exogenous inputs or the regulated outputs are continuous signals, a multirate model can be used to approximate these continuous signals to any degree of accuracy. As a result, it is important to be able to design controllers for multirate sampled data systems that perform optimally in some sense.

In this paper the notion of optimality is with respect to ℓ^∞ performance. In particular, we are interested in minimizing the ℓ^∞-induced norm of the closed loop map. In the linear time invariant (LTI), case this amounts to minimizing the corresponding ℓ^1 norm. This ℓ^1 problem can be solved using input-output techniques and duality theory (e.g., [5]). For linear multirate sampled data (LMRSD) systems the problem is solved in [3] using again an input-output viewpoint and lifting techniques developed in [7, 8, 9] that convert the problem to an LTI however nonstandard, problem.

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Although the problem of ℓ^∞-gain minimization is solved in the input-output framework for both LTI and LMRSD systems certain characteristics of their solutions may not be desirable. In particular, considering the ℓ^1-optimal control problem with full state feedback it was shown [4] that, unlike the $\mathcal{H}^\infty$-optimal case, optimal as well as near-optimal controllers can be dynamic and of arbitrarily high order. This result motivated a new, state-space, approach to the ℓ^1 problem when the state is available for feedback. Recent work in [12, 13] towards this direction has shown that static *nonlinear* state feedback performs as well as linear dynamic feedback. In other words, full state feedback ℓ^1-optimal control need not require dynamics if nonlinear controllers are admissible. Moreover, a constructive, finite-step, algorithm for near-optimal nonlinear state feedback is furnished. The approach in the work of [12, 13] is to construct controlled invariant sets in the context of viability theory and differential inclusions (e. g., [1, 2, 6, 10, 11]). It is precisely this work that we generalize to the multirate case in this paper. We show that a memoryless nonlinear controller can be constructed to achieve near-optimal performance.

We note that the method of constructing controlled invariant sets has been used extensively throughout the control literature from a variety of contexts (see [13] and references therein) including dynamic programming, systems with control constraints, construction of reachable sets, and time-varying system analysis.

The remainder of this paper is organized as follows. Section 2 presents some background material. Section 3 presents the problem formulation. Section 4 discusses the notion of a multirate controlled invariance kernel. Section 5 introduces machinery necessary for the construction of an ℓ^1-optimal multirate controller, and outlines an algorithm to construct such a controller. Section 6 presents an explicit formulation of this algorithm and an example, Section 7 contains an example illustrating an application of this algorithm and Section 8 contains some concluding remarks.

2. MATHEMATICAL PRELIMINARIES

First, we give some basic notation: $\mathcal{R}^+$ denotes the set of nonnegative real numbers and $\mathcal{Z}^+$ denotes the set of nonnegative integers. For $M \in \mathcal{R}^{m\times n}$, let $M_{(i,j)}$ denote the ijth element of M, let $M_{(i,:)}$ denote the ith row of M, and let $M_{(:,j)}$ denote the jth column of M. Also, let $M_{(i,j:)}$ denote the portion of the ith row of M which includes the jth through the right-most column. Define $|M_{(i,:)}| := \sum_{j=1}^{n} |M_{(i,j)}|$, and $|M| = \max_i |M_{(i,:)}|$. Similarly for $x \in \mathcal{R}^n$, let x_i denote the ith component of x and define $|x| = \max_i |x_i|$. The appropriate definition of $|\cdot|$ will be apparent from context. Let $\ell_n^\infty(\mathcal{Z}^+)$ denote the set of bounded one-sided sequences in $\mathcal{R}^n$. For $f = \{f(0), f(1), f(2), \ldots\} \in \ell_n^\infty(\mathcal{Z}^+)$, define $\|f\| := \sup_{t\in\mathcal{Z}^+} |f(t)|$. A causal operator $H : \ell_n^\infty(\mathcal{Z}^+) \to \ell_m^\infty(\mathcal{Z}^+)$ is called stable if $\|H\| := \sup_{f\in\ell_n^\infty, f\neq 0} \frac{\|Hf\|}{\|f\|} < \infty$.

A set-valued map $F : X \rightsquigarrow Y$ is a mapping from individual points $x \in X$ to sets $F(x) \subset Y$. The domain of a set-valued map F is defined as $dom(F) = \{x \in X : F(x) \text{ is non-empty}\}$. Finally, we give the definition of lower and upper semicontinuity of a set-valued map which is required in later developments.

Definition 2.1. ([1], p. 56) Let X and Y be Banach spaces. A set-valued map $F : X \rightsquigarrow Y$ is called *lower semicontinuous* if for any $x \in dom(F)$, $y \in F(x)$, and sequence $x_n \in dom(F)$ converging to x, there exists a sequence of elements $y_n \in F(x_n)$ converging to y.

A set-valued map $F : X \rightsquigarrow Y$ is called *upper semicontinuous* if 1) $dom(F)$ is closed and 2) for any $x \in dom(F)$ and any $\varepsilon > 0$, there exists a $\delta > 0$ such that $x' \in dom(F)$ and $||x' - x|| < \Delta$ together imply

$$\sup_{y' \in F(x')} \inf_{y \in F(x)} ||y' - y|| < \varepsilon.$$

In our developments the spaces X and Y in the above definitions will be product spaces of the real numbers $\mathcal{R}$ with itself. It should also be noted that some elements of viability theory will be adapted for use in this paper with multirate systems. For a more complete treatment of viability theory, the interested reader should consult [1, 2].

Let $\mathbf{1}$ denote a column vector of appropriate length with unit elements. For $M \in \mathcal{R}^{z \times n}$ and $m \in \mathcal{R}^n$, let $Set(M, n)$ denote the subset of $\mathcal{R}^n$ associated with M and m, defined by the constraints

$$Set(M) = \{x : M\, x \leq m\}.$$

This notation is used to develop the definition of the *Rack* operator which appears below.

Definition 2.2. Let $M \in \mathcal{R}^{r \times (n+1)}$ and $m \in \mathcal{R}^{n+1}$. Define $Rack[M, m]$ as the set of matrices $\widetilde{M}$ and vectors $\widetilde{m}$ such that

$$v \in Set(\widetilde{M}, \widetilde{m}) \subset \mathcal{R}^n$$

$$\Leftrightarrow$$

$$\begin{pmatrix} v \\ w \end{pmatrix} \in Set(M, m) \subset \mathcal{R}^{n+1}, \quad \text{for some } w \in \mathcal{R}.$$

The *Rack* operator, then, allows a group of constraints on n variables to be rewritten as a group of constraints of the first $n - 1$ variables. Often, it will be necessary to apply the *Rack* operator multiple times upon a single matrix in order to remove multiple variables from the constraints. Accordingly, the notation $Rack^k[M, m]$ will be used to denote k such applications of the *Rack* operator when the removal of k variables from the constraints is desired.

In the sequel the functions $\phi_M^+(\xi)$ and $\phi_M^-(\xi)$, as defined below, will be used to define ℓ^1-optimal control laws.

Definition 2.3. Let $M \in \mathcal{R}^{r\times(n+1)}$ be a matrix and $m \in \mathcal{R}^r$ be an associated vector, such that M and m describe a set of inequalities

$$M\xi \leq m.$$

Define $M_{\rm I} = M_{(:,1:n)}$ and $M_{\rm II} = M_{(:,n+1)}$. For each row of $M_{\rm II}$, let

$$\begin{aligned} Z^+ &= \{j : (M_{\rm II})_j > 0\} \\ Z^- &= \{j : (M_{\rm II})_j < 0\} \\ Z^0 &= \{j : (M_{\rm II})_j = 0\}. \end{aligned} \tag{1}$$

If Z^+ and Z^- are non-empty, the functions $\phi_M^+ : \mathcal{R}^n \to \mathcal{R}$ and $\phi_M^- : \mathcal{R}^n \to \mathcal{R}$ are defined such that

$$\begin{aligned} \phi_M^+(\xi) &= \min_{j^+\in Z_i^+} \frac{(m)_{j^+} - (M_{\rm I})_{(j^+,:)}\xi}{(M_{\rm II})_{(j^+)}} \\ \phi_M^-(\xi) &= \max_{j^-\in Z_i^-} \frac{(m)_{j^-} - (M_{\rm I})_{(j^-,:)}\xi}{(M_{\rm II})_{(j^-)}}. \end{aligned} \tag{2}$$

The functions $\phi_M^+(\xi)$ and $\phi_M^-(\xi)$ are intimately connected with the *Rack* operator, as may be seen by comparing Definition 2.3 with the Fourier–Motzkin algorithm contained in [13]. In fact, these functions provide upper and lower bounds upon a variable which has been removed by use of the *Rack* operator, such that

$$v \in Set(\widetilde{M}, \widetilde{m}) \subset \mathcal{R}^n \quad \text{and} \quad w \in \{w : \phi_M^-(v) \leq w \leq \phi_M^+(v)\} \subset \mathcal{R}$$
$$\Updownarrow$$
$$\begin{pmatrix} v \\ w \end{pmatrix} \in Set(M, m) \subset \mathcal{R}^{n+1}.$$

Note the form of the functions ϕ_M^+ and ϕ_M^- depend upon both M and m, although only the matrix M is explicitly indicated by the notation. However, in practice, there will be a unique vector m corresponding to each M used to formulate the functions ϕ_M^+ and ϕ_M^-.

3. PROBLEM FORMULATION

In this paper, the ℓ^1-optimal control problem for a linear multirate system with state feedback available is considered. The system equations are given by

$$\begin{aligned} x(t+1) &= Ax(t) + Ew(t) + Bu(t) \\ z(t) &= C_1x(t) + D_{11}w(t) + D_{12}u(t) \end{aligned} \tag{3}$$

where $x \in \mathcal{R}^n$ contains the state of the system, $w(t) \in \mathcal{R}^+$ contains exogenous inputs, $u(t) \in \mathcal{R}^m$ contains control inputs, $z(t) \in \mathcal{R}^p$ contains regulated outputs. The measured inputs of the system (i.e. the states) are sampled at rates of l_1T, $l_2T, \ldots, l_nT$.

The control inputs are delivered to the system at rates of k_1T, $k_2T, \ldots, k_mT$. It is assumed that T is the least common sampling interval among all of the inputs and the outputs. The noise enters the system discretely at a rate of T time units. The assumption that the noise enters at the fastest sampling rate simplifies the solution of the ℓ^1-optimal control problem, but is can be removed through straightforward extensions of the algorithms which appear in this paper. It is also assumed that the sampling intervals of the inputs and the outputs are all synchronized, such that the jump discontinuities in the inputs and the outputs occur at the same time instant. State feedback is assumed to be available such that the measured outputs (denoted as $y \in \mathcal{R}^n$) are the states.

The controllers, $\mathcal{K}_{\text{multi}}$, which are *admissible* for this systems are memoryless multirate controllers which are, in general, a nonlinear function of the state. By memoryless, it is meant that the controllers may be defined without introducing additional state variables to the system. And, multirate refers to the above stipulation that the inputs and the regulated outputs may appear at different rates.

Given an admissible controller, $\mathcal{K}_{\text{multi}}$, define $T_{zw}(\mathcal{K}_{\text{multi}})$ to be the forced dynamics from w to z with zero initial conditions. Similarly define $T_{xw}(\mathcal{K}_{\text{multi}})$ and $T_{uw}(\mathcal{K}_{\text{multi}})$.

Definition 3.1. An admissible multirate controller, $\mathcal{K}_{\text{multi}}$, is said to be internally stabilizing with a performance (resp., strict performance) of γ if 1) the unforced dynamics ($w = 0$) are globally exponentially stable and 2) the forced dynamics with zero initial conditions satisfy $||T_{zw}(\mathcal{K}_{\text{multi}})|| \leq \gamma$, (resp., $||T_{zw}(\mathcal{K}_{\text{multi}})|| < \gamma$), with both $||T_{xw}(\mathcal{K}_{\text{multi}})||$, $||T_{zw}(\mathcal{K}_{\text{multi}})|| < \infty$.

The optimum performance problem can now be postulated as

$$\gamma_{\text{opt}} = \inf_{\mathcal{K}_{\text{multi}}} \{||T_{zw}(k)|| : \mathcal{K}_{\text{multi}} \text{ is admissible and internally stabilizing}\}.$$

We point out that arbitrary time variation does not offer any advantage over multirate if the controller is linear [3]. Moreover, it can be deduced from the developments of Section 5 and Section 6 that a memoryless nonlinear controller can at least match the performance of any linear one. In fact, if may perform better [15]. Finally, it also can be concluded from the results of Section 5 and Section 6 that a dynamic controller does not outperform a memoryless periodic one. Hence, the class of admissible controllers is not restrictive.

4. MULTIRATE CONTROLLED INVARIANCE

In this section, the concept of a multirate controlled difference inclusion, which may be used to represent a dynamic system, is introduced. For a particular multirate controlled difference inclusion, the structures which are of particular interest are multirate controlled invariant sets. If a multirate system begins within such a set, then it will be confined to that set for all time under the action of the associated controlled difference inclusion. This invariance property will be exploited in the construction of a controller which solves the stated ℓ^1-optimal control problem. Due to the requirements of this control law construction method which will be detailed in Sections 5 and 6, it is necessary to consider simultaneously the behavior of the multirate system at each step of a time interval of R time steps (i.e. RT time units), where $R = LCM(l_1, l_2, \ldots, l_n, k_1, k_2, \ldots, k_n)$. As a consequence, the definition of a controlled difference inclusion must be appropriately adapted in order to be used to model multirate systems. Specifically, it must be altered such that the behavior of the multirate system for R time steps is described. This requirement is met by the following definition.

Definition 4.1. Let $F : \mathcal{R}^n \times \mathcal{R}^{R_m} \rightsquigarrow \mathcal{R}^n$ be a set valued map. Define

$$\widetilde{F}(x) = \left\{ \bigcup_{\substack{u^i \in \mathcal{R}^m \\ i \in \{0,\ldots,R-1\}}} F(x, u^0, \ldots, u^{R-1}) \right\}.$$

Then, $x(j + R) \in \widetilde{F}(x(j))$ is the *multirate controlled difference inclusion* defined by F.

In the above definition, the variables $u^0, \ldots, u^{R-1}$ represent the control inputs at times $R_j, \ldots, R_{j+(R-1)}$. Also, the time interval described by a multirate controlled difference inclusion will always begin and end at time steps at which the system has access both to all the states and to all the controls. Note that the shortest length of time between such time steps is in fact R time steps. Another important detail of the above definition is that the output of the multirate system can only be considered every R steps when modeled with a multirate controlled difference inclusion. However, when applied to an ℓ^1-optimal control problem, multirate controlled difference inclusions clearly also must satisfy the required bounds on the outputs of intermediate steps. This will be accomplished by appropriately defining the set-valued map $F(x, u^0, \ldots, u^{R-1})$. Finally, note that, while the above definition accomodates multirate controllers and systems, it does not explicitly restrict the system or the controller to be multirate. This over-generality will also be addressed in the sequel by appropriately defining the set valued map $F(x, u^0, \ldots, u^{R-1})$.

As previously indicated, the concept of the controlled invariance of a multirate controlled difference inclusion is integral to the construction of an ℓ^1-optimal control law. The essential idea is to define a set which will insure that the required output ℓ^∞-norm bounds are met and to then search for the largest subset to which the multirate system can be confined under some admissible control law for all time. If such a set exists, then an ℓ^1-optimal controller can be constructed. Formally, a controlled invariant set for a multirate controlled difference inclusion satisfies the following definition.

Definition 4.2. Consider the multirate controlled difference inclusion defined by F. A set $K \subset \mathcal{R}^n$ is *multirate controlled invariant* under F if $\forall\, x \in K$, there exists $u^i \in \mathcal{R}^m$, $i \in \{0, \ldots, R-1\}$, such that $F(x, u^0, \ldots, u^{R-1}) \subset K$.

Clearly, it is desirable to find the "largest" multirate controlled invariant set of a particular multirate controlled difference inclusion. Therefore, an important type of multirate controlled invariant set is the multirate controlled invariance kernel, which is the largest multirate controlled invariant set in the sense given by the below definition.

Definition 4.3. Consider the multirate controlled difference inclusion defined by F. Let the set K be a subset of $\mathcal{R}^n$. The *multirate controlled invariance kernel* of K for F, denoted as $\text{CINV}(K)_R$, is the largest closed subset of K such that for all $x \in \text{CINV}(K)_R$, there exist $u^i \in \mathcal{R}^m$, $i \in \{0,\ldots,R-1\}$, such that $F(x,u^0,\ldots,u^{R-1}) \subset \text{CINV}(\text{K})_R$. Here, the term largest implies that $\text{CINV}(K)_R$ contains all other closed subsets of K with the above invariance property.

An algorithm for the construction of the multirate controlled invariance kernel is given in the following proposition, which follows almost immediately from the version of the Controlled Invariance Kernel Algorithm contained in [4].

Proposition 4.1. Let $F : \mathcal{R}^n \times \mathcal{R}^{Rm} \rightsquigarrow \mathcal{R}^n$ be a lower semicontinuous set valued map. Also, assume that the set

$$\bigcup_n F\left(x_n, u_n^0, \ldots, u_n^{R-1}\right)$$

is bounded if and only if the sequence $\{u_n^i\} \in \mathcal{R}$, $i \in \{0,\ldots,R-1\}$ and $x_n \in \mathcal{R}^n$ are bounded. Let $K \subset \mathcal{R}^n$ be a compact set. Define $K_0 = K$, and recursively define the subsets K_{R_j} of K, for $j = 1,2,\ldots$, by

$$\begin{aligned} K_{R_j} &= \Big\{x \in K_{R(j-1)} : F(x,u^0,\ldots,u^{R-1}) \subset K_{R(j-1)}, \\ &\qquad\qquad \text{with } u^i \in \mathcal{R}^m,\ i \in \{0,\ldots,R-1\}\Big\}. \end{aligned}$$

Then

$$\text{CINV}(K)_R = \bigcap_{j=0}^{\infty} K_{R_j}.$$

The construction of a multirate controlled invariance kernel $\text{CINV}(K)_R$ is integral to the construction of the ℓ^1-optimal control law developed in this section for multirate systems. It is important to note that in the most general sense, the definition of multirate controlled difference inclusions allows the control input to be non-causal and to depend upon unavailable state information. As discussed in the following section, this potential difficulty can be avoided by imparting to the multirate controlled difference inclusion a form which depends upon the particular multirate system of interest.

5. FORMULATION OF MULTIRATE CONTROLLED DIFFERENCE INCLUSION

In this section, the multirate controlled invariant set $\text{CINV}(\text{OBJECT}_\gamma^0)$ is defined and its role in the construction of an admissible ϵ-suboptimal multirate controller is described. The following assumptions are made for the remaining discussion of multirate systems in order to simplify the construction of the controller and the arguments of the proofs which follow.

Assumption 5.1.

1. $rank(E) = rank(C_1(t)) = n$
2. $rank(B(t)) = m$.

The first two assumptions simplify greatly the construction of the control law. It should be noted that is possible to remove the rank assumption on E with arbitrarily small perturbations to E. The rank assumption on each C_1 may also be removed, but this must be done by introducing new, non-trivial outputs in order to avoid numerical difficulties and to insure a reasonable bound on the plant states. The final assumption insures that there will be no control redundancies, and it may be removed by arbitrarily small perturbations to B.

For most remainder of the discussion of the construction of ℓ^1-optimal control laws for multirate systems, it also will be assumed that *both the states and control input only have rates of T and $2T$*, such that $R = 2$. The states and control variables with the same sampling rates will be grouped together, such that x_1 (u_1) contains all states (control inputs) which appear at rates of T, and x_2 (u_2) contains all states (control inputs) which appear at rates of $2T$. Also, define $n_{x_1} := \dim(x_1)$, $x_{x_2} := \dim(x_2)$, $n_{u_1} := \dim(u_1)$, $n_{u_2} := \dim(u_2)$. This assumption that the multirate system possesses only sampling rates of T and $2T$ will greatly simplify and clarify the presentation of the multirate control law construction algorithm. But the algorithms presented here may be extended to the general multirate problem, and this extension process is described at the close of Section 6. In the sequel, we will write x and u as

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad u = \begin{pmatrix} u_2 \\ u_1 \end{pmatrix}$$

such that $l_1 = \cdots = l_{n_{x_1}} = k_1 = \cdots = k_{m_{u_1}} = 1$ and $l_{n_{x_1}+1} = \cdots = l_n = k_{m_{u_1}+1} = \cdots = k_m = 2$. Note that u_2 appears above u_1 in u, opposite to the usual manner. This is done to simplify the formulation of the algorithms which follow.

The first step in constructing an ℓ^1-optimal controller is to use the state equations given by (3) to formulate a multirate controlled difference inclusion which will be suitable for use in Proposition 4.1. As previously indicated, this multirate controlled difference inclusion must be peculiarly defined in order to insure that the resulting controller is causal and that only available state information is used to produce control inputs.

To understand these difficulties, suppose a multirate system begins at an even time step $2j$. Then, defining $x^t := x(t)$, it is clear from (3) that $x^{2j+1} = Ax^{2j} + Bu^{2j} + Ew^{2j}$. It is important to note that since both all states and all control inputs are available at even time steps, as indicated in Section 4, each element of x^{2j} is known and each element of u^{2j} may be prescribed. Therefore, as in the LTI single rate case, the state dynamics from time step $2j$ to time step $2j+1$ can be represented by the following controlled difference inclusion:

$$x^{2j+1} \in \left\{ Ax^{2j} + Bu^{2j} + Ew^{2j}, \text{ for some } u \in \mathcal{R}^m : |w^{2j}| \leq \frac{1}{\gamma} \right\}. \tag{4}$$

Here, as an ℓ^1-optimal controller is sought, a bound on the ℓ^∞-norm of the exogenous noise is assumed, such that $|w^{2j}| \leq \frac{1}{\gamma}$. At even time steps $2j+2$, the state may be written as

$$x^{2j+2} = Ax^{2j+1} + Bu^{2j+1} + Ew^{2j+1}. \tag{5}$$

If the state dynamics from time step $2j+1$ to time $2j+2$ were represented by a controlled difference inclusion in the simple form of (4), the standard LTI single rate technique of [13] could be used to construct a control law. However, unlike (4), some of the elements of the state and the control input are not known at time steps $2j+1$ (i.e. x_2^{2j+1} and u_2^{2j+1}). And the resulting control law, therefore, could be non-causal or utilize unavailable state information. In order to properly construct a controlled difference inclusion for a multirate system, its structure must inherently insure that the resulting controller both will be causal and will not use unavailable state information.

To preserve the causality of the controller, when the controlled difference inclusion is defined to describe the transition from time step $2j+1$ to time step $2j+2$, the unavailable control inputs at time $2j+1$ (i.e. u_2^{2j+1}) may be equated to u_2^{2j}. This insures that the u_2 may only be updated at even time steps. The resulting controlled difference inclusion would then have the following form:

$$x^{2j+1} \in \left\{ Ax^{2j+1} + \left[\begin{array}{cc|cc} B_{\mathrm{I}} & 0 & 0 & B_{\mathrm{II}} \end{array}\right] \begin{pmatrix} u^{2j} \\ u^{2j+1} \end{pmatrix} + Ew^{2j+1}, \right.$$
$$\left. \text{for some } u \in \mathcal{R}^m : |w^{2j+1}| \leq \frac{1}{\gamma} \right\}$$

where the $B = [\, B_{\mathrm{I}}\ B_{\mathrm{II}} \,]$ system matrix has been split according to the dimensions of u_1 and u_2. If the state were fully available at time step $2j+1$, the above controlled difference inclusion could be used to formulate a controller in a manner analogous to the LTI single rate case. But if B_{II} is non-zero, then the controller will still depend in general upon x_2^{2j+1}, which is unavailable. We will, however, in the sequel use this technique of explicitly including u^{2j} in the equation for x^{2j+2} in order to preserve the causality of the controller.

The problem of insuring that the equation for x^{2j+2} is written only in terms of available information is more complicated than insuring causality. However, a solution may be obtained by considering the problem of removing the dependency of the controller upon unknown states (i.e. x_2^{2j+1}) as the problem of "estimating" x_2^{2j+1}. In this context, an estimate for x_2^{2j+1} is a set which contains all possible values of x_2^{2j+1}, given the available information. The best estimate is then the smallest such set, with a single point corresponding to an estimate with zero error. Once an estimate for x_2^{2j+1} has been obtained, we will then seek a control law which will work for any x_2^{2j+1} contained in this estimating set. In constructing this estimating set, elements of the set-valued estimator developed in [14] will be used.

The first step is to insuring that the controller will not depend on x^{2j+2} is to produce a set-valued estimate of x_2^{2j+1}. This may be done by modifying the set-valued estimator which appears in [14]. Recall that this algorithm describes a procedure to construct an online, set-valued estimator. At each time step, this set valued estimator produces the set of all possible states, which are consistent with all the available measurements. In general, these measurements could be noise corrupted combinations of the state. It was shown in [14] that the centers of these set valued estimates were optimal estimates in an induced-norm sense. Such a point estimate is not needed in the present case. However, a simplified version of the algorithm of [14] will be used to produce an estimating set for x_2^{2j+1}. The simplifications result from the fact that there is no measurement noise and the fact that due to the assumed two-rate structure of the system, only a single step of the algorithm of [14] is required. (In the case of the general multirate problem, multiple steps of the algorithm of [14] will be required, as will be discussed subsequently.) One significant modification will be made in constructing this estimating set, due to the online nature of the algorithm of [14]. This online algorithm requires specific values for the measurements and the control inputs at each time step. Since an estimating set must be produced off line and since an explicit control law has yet to be chosen, specific values for the measurements and the control inputs will not be available, and, thus, they will be represented symbolically.

We now proceed construct a set-valued estimate for x_2^{2j+1}, by beginning at time step $2j$ at which we have noise free measurements of the entire state. Then, we move one time step forward to construct the set-valued estimate for the state at time step $2j+1$. This can be done by noticing that $x^{2j+1} = Ax^{2j} + Bu^{2j} + Ew^{2j}$. In addition, in the context of ℓ^1-optimal control, a bound on the ℓ^∞-norm of the exogenous noise exists, such that it will be assumed that $|w^{2j}| \le \frac{1}{\gamma}$. This information may be grouped together to form an equivalent matrix inequality in the following manner:

$$\underbrace{\begin{bmatrix} -I & A & B & E \\ I & -A & -B & -E \\ 0 & 0 & 0 & \gamma I \\ 0 & 0 & 0 & -\gamma I \end{bmatrix}}_{:=\overline{M}^{\text{est}}} \begin{pmatrix} x^{2j+1} \\ x^{2j} \\ u^{2j} \\ w^{2j} \end{pmatrix} \le \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}}_{:=\overline{m}^{\text{est}}}. \tag{6}$$

In order to form the estimating set for x_2^{2j+1} we will apply the *Rack* operator to the above set of inequalities in order to produce a set of equivalent inequalities containing only x^{2j+1}, x^{2j}, and u^{2j}. Accordingly, if we write

$$(M^{\text{est}}, m^{\text{est}}) \in Rack^q[\overline{M}^{\text{est}}, \overline{m}^{\text{est}}]$$

then

$$\underbrace{\begin{bmatrix} M_{\text{I}}^{\text{est}} & M_{\text{II}}^{\text{est}} & M_{\text{III}}^{\text{est}} & M_{\text{IV}}^{\text{est}} \end{bmatrix}}_{=M^{\text{est}}} \begin{pmatrix} x_1^{2j+1} \\ x_2^{2j+1} \\ x^{2j} \\ u^{2j} \end{pmatrix} \le m^{\text{est}} \tag{7}$$

is the aforementioned set of inequalities equivalent to (6), which defines the estimating set for x_2^{2j+1}. This may be seen by noticing that, since x^{2j+1}, x^{2j}, and u^{2j} are all known at time $t = 2j + 1$, the only unknown quantity in (7) is x_2^{2j+1}. Therefore, the intersection of all of the inequalities contained in (7), provides limits on the possible values of x_2^{2j+1}. Since (6) contains all available information relevant to producing an estimating set for x_2^{2j+1}, the best possible estimating set for a noise level of $|w^{2j}| \le \frac{1}{\gamma}$ is given by $x_2^{2j+1} \in Est_\gamma\left(x_1^{2j+1}, x^{2j}, u^{2j}\right)$, where

$$Est_\gamma\left(x_1^{2j+1}, x^{2j}, u^{2j}\right) = \left\{ x_2^{2j+1} : \begin{bmatrix} M_{\mathrm{I}}^{\mathrm{est}} & M_{\mathrm{II}}^{\mathrm{est}} & M_{\mathrm{III}}^{\mathrm{est}} & M_{\mathrm{IV}}^{\mathrm{est}} \end{bmatrix} \begin{pmatrix} x_1^{2j+1} \\ x_2^{2j+1} \\ x^{2j} \\ u^{2j} \end{pmatrix} \le m^{\mathrm{est}} \right\}. \tag{8}$$

The above construction process for $Est_\gamma(x_1^{2j+1}, x^{2j}, u^{2j})$ is summarized in the following algorithm.

Algorithm 5.1. (Construction of Estimating Set for x_2^{2j+1} with $|w| \le \frac{1}{\gamma}$.)

1. Specify $\gamma > 0$.
2. Let
$$\overline{M}^{est} = \begin{bmatrix} -I & A & B & E \\ I & -A & -B & -E \\ 0 & 0 & 0 & \gamma I \\ 0 & 0 & 0 & -\gamma I \end{bmatrix} \quad \text{and} \quad m^{est} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$
3. Let
$$\left(M^{est}, m^{est}\right) \in Rack^q\left[\overline{M}^{est}, \overline{m}^{est}\right].$$

Then the estimating set for x_2^{2j+1} for a noise level of $|w^{2j}| \le \frac{1}{\gamma}$ is given by

$$Est_\gamma\left(x_1^{2j+1}, x^{2j}, u^{2j}\right) = \left\{ x_2^{2j+1} : M^{est} \begin{pmatrix} x_1^{2j+1} \\ x_2^{2j+1} \\ x^{2j} \\ u^{2j} \end{pmatrix} \le m^{est} \right\}.$$

Now that an explicit form for the estimating set for x_2^{2j+1} is available, it is possible to insure that the controller will not depend upon x_2^{2j+1} by insuring that the controller will work for any $x_2^{2j+1} \in Est_\gamma(x_1^{2j+1}, x^{2j}, x^{2j})$. This will be accomplished by allowing x_2^{2j+1} in the controlled difference inclusion defined below to assume any value in $Est_\gamma(x_1^{2j+1}, x^{2j}, u^{2j})$. By also writing this controlled difference inclusion to insure that causality is maintained, as was done previously, the set-valued map F_γ is obtained, where

$$\begin{aligned} &F_\gamma\left(x^{2j}, u^{2j}, u^{2j+1}\right) \\ &= \Big\{Ax^{2j+1} + \left[\begin{array}{cc|cc} B_{\mathrm{I}} & 0 & 0 & B_{\mathrm{II}} \end{array}\right]\begin{pmatrix} u^{2j} \\ u^{2j+1}\end{pmatrix} + Ew^{2j+1} : \\ &\qquad \left|C_1x^{2j} + D_{11}w^{2j} + D_{12}u^{2j}\right| \le 1,\ \left|C_1w^{2j+1} + D_{11}w^{2j+1} + D_{12}u^{2j+1}\right| \le 1, \\ &\qquad \forall\, x_2^{2j+1} \in Est_\gamma\left(x_1^{2j+1}, x^{2j}, u^{2j}\right),\ \forall\,|w^{2j}| \le \frac{1}{\gamma},\ \text{and}\ \forall\,|w^{2j+1}| \le \frac{1}{\gamma}\Big\} \end{aligned}$$

where

$$x_1^{2j+1} = [A_{11}\ A_{12}]\, x^{2j} + [B_{11}\ B_{12}]\, u^{2j} + E_1\, w^{2j} \tag{9}$$

with $[A_{11}\ A_{12}]$ (resp. $[B_{11}\ B_{12}]$) representing the first n_{x_1} (resp. n_{u_1}) rows of A (resp. B). Note that the multirate controlled difference inclusion defined by F_γ is equivalent to two time steps of the system equations (3) for $\|w\| \le 1/\gamma$ and $\|u\| \le 1$. Also, note that F_γ does not depend on u_2^{2j+1}, since u_2 is only available at times $t = 2j$, $j \in \{0, 1, \ldots\}$. And, since x_2^{2j+1} may take any value in $Est_\gamma(x_1^{2j+1}, x^{2j}, u^{2j})$, the multirate controlled difference inclusion is effectively independent of x_2^{2j+1}.

Now, define the set $\textsc{Object}_\gamma^0$ such that

$$\textsc{Object}_\gamma^0 = \Big\{x \in \mathcal{R}^n : |C_1x + D_{11}w + D_{12}u| \le 1, \quad \text{for some } u \in \mathcal{R}^m \text{ and } \forall\,|w| \le \frac{1}{\gamma}\Big\}. \tag{10}$$

Then, if $\textsc{Object}_\gamma^0$ is non-empty, it may be shown straightforwardly that F_γ satisfies the hypotheses of Proposition 4.1. As a result, Proposition 4.1 may be used to construct the multirate controlled invariance kernel $\textsc{Cinv}(\textsc{Object}_\gamma^0)$, when it exists, of $\textsc{Object}_\gamma^0$. A simple recursive argument following the general structure of Proposition 4.1 may also be used to show that the convexity and compactness of $\textsc{Object}_\gamma^0$ implies the convexity and compactness of $\textsc{Cinv}(\textsc{Object}_\gamma^0)$, when it exists.

As indicated in Section 4, the concept of the multirate controlled invariance kernel is integral to the formation of an ℓ^1-optimal controller. Specifically, the controlled invariance kernel of interest is $\textsc{Cinv}(\textsc{Object}_\gamma^0)$. Clearly, if the state is confined at time steps $2j$ to $\textsc{Object}_\gamma^0$, then *the ℓ^∞-norm of the output at time steps $2j$ will be less than or equal to one.* The ℓ^∞-norm of the output at all intermediate times will also be less than one due to the definition of the multirate controlled difference inclusion F_γ. This ability to bound the ℓ^∞-norm of the output at all times, suggests the following two step algorithm for the construction of an optimal control law.

The first step of the algorithm is to construct the multirate controlled invariance kernel $\textsc{Cinv}(\textsc{Object}_\gamma^0)$ for a particular $\gamma > 0$, using the algorithm described in Proposition 4.1. Practically, $\textsc{Cinv}(\textsc{Object}_\gamma^0)$ will be difficult to form if the infinite intersection $\bigcap_{j=0}^\infty K_{2j}$ does not converge within a finite and suitably small number of steps. An alternative is to truncate the invariance kernel algorithm at a point when additional iterations produce only an incremental change which is small in some sense. This issue will be commented upon in the Section 6. If it is determined that CINV(OBJECT$_\gamma^0$) is empty, then γ has been chosen too small. In fact, it can be shown that if CINV(OBJECT$_\gamma^0$) does not exist for a particular γ, then it is not possible to find a controller with a performance level of γ. Therefore, if CINV(OBJECT$_\gamma^0$) does not exist, γ should be increased, and the algorithm should be re-run.

If γ is not too small, then the second step of the algorithm may be run. This second step is to determine the set of all controls by which the state can be confined within the multirate controlled invariant set CINV(OBJECT$_\gamma^0$). A memoryless multirate controller may then be chosen from this set of potential controls. By construction, this controller will have a performance level of γ. If this performance level is not small enough or a performance level closer to the optimal value is desired, then γ should be decreased by an appropriate value and the algorithm should be re-run from the first step. An explicit description of this two step process is given in the following section.

6. EXPLICIT MULTIRATE CONTROL CONSTRUCTION ALGORITHMS

In this section, explicit algorithms are discussed which may be used to construct memoryless multirate controllers for the two-rate problem in which both the state and the control may only appear at rates of T and $2T$. The extension of this process to the general multirate problem is described at the close of this section. These algorithms require an explicit form of the multirate controlled invariance kernel CINV(OBJECT$_\gamma^0$). As previously discussed, confinement to CINV(OBJECT$_\gamma^0$) is desirable, since, if the state belongs to CINV(OBJECT$_\gamma^0$), then a bound on the ℓ^∞-norm of the output will be insured. The control law is then defined such that it ensures the invariance of CINV(OBJECT$_\gamma^0$).

In Section 6.1, an explicit algorithm is developed to construct CINV(OBJECT$_\gamma^0$). The case of zero D system matrices is addressed first, and then the results are generalized to the case of non-zero D system matrices. In Section 6.2, a control algorithm is formulated which insures that CINV(OBJECT$_\gamma^0$) remains controlled invariant.

6.1. Construction of the multirate controlled invariance kernel

In order to construct CINV(OBJECT$_\gamma^0$), the methodology of Proposition 4.1 will be utilized. Since the state must be confined to OBJECT$_\gamma^0$, the set K_0 in Proposition 4.1 will be set equal to OBJECT$_\gamma^0$. Then, matrices M_{2j} and associated vectors m_{2j} will be computed, such that $K_{2j} = Set(M_{2j}, m_{2j})$, where the K_{2j} are as defined in Proposition 4.1.

6.1.1. Case of $D_{11} = D_{12} = 0$

The first case of the construction of CINV(OBJECT$_\gamma^0$) addressed is that of a system with a D system matrix which is zero (i.e. $D_{11} = D_{12} = 0$). The first step in constructing CINV(OBJECT$_\gamma^0$) for this case is to substitute $D_{11} = 0$ and $D_{12} = 0$

into (10) and (9) in order to obtain simplified versions of OBJECT_γ^0 and F_γ. The result is that

$$\text{OBJECT}_\gamma^0 = \left\{x^{2j+2} : |C_1 x^{2j+2}| \le 1\right\}$$

such that

$$x^{2j+2} \in \text{OBJECT}_\gamma^0 \Longleftrightarrow \begin{bmatrix} C_1 \\ -C_1 \end{bmatrix} x^{2j+2} \le 1.$$

Also

$$F_\gamma\left(x^{2j}, u^{2j}, u^{2j+1}\right) = \Big\{Ax^{2j+1} + \left[\begin{array}{cc|cc} B_{\text{I}} & 0 & 0 & B_{\text{II}} \end{array}\right] \begin{pmatrix} u^{2j} \\ u^{2j+1} \end{pmatrix} + Ew^{2j+1} :$$
$$\left|C_1 x^{2j} \le 1, \right|, \ \left|C_1 x^{2j+1}\right| \le 1, \ \forall\, x_2^{2j+1} \in Est_\gamma\left(x_1^{2j+1}, x^{2j}, u^{2j}\right), \ \forall\, |w^{2j}| \le \frac{1}{\gamma},$$
$$\text{and } \forall\, |w^{2j+1}| \le \frac{1}{\gamma}\Big\}$$

where

$$x_1^{2j+1} = [A_{11}\ A_{12}]\, x^{2j} + [B_{11}\ B_{12}]\, u^{2j} + E_1 w^{2j}. \tag{11}$$

Following Proposition 4.1, we start by initializing $K_0 = \text{OBJECT}_\gamma^0$. Defining

$$M_0 = \begin{pmatrix} C_1 \\ -C_1 \end{pmatrix} \quad \text{and} \quad m_0 = 1$$

it follows that $Set(M_0, m_0) = K_0 = \text{OBJECT}_\gamma^0$.

To construct K_2, it is clear that $x^{2j} \in K_2$ necessarily requires that

$$[A_{11}\ A_{12}]\, x^{2j+1} + [B_{11}\ B_{12}]\, u^{2j+1} + E_1 w^{2j+1} \subset K_0, \quad \forall\, w^{2j+1} \le \frac{1}{\gamma}.$$

This condition may be rewritten in matrix notation as

$$\underbrace{[M_0 A \ \ M_0[B_{\text{I}}\ 0] \ \ M_0[0\ B_{\text{II}}] \ \ M_0 E]}_{:=\overline{M}_0} \begin{pmatrix} x^{2j+1} \\ u^{2j} \\ u^{2j+1} \\ w^{2j+1} \end{pmatrix} \le m_0, \quad \forall\, w^{2j+1} \le \frac{1}{\gamma}. \tag{12}$$

In order to remove the noise variables from the above inequality, the matrix $M_{\frac{1}{2}}$ is defined using a row-by-row analysis. For each row of $\overline{M}_0$, there are three possibilities:

1. If $\frac{1}{\gamma}\left|(M_0 E)_{(l,:)}\right| > (m_0)_l$, the K_2 is empty, and so is $\text{CINV}(\text{OBJECT}_\gamma^0)$.
2. If $\frac{1}{\gamma}\left|(M_0 E)_{(l,:)}\right| = (m_0)_l$, then $\gamma \le \gamma_{\text{opt}}$.
3. If $\frac{1}{\gamma}\left|(M_0 E)_{(l,:)}\right| < (m_0)_l$, then $\gamma < \gamma_{\text{opt}}$.

If $\gamma > \gamma_{\text{opt}}$ for all rows, then define $M_{\frac{1}{2}}$ using a row-by-row analysis such that

$$M_{\frac{1}{2}(l,:)} = \frac{(M_0 A \;\; M_0[B_{\text{I}}\;0] \;\; M_0[0\;B_{\text{II}}])_{(l,:)}}{m(0)_l - \frac{1}{\gamma}\left|(M_0 E)_{(l,:)}\right|}.$$

In order for $x^{2j} \in K_2$ it also will be necessary that $|C_1 x^{2j+1}| \leq 1$. This is equivalent to

$$M_0 x^{2j+1} \leq \mathbf{1}.$$

Splitting $M_{\frac{1}{2}}$ and M_0 according to the columns which correspond to the known and unknown states, the two control variables, and the noise, this new constraint can be appended to those defined by $M_{\frac{1}{2}}$, to yield

$$\underbrace{\begin{bmatrix} (M_0)_{\text{I}} & (M_0)_{\text{II}} & 0 & 0 \\ (M_{\frac{1}{2}})_{\text{I}} & (M_{\frac{1}{2}})_{\text{II}} & (M_{\frac{1}{2}})_{\text{III}} & (M_{\frac{1}{2}})_{\text{IV}} \end{bmatrix}}_{:=\overline{M}_{\frac{1}{2}}=\left[(\overline{M}_{\frac{1}{2}})_{\text{I}}\;(\overline{M}_{\frac{1}{2}})_{\text{II}}\;(\overline{M}_{\frac{1}{2}})_{\text{III}}\;(\overline{M}_{\frac{1}{2}})_{\text{IV}}\right]} \begin{pmatrix} x_1^{2j+1} \\ x_2^{2j+1} \\ u^{2j} \\ u^{2j+1} \end{pmatrix} \leq \underbrace{\begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix}}_{:=\overline{m}_{\frac{1}{2}}}. \tag{13}$$

At this point, we would like to remove u^{2j+1} from the above inequality. In the case of the LTI single rate problem, in which all states are sampled at all time steps, this would be done by applying the *Rack* operator m times (once for each element of u^{2j+1}). This can not be done in the case of the multirate systems being considered here. The problem stems from the fact that using the *Rack* operator to remove control inputs from a set of constraints implicitly allows those controls to depend upon each of the remaining variables. Thus, if the *Rack* operator was used to remove the columns corresponding to u^{2j+1}, there would be an implicit assumption that u^{2j+1} could depend upon x_2^{2j+1}. This would allow the controller to depend upon state information which is not available at time step $2j+1$. To avoid this type of difficulty, it will be necessary to remove x_2^{2j+1} from (13) before the *Rack* operator is applied. This will be done in a manner similar to that which was used to remove w^{2j+1} in order to create $M_{\frac{1}{2}}$. That is, a row-by-row analysis of $\overline{M}_{\frac{1}{2}}$ will be used to identify a scalar region over which each row of $\left(\overline{M}_{\frac{1}{2}}\right)_{\text{II}}$ varies. Then we insure that (13) holds for all possible values of $x_2^{(2j+1)} \in Est_\gamma\left(x_1^{2j+1}, x^{2j}, u^{2j}\right)$. Note that this will satisfy the inherent requirement of F_γ that $F_\gamma \subset K_0$ for all possible values of x_2^{2j+1}.

The first step of removing the x_2^{2j+1} variables is to identify the combinations of the x_2^{2j+1} variables which must be estimated. Since each row of (13) has to hold independently, it is necessary to determine the range over which each nonzero row of $\left(\overline{M}_{\frac{1}{2}}\right)_{\text{II}}$ varies. To this end, let

1. $\overline{M}^0_{\frac{1}{2}}$ contain each row $\left(\overline{M}_{\frac{1}{2}}\right)_{(j,:)}$ with $\left|\left((\overline{M}_{\frac{1}{2}})_{\text{II}}\right)_{(j,:)}\right| = 0$,

2. $\overline{M}^{\emptyset}_{\frac{1}{2}}$ contain each row $\left(\overline{M}_{\frac{1}{2}}\right)_{(j,:)}$ with $\left|\left((\overline{M}_{\frac{1}{2}})_{\text{II}}\right)_{(j,:)}\right| \neq 0$.

Using this notation and splitting $\overline{M}^{\emptyset}_{\frac{1}{2}}$ and $\overline{M}^{0}_{\frac{1}{2}}$ in the obvious ways, (13) may be rewritten in the following form

$$\begin{bmatrix} (\overline{M}^{0}_{\frac{1}{2}})_{\text{I}} & 0 & (\overline{M}^{0}_{\frac{1}{2}})_{\text{III}} & (\overline{M}^{0}_{\frac{1}{2}})_{\text{IV}} \\ (\overline{M}^{\emptyset}_{\frac{1}{2}})_{\text{I}} & (\overline{M}^{\emptyset}_{\frac{1}{2}})_{\text{II}} & (\overline{M}^{\emptyset}_{\frac{1}{2}})_{\text{III}} & (\overline{M}^{\emptyset}_{\frac{1}{2}})_{\text{IV}} \end{bmatrix} \begin{pmatrix} x_1^{2j+1} \\ x_2^{2j+1} \\ u^{2j} \\ u^{2j+1} \end{pmatrix} \leq 1. \tag{14}$$

The vector *est*, which contains each variable which must be estimated, may then be defined as

$$est := \left(\overline{M}^{\emptyset}_{\frac{1}{2}}\right)_{\text{II}} x_2^{2j+1}.$$

Let the dimension of *est* be $\dim(est) = e$.

We now need to determine scalar regions over which each est_i, $i \in \{1, \dots, e\}$ varies. To accomplish this, we first form an individual estimating set for each est_i and the available information (i. e. x_1^{2j+1}, x^{2j}, u^{2j}). Let Est_i be the estimating set for each est_i, for all $i = 1, \dots, e$. Then, using the notation of (8) we may write

$$(M_{Est_i}, m_{Est_i})$$

$$\in \; Rack^{n_{x_2}} \left[\begin{bmatrix} 0 & M_{\text{I}}^{est} & M_{\text{III}}^{est} & M_{\text{IV}}^{est} & M_{\text{II}}^{est} \\ 1 & 0 & 0 & 0 & \left((\overline{M}^{\emptyset}_{\frac{1}{2}})_{\text{II}}\right)_{(i,:)} \\ -1 & 0 & 0 & 0 & \left((\overline{M}^{\emptyset}_{\frac{1}{2}})_{\text{II}}\right)_{(i,:)} \end{bmatrix}, \begin{pmatrix} m^{est} \\ 0 \\ 0 \end{pmatrix} \right]$$

and

$$Est_i = \left\{ est_i : M_{Est_i} \begin{pmatrix} est_i \\ x_1^{2j+1} \\ x^{2j} \\ u^{2j} \end{pmatrix} \leq m_{Est_i} \right\}.$$

As the estimating set of est_i, Est_i, is the set of all possible values of est_i, given the available information, just as Est_γ contained all possible values of x_2^{2j+1}. Each row of Est_i defines an inequality constraint upon est_i, and, therefore, each row of Est_i corresponds to either an upper bound or a lower bound on the value of est_i. Taking the intersection of all of the upper bounds produces the desired upper bound on est_i. However, since it will not be known a *priori* which individual upper bound is the most restrictive, this upper bound will not have a constant functional form. To produce upper bounds with a constant functional form, we will divide the space of the measured data (i. e. x_1^{2j+1}, x^{2j}, and u^{2j}) into regions in which the upper bounds on all the est_i variables have a constant functional form. To that end, define the regions $Region(l^R)$, $l^R \in \{1, \dots, L^R\}$, $L^R \in \mathcal{Z}^+$ such that each est_i may be bounded by a function with a constant form within each $Region(l^R)$, such that we may write

$$est_i \le a_i^{l^R} + b_i^{l^R} \left(x_1^{2j+1} \; x^{2j} \; u^{2j} \right)', \quad \forall i \in \{1, \dots, e\} \tag{15}$$

for all $(x_1^{2j+1}, x^{2j}, u^{2j}) \in Region(l^R)$, where each $a_i^{l^R}$ is a scalar constant and each $b_i^{l^R}$ is a constant vector. These regions may be obtained from the inequality constraint descriptions of each Est_i by utilizing algebraic manipulations and linear programming methods.

By sectioning the space of the measured data into regions, it is possible to remove the unmeasured variables, the x_2^{2j+1} variables, from (14) in a row-by-row manner similar to that by which the noise variables were removed from (12). Over each region corresponding to some index l^R, we want to insure that (14) will hold for any $est \in Est$. Therefore, for each row of $\overline{M}_{\frac{1}{2}}^{\emptyset}$, let

$$mr_1^{\emptyset}(l^R)_i = 1 - a_i^{l^R}$$

and

$$MR_1^{\emptyset}(l^R)_{(i,:)} = \left[\left((M_{\frac{1}{2}}^{\emptyset})_{\mathrm{I}} \right)_{(i,:)} \; \left((M_{\frac{1}{2}}^{\emptyset})_{\mathrm{III}} \right)_{(i,:)} \; \left((M_{\frac{1}{2}}^{\emptyset})_{\mathrm{IV}} \right)_{(i,:)} \right] + [b_i^{l^R} \; 0].$$

Also, let

$$mr_1^0 = 1 \quad \text{and} \quad MR_1^0 = \left[\left(M_{\frac{1}{2}}^{\emptyset} \right)_{\mathrm{I}} \; \left(M_{\frac{1}{2}}^{\emptyset} \right)_{\mathrm{III}} \; \left(M_{\frac{1}{2}}^{\emptyset} \right)_{\mathrm{IV}} \right].$$

Note that MR_1^0 and mr_1^0 will be the same over each $Region(l^R)$. Using the above notation, the following inequality holds for each $Region(l^R)$:

$$\underbrace{\begin{bmatrix} MR_1^0(l^R) \\ MR_1^{\emptyset}(l^R) \end{bmatrix}}_{:=MR_1(l^R)} \begin{pmatrix} x_1^{2j+1} \\ x^{2j} \\ u^{2j} \\ u^{2j+1} \end{pmatrix} \le \underbrace{\begin{pmatrix} mr_1^0(l^R) \\ mr_1^{\emptyset}(l^R) \end{pmatrix}}_{:=mr_1(l^R)}. \tag{16}$$

Once $MR_1(l^R)$ and $mr_1(l^R)$ have been constructed for each $Region(l^R)$, the constraints no longer depend upon x_2^{2j+1}. Therefore, the u^{2j+1} control variables may be removed by using the *Rack* operator without forcing the control law to depend upon x_2^{2j+1}. For each region corresponding to some index l^R, define the matrix $M_1(l^R)$ and the vector $m_1(l^R)$, such that

$$(M_1(l^R), m_1(l^R)) \in Rack^m \left[MR_1(l^R), mr_1(l^R) \right]. \tag{17}$$

Dividing $M_1(l^R)$ in the obvious manner results in the following set of constraints

$$\left[(M_1(l^R))_{\mathrm{I}} \; (M_1(l^R))_{\mathrm{II}} \; (M_1(l^R))_{\mathrm{III}} \right] \begin{pmatrix} x_1^{2j+1} \\ x^{2j} \\ u^{2j} \end{pmatrix} \le m_1(l^R).$$

Note that the *Rack* operator is not applied $2m$ times to (13), as this also would remove the columns which correspond to u^{2j}. As previously discussed, such a use of the *Rack* operator would implicitly allow u^{2j} to depend upon x_1^{2j+1}, yielding a non-causal control law. To remove u^{2j} from the above constraints using the *Rack* operator (and, thereby, yielding constraints only in terms of the states) without producing a non-causal controller, x^{2j+1} must be written in terms of x^{2j}, u^{2j}, and w^{2j} by using the system equations. This yields, for each region corresponding to some index l^R, the following inequality

$$\begin{aligned}&\left[(M_1(l^R))_{\mathrm{I}}\,[A_{11}\ A_{12}] + (M_1(l^R))_{\mathrm{II}}\right] x^{2j}\\ &+ \left[(M_1(l^R))_{\mathrm{I}}\,[B_{11}\ B_{12}] + (M_1(l^R))_{\mathrm{III}} \mid (M_1(l^R))_{\mathrm{I}}\,E_1\right] \begin{pmatrix} u^{2j} \\ --- \\ w^{2j} \end{pmatrix} \le m_1(l^R).\end{aligned} \tag{18}$$

To this inequality, the final condition necessary to insure that $x^{2j} \in K_2$, namely that $|C_1 x^{2j}| \le 1$, must be added. This condition is equivalent to

$$M_0\, x^{2j} \le \mathbf{1}.$$

Appending this inequality to (18) yields

$$\overline{M}_1(l^R)_{\mathrm{I}} \begin{pmatrix} x^{2j} \\ --- \\ u^{2j} \end{pmatrix} + \overline{M}_1(l^R)_{\mathrm{II}}\, w^{2j} \le \overline{m}_1(l^R) \tag{19}$$

where

$$\begin{aligned}&\overline{M}_1(l^R)_{\mathrm{I}}\\ &= \begin{bmatrix} M_0 & 0 \\ (M_1(l^R))_{\mathrm{I}}\,[A_{11}\ A_{12}] + (M_1(l^R))_{\mathrm{II}} & (M_1(l^R))_{\mathrm{I}}\,[B_{11}\ B_{12}] + (M_1(l^R))_{\mathrm{III}} \end{bmatrix}\end{aligned}$$

$$\overline{M}_1(l^R)_{\mathrm{II}} = \begin{bmatrix} 0 \\ (M_1(l^R))_{\mathrm{I}}\,E_1 \end{bmatrix}$$

$$\overline{m}_1(l^R) = \begin{pmatrix} \mathbf{1} \\ m_1(l^R) \end{pmatrix}.$$

The next step is to remove w^{2j} from (19) for each *Region*(l^R). This will be done in a row-by-row manner similar to that used to remove w^{2j+1} from (12), although there is one critical difference. In the latter case, it was simply assumed that $|w^{2j+1}| \le \frac{1}{\gamma}$. In the present instance, inside of each region associated with some index l^R, the noise is actually more restricted. This is due to the fact that inequalities defining each region (i.e. (15)) restrict the value that x^{2j+1} may have relative to the values of x^{2j} and u^{2j}. This clearly restricts the value that the noise w^{2j} may hold.

The unique relationship that each *Region*(l^R) has with some region in the space of the w^{2j} variables also may be seen from the definition of the *Rack* operator, which was first used to construct each Est_i. By the definition of the *Rack* operator, for each x^{2j+1}, x^{2j}, and u^{2j} which satisfy the conditions of Est_i, there corresponds some w^{2j} with $|w^{2j}| \leq \frac{1}{\gamma}$. Tracing the consequences of this fact through the formulation of each *Region*(l^R), the unique relationship of each *Region*(l^R) with some region of space of the w^{2j} variables may easily be seen.

In order to remove w^{2j} from (19) it is necessary to determine a scalar bound on each row of $(\overline{M}_1(l^R))_{\mathrm{II}}$. Due to the unique relationship that each *Region*(l^R) has with some region in the space of the w^{2j} variables, such a bound may be found by using algebraic manipulations of the constraints which define each *Region*(l^R) and the state equations (3). The result is that for each *Region*(l^R), we may write

$$\left((\overline{M}_1(l^R))_{\mathrm{II}}\right)_{(i,:)} \leq w_{\max}(l^R)_i, \quad \forall i.$$

It is now possible to remove w^{2j} from (19) by defining the matrix $M_{1\frac{1}{2}}(l^R)$ such that

$$M_{1\frac{1}{2}}\left(l^R\right)_{(i,:)} = \frac{\left((\overline{M}_1(l^R))_{\mathrm{I}}\right)_{(i,:)}}{(\overline{m}_1(l^R))_i - \frac{1}{\gamma}\, w_{\max}\left(l^R\right)_i}.$$

The next step is to remove u^{2j} from the constraints defined by $M_{1\frac{1}{2}}(l^R)$ in order to yield a set of constraints, which depend only upon x^{2j}. These constraints will define K_2, to whose construction the entire above process has been leading. To effect this removal, one must essentially determine the set of states for which a control law exists, which will insure that the constraints defined by $M_{1\frac{1}{2}}(l^R)$ are satisfied, no matter which *Region*(l^R) occurs. Since it is not known *a priori* which *Region*(l^R) region will occur, the control must be chosen such that *Set* $\left(M_{1\frac{1}{2}}(l^R), \mathbf{1}\right)$ is satisfied for all *Region*(l^R). To insure this, define matrix $\overline{M}_{1\frac{1}{2}}$ such that

$$Set\left(\overline{M}_{1\frac{1}{2}}, \mathbf{1}\right) = \bigcap\nolimits_{l^R} Set\left(\overline{M}_{1\frac{1}{2}}(l^R), \mathbf{1}\right).$$

This intersection may be effected by simply collecting all the constraints defined by each $M_{1\frac{1}{2}}(l^R)$ into a single matrix inequality. The redundant constraints could then be removed by using a linear program in order to simplify later computations.

Before finally constructing K_2, note that by definition $x^{2j} \in K_2$ only if $x^{2j} \in K_0$. Therefore, if M_2 and m_2 are defined such that

$$M_2 = \begin{pmatrix} M_0 \\ N_2 \end{pmatrix} \quad \text{and} \quad m_2 = \begin{pmatrix} m_0 \\ n_2 \end{pmatrix}$$

where

$$(N_2, n_2) \in Rack^m \left[\overline{M}_{1\frac{1}{2}}, \mathbf{1}\right]$$

then $Set(M_2, m_2) = K_2$. Given M_2, the above process may be repeated recursively to yield K_{2j}, $j \in \{2, 3, \ldots\}$. As indicated in Proposition 4.1, the intersection of this infinite sequences of sets equals $\mathrm{CINV}(\mathrm{OBJECT}^0_\gamma)$.

The above iterative construction process is summarized in the following algorithm.

Algorithm 6.1 ($D_{11} = D_{12} = 0$ Control)

1. Specify $\gamma > 0$ and initialize
$$M_0 = \begin{pmatrix} C_1 \\ -C_1 \end{pmatrix} \quad \text{and} \quad m_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$
2. Let the index $j = 0$.
3. For each row of M_j,
   (a) If $\frac{1}{\gamma}\left|(M_j E)_{(l,:)}\right| \geq (m_j)_l$, increase γ and restart.
   (b) Otherwise, set
$$\begin{aligned} \left(M_{j\frac{1}{2}}\right)_{(l,:)} &= \frac{(M_j A \ \ M_j[B_{\mathrm{I}}\ 0] \ \ M_j[0\ B_{\mathrm{II}}])_{(l,:)}}{(m_j)_l - \frac{1}{\gamma}\left|(M_j E)_{(l,:)}\right|} \\ \left(m_{j\frac{1}{2}}\right)_l &= \mathbf{1}. \end{aligned}$$
4. Define $\overline{M}_{j\frac{1}{2}}$ and $\overline{m}_{j\frac{1}{2}}$ as
$$\begin{aligned} \overline{M}_{j\frac{1}{2}} &= \begin{bmatrix} [M_0\ 0] \\ M_{j\frac{1}{2}} \end{bmatrix} \\ \overline{m}_{j\frac{1}{2}} &= \mathbf{1}. \end{aligned}$$
5. Let
   (a) $\overline{M}^0_{j\frac{1}{2}}$ contain each row $\left(\overline{M}_{j\frac{1}{2}}\right)_{(i,:)}$ with $\left|\left((\overline{M}_{j\frac{1}{2}})_{\mathrm{II}}\right)_{(i,:)}\right| = 0$.
   (b) $\overline{M}^{\emptyset}_{j\frac{1}{2}}$ contain each row $\left(\overline{M}_{j\frac{1}{2}}\right)_{(i,:)}$ with $\left|\left((\overline{M}_{j\frac{1}{2}})_{\mathrm{II}}\right)_{(i,:)}\right| \neq 0$.
6. Use the set valued estimate $Est_\gamma\left(x_1^{2j+1}, x_2^{2j}, u^{2j}\right)$ to form the regions $Region(l^R)$, $l^R \in \{1, \ldots, L^R\}$, $L^R \in \mathcal{Z}^+$ within each $Region(l^R)$, such that we may write
$$\begin{aligned} &est_i \leq a_i^{l^R} + b_i^{l^R}\left(x_1^{2j+1}\ x^{2j}\ u^{2j}\right)', \\ &\forall\, i \in \{1, \ldots, e\},\ \forall\left(x_1^{2j+1}, x^{2j}, u^{2j}\right) \in Region(l^R). \end{aligned}$$
7. For each l^R, let
$$mr^{\emptyset}_{j+1}(l^R)_i = 1 - a_i^{l^R}$$
and
$$MR^{\emptyset}_{j+1}(l^R)_{(i,:)} = \left[\left((\overline{M}^{\emptyset}_{j\frac{1}{2}})_{\mathrm{I}}\right)_{(i,:)} \left((\overline{M}^{\emptyset}_{j\frac{1}{2}})_{\mathrm{III}}\right)_{(i,:)} \left((\overline{M}^{\emptyset}_{j\frac{1}{2}})_{\mathrm{IV}}\right)_{(i,:)}\right] - [b_i^{l^R}\ 0]$$
$$mr^0_{j+1} = 1 \quad \text{and} \quad MR^0_{j+1} = \left[\left(\overline{M}^{\emptyset}_{j\frac{1}{2}}\right)_{\mathrm{I}} \left(\overline{M}^{\emptyset}_{j\frac{1}{2}}\right)_{\mathrm{III}} \left(\overline{M}^{\emptyset}_{j\frac{1}{2}}\right)_{\mathrm{IV}}\right].$$

8. For each l^R let

$$\underbrace{\begin{bmatrix} MR^0_{j+1}(l^R) \\ MR^{\emptyset}_{j+1}(l^R) \end{bmatrix}}_{:=MR_{j+1}(l^R)} \begin{pmatrix} x_1^{2j+1} \\ x^{2j} \\ u^{2j} \\ u^{2j+1} \end{pmatrix} \leq \underbrace{\begin{pmatrix} mr^0_{j+1}(l^R) \\ mr^{\emptyset}_{j+1}(l^R) \end{pmatrix}}_{:=mr_{j+1}(l^R)}. \tag{20}$$

9. For each l^R, define M_{j+1} and m_{j+1} by

$$(M_{j+1}(l^R),\, m_{j+1}(l^R)) \in Rack^m \left[MR_j(l^R),\, mr_j(l^R)\right].$$

10. For each l^R, define $\overline{M}_{j+1}(l^R)$ and $\overline{m}_{j+1}(l^R)$ as

$$\begin{aligned} \overline{M}_{j+1}(l^R) &= [(\overline{M}_{j+1}(l^R))_{\mathrm{I}a}\ (\overline{M}_{j+1}(l^R))_{\mathrm{I}b}\ (\overline{M}_{j+1}(l^R))_{\mathrm{II}}] \\ \overline{m}_{j+1}(l^R) &= \begin{pmatrix} \mathbf{1} \\ m_{j+1}(l^R) \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} (\overline{M}_{j+1}(l^R))_{\mathrm{I}a} &= \begin{bmatrix} M_0 \\ (M_{j+1}(l^R))_{\mathrm{I}}\,[A_{11}\ A_{12}] + (M_{j+1}(l^R))_{\mathrm{II}} \end{bmatrix} \\ (\overline{M}_{j+1}(l^R))_{\mathrm{I}b} &= \begin{bmatrix} 0 \\ (M_{j+1}(l^R))_{\mathrm{I}}\,[B_{11}\ B_{12}] + (M_{j+1}(l^R))_{\mathrm{III}} \end{bmatrix} \\ (\overline{M}_{j+1}(l^R))_{\mathrm{II}} &= \begin{bmatrix} 0 \\ (M_{j+1}(l^R))_{\mathrm{I}}\,E_1 \end{bmatrix}. \end{aligned}$$

11. Use the set valued estimate $Est_\gamma\left(x_1^{2j+1}, x^{2j}, u^{2j}\right)$ to determine bounds $w_{\max}(l^R)$ such that

$$\left(\overline{M}_{j+1}(l^R)_{\mathrm{II}}\right)_{(i,:)} \leq w_{\max}(l^R)_i, \quad \forall i.$$

12. For each l^R and each row of $\left(\overline{M}_{j+1}(l^R)\right)_{\mathrm{I}}$, define

$$M_{j+1\frac{1}{2}}(l^R)_{(l,:)} = \frac{\left((\overline{M}_{j+1}(l^R))_{\mathrm{I}}\right)_{(l,:)}}{(\overline{m}_{j+1}(l^R))_l - \frac{1}{\gamma}\, w_{\max}(l^R)_l}.$$

13. Construct $\overline{M}_{j+1\frac{1}{2}}$ such that

$$Set\left(\overline{M}_{j+1\frac{1}{2}}, \mathbf{1}\right) = \bigcap_{l^R \in Z_j^{\text{Nonempty}}} Set\left(M_{j+1\frac{1}{2}}, \mathbf{1}\right).$$

14. Set

$$M_{j+2} = \begin{pmatrix} M_j \\ N_j \end{pmatrix} \quad \text{and} \quad m_{j+2} = \begin{pmatrix} m_j \\ n_j \end{pmatrix}$$

where

$$(N_{j+2}, n_{j+2}) \in Rack^m \left[\overline{M}_{j+1\frac{1}{2}}, \mathbf{1}\right].$$

15. Let $j = j + 2$ and return to Step 3.

The algorithm will restart whenever $\gamma \leq \gamma_{\text{opt}}$. Otherwise,

$$\text{CINV}(\text{OBJECT}_\gamma^0) = \left\{\bigcap\nolimits_{l=0}^{\infty} Set(M_{2j}, m_{2j})\right\}.$$

Note that the construction of the M_{2j} and the m_{2j} does not require the solution of linear programs. However, linear programs are required for a computationally efficient implementation by removing redundant constraints in the matrix description of the various sets in the above algorithm.

6.1.2. Case of D_{11}, $D_{12} \neq 0$

Suppose now that $D_{11} \neq 0$ or $D_{12} \neq 0$. Define

$$V(\zeta) = \left\{\zeta \in \mathcal{R}^p : \left|\zeta + \frac{1}{\gamma} D_{11} w\right| \leq 1, \ \forall\, |w| \leq 1\right\}.$$

Assuming V is non-empty, use a row-by-row analysis to construct $M_{-\frac{1}{2}}$, such that

$$V = Set\left(M_{-\frac{1}{2}}, \mathbf{1}\right).$$

Then

$$\text{OBJECT}_\gamma^0 = \left\{x \in \mathcal{R}^n : M_{-\frac{1}{2}}(C_1 x + D_{12} u) \leq \mathbf{1}, \text{ for some } u \in \mathcal{R}^m\right\}.$$

Defining M_0 and m_0 such that

$$(M_0, m_0) \in Rack^m \left[[M_{-\frac{1}{2}} C_1 \quad M_{-\frac{1}{2}} D_{12}], \mathbf{1}\right]$$

it follows that $Set(M_0, m_0) = \text{OBJECT}_\gamma^0$. Intialize $K_0 = Set(M_0, m_0)$. Note that if $D_{11} = D_{12} = 0$, then $M_{-\frac{1}{2}} = \begin{pmatrix} I_{p\times p} \\ -I_{p\times p} \end{pmatrix}$. In this case, $M_0 = \begin{bmatrix} C_1 \\ -C_1 \end{bmatrix}$ and $m_0 = \mathbf{1}$, which are precisely the definitions given in the $D_{11}(j) = D_{12}(j) = 0$ Control Algorithm.

To construct the sets K_j, $j \in \{1, 2, \ldots\}$, an algorithm which is nearly identical to the $D_{11}(j) = D_{12}(j) = 0$ Control Algorithm may be used. Other than the above alteration to the definition of M_0, alterations need only be made in Algorithm 6.1 at the points at which constraints were added to insure that $||z|| \leq 1$. These conditions were of the form of $|C_1 x^{2j+1}| \leq 1$ and $|C_1 x^{2j}| \leq 1$. Clearly, these constraints will not insure that the ℓ^∞-norm of the regulated outputs remains less than one when the D system matrices are non-zero, as they were intended to do. Therefore, in the case of non-zero D system matrices, the above conditions must be replaced, with, respectively

$$\left| M_{-\frac{1}{2}}[(C_1)_{\rm I} \;\; (C_1)_{\rm II}] \begin{pmatrix} x_1^{2j+1} \\ x_2^{2j+1} \end{pmatrix} + M_{-\frac{1}{2}}[(D_{12})_{\rm I} \;\; 0 \;|\; 0 \;\; (D_{12})_{\rm II}] \begin{pmatrix} u_1^{2j+1} \\ u^{2j} \end{pmatrix} \right| \leq 1$$

and

$$\left| M_{-\frac{1}{2}} C_1 x^{2j} + M_{-\frac{1}{2}} D_{12} u^{2j} \right| \leq 1.$$

Note that the matrix D_{12} has been split in the first expression according to the sizes of u_1 and u_2, in order to insure that the controller is causal. Also, the matrix C_1 has been split according to the sizes of x_1 and x_2. As a result of the above changes, (13) becomes

$$\underbrace{\begin{bmatrix} M_{-\frac{1}{2}}(C_1)_{\rm I} & M_{-\frac{1}{2}}(C_1)_{\rm II} & M_{-\frac{1}{2}}[(D_{12})_{\rm I} \;\; 0] & M_{-\frac{1}{2}}[0 \;\; (D_{12})_{\rm II}] \\ \left(M_{\frac{1}{2}}\right)_{\rm I} & \left(M_{\frac{1}{2}}\right)_{\rm II} & \left(M_{\frac{1}{2}}\right)_{\rm III} & \left(M_{\frac{1}{2}}\right)_{\rm IV} \end{bmatrix}}_{:=\overline{M}_{\frac{1}{2}} = \left[(\overline{M}_{\frac{1}{2}})_{\rm I} \; (\overline{M}_{\frac{1}{2}})_{\rm II} \; (\overline{M}_{\frac{1}{2}})_{\rm III} \; (\overline{M}_{\frac{1}{2}})_{\rm IV}\right]} \begin{pmatrix} x_1^{2j+1} \\ x_2^{2j+1} \\ u^{2j} \\ u^{2j+1} \end{pmatrix} \leq 1 \tag{21}$$

and (19) becomes

$$\underbrace{\begin{bmatrix} M_{-\frac{1}{2}} C_1 & M_{-\frac{1}{2}} D_{12} \\ (M_1(l^R))_{\rm I}\,[A_{11} \; A_{12}] + (M_1(l^R))_{\rm II} & (M_1(l^R))_{\rm I}\,[B_{11} \; B_{12}] + (M_1(l^R))_{\rm III} \end{bmatrix}}_{:=(\overline{M}_1(l^R))_{\rm I}} \begin{pmatrix} x^{2j} \\ u^{2j} \end{pmatrix}$$
$$+ \underbrace{\begin{bmatrix} 0 \\ (M_1(l^R))_{\rm I}\, E_1 \end{bmatrix}}_{:=(\overline{M}_1(l^R))_{\rm II}} w^{2j} \leq \begin{pmatrix} 1 \\ m_1 \end{pmatrix}. \tag{22}$$

With these changes, Algorithm 6.1 may be used to construct the sets K_j, $j = 2, 4, 6 \ldots$

6.2. Using the multirate controlled invariance kernel to construct memoryless controllers

As previously indicated, once $\textsc{Cinv}(\textsc{Object}^0_\gamma)$ has been constructed, it can be used to construct a memoryless multirate controller which meets the specified output ℓ^∞-norm bounds. Consider first the case where both u_1 and u_2 are a scalar control inputs. That is where the system has only two control inputs: u_1 which is sampled at a rate of T time units and u_2 which is sampled at a rate of $2T$ time units. Given that $\textsc{Cinv}(\textsc{Object}_\gamma^0)$ has been constructed using the algorithms in the previous sections, it can be written in the form

$$\textsc{Cinv}(\textsc{Object}_\gamma^0) = \mathit{Set}(M_\infty, m_\infty).$$

In order to construct a control law with the desired performance level, we will identify the range of values of the control u which will insure that the state is transfered from $\mathit{Set}(M_\infty, m_\infty)$ to $\mathit{Set}(M_\infty, m_\infty)$, under the action of F_γ. In order to identify this range of control values for u_1 at times steps $j = 1, 3, \ldots$, Step 3 through Step 8 of Algorithm 6.1 are used to construct matrices $MR_\infty(l^R)$ and vectors $mr_\infty(l^R)$ for all $\mathit{Region}(l^R)$, by replacing M_j in Step 3 of Algorithm 6.1 with M_∞. Note that since unique $MR_\infty(l^R)$ and $mr_\infty(l^R)$ exist for each $\mathit{Region}(l^R)$, a unique control law must be constructed for u_1^{2j+1} for each $\mathit{Region}(l^R)$.

Using Definition 2.3, the functions $\phi^+_{MR_\infty(l^R)} : \mathcal{R}^n \to \mathcal{R}$ and $\phi^-_{MR_\infty(l^R)} : \mathcal{R}^n \to \mathcal{R}$ may be defined, for all $\mathit{Region}(l^R)$, such that

$$\begin{aligned}
\phi^+_{MR_\infty(l^R)}(x) &= \min_{l^+ \in Z_i^+} \frac{(mr_\infty(l^R))_{l^+} - \left((MR_\infty(l^R))_{\mathrm{I}}\right)_{(l^+,:)} x}{((MR_\infty(l^R))_{\mathrm{II}})_{l^+}} \\
\phi^-_{MR_\infty(l^R)}(x) &= \min_{l^- \in Z_i^-} \frac{(mr_\infty(l^R))_{l^-} - \left((MR_\infty(l^R))_{\mathrm{I}}\right)_{(l^-,:)} x}{((MR_\infty(l^R))_{\mathrm{II}})_{l^-}}.
\end{aligned}$$

Note that, for each index l^R, Z_i^+ is non-empty if and only if Z_i^- is non-empty.

For each $\mathit{Region}(l^R)$, both $\phi^+_{MR_\infty(l^R)}(x)$ and $\phi^-_{MR_\infty(l^R)}(x)$ are continuous functions. And, by construction, if either is used as a control law for u_1^{2j+1} inside of each $\mathit{Region}(l^R)$, $\textsc{Cinv}(\textsc{Object}_\gamma^0)$ will be multirate controlled invariant under F_γ, assuming that an appropriate control law is used for u_1^{2j} and u_2^{2j}. In fact, from the definition of F_γ, it is apparent that any convex combination of the above functions is also a valid control law for u_1^{2j+1}. Accordingly, we define for u_1^{2j+1} in each $\mathit{Region}(l^R)$ the control law

$$\widetilde{g}_1(l^R; x) = \alpha\, \phi^+_{MR_\infty(l^R)}(x) + (1-\alpha)\, \phi^-_{MR_\infty(l^R)}(x), \quad \alpha \in [0,1]. \tag{23}$$

If the assumption of Definition 2.3 that Z^+ and Z^- are non-empty does not hold for some index l^R, then the *Rack* operation produces no constraints on the control in $\mathit{Region}(l^R)$. Therefore, $\widetilde{g}_1(l^R; x)$ may be set equal to any value, such that the assumption of the non-emptiness of Z^+ and Z^- is non-restrictive.

To form a control law which holds for any $\mathit{Region}(l^R)$, $\widetilde{g}_1(x)$ may be defined as

$$\widetilde{g}_1(x^{2j+1}, x^{2j}, u^{2j}) = \left\{\widetilde{g}_1(l^R; x), \text{ where } (x^{2j+1}, x^{2j}, u^{2j}) \in \mathit{Region}(l^R)\right\}.$$

Following the pattern of the LTI control construction method, [13], to determine also the range of control values for u_2 at times steps $j = 1, 3, \ldots$, we would apply the *Rack* operator to $MR_\infty(l^R)$ and $mr_\infty(l^R)$ in each $\mathit{Region}(l^R)$ to yield

$$(\overline{MR}_\infty(l^R), \overline{mr}_\infty(l^R)) \in \mathit{Rack}\left[MR_\infty(l^R), mr_\infty(l^R)\right].$$

Then we could attempt to use Definition 2.3 to construct functions $\phi^+_{\overline{MR}_\infty(l^R)}$ and $\phi^-_{\overline{MR}_\infty(l^R)}$. However, Z^+ and Z^- will be empty in each *Region*(l^R) due to the fact that all the columns corresponding to u_2^{2j+1} are zero. Therefore, no constraints are placed upon the control inputs u_2^{2j+1} and no control functions (i.e. ϕ^+ and ϕ^-) can be constructed. This simply reflects the fact that u_2 cannot be altered at time steps $j = 1, 3, \ldots$

Now, we proceed to construct the range of control values for u_1 at $j = 0, 2, 4, \ldots$ First, we define

$$(M_{\infty+1}(l^R), m_{\infty+1}(l^R)) \in Rack\left[\overline{MR}_\infty(l^R), \overline{mr}_\infty(l^R)\right].$$

Then, following Step 9 of Algorithm 6.1, the set $\overline{M}_{\infty+1}(l^R)$ is constructed by removing any element of x_1^{2j+1}, replacing it with an equivalent expression which contains only x^{2j}, u^{2j}, and w^{2j}, and adding the output constraint. Then, following Step 10 through Step 12 of Algorithm 6.1, the noise variable w^{2j} is removed to yield $\overline{M}_{\infty+1\frac{1}{2}}(l^R)$. Then, we may define the functions $\phi^+_{\overline{M}_{\infty+1\frac{1}{2}}(l^R)}$ and $\phi^-_{\overline{M}_{\infty+1\frac{1}{2}}(l^R)}$ using Definition 2.3, construct control functions $\widetilde{h}_1(l^R; x)$ analogously to $\widetilde{g}_1(l^R; x)$, and define $\widetilde{h}_1$ analogously to $\widetilde{g}_1$. Then, $\widetilde{h}_1$ provides a control law for u_1 at $j = 0, 2, 4, \ldots$

To construct the range of control values for u_2 at $j = 0, 2, 4, \ldots$, first construct

$$\left(\widetilde{M}_{\infty+1\frac{1}{2}}, \widetilde{m}_{\infty+1\frac{1}{2}}\right) \in Rack\left[\overline{M}_{\infty+1\frac{1}{2}}(l^R), \overline{m}_{\infty+1\frac{1}{2}}(l^R)\right].$$

Then, using Definition 2.3, we may define the functions $\phi^+_{\widetilde{M}_{\infty+1\frac{1}{2}}(l^R)}$ and $\phi^-_{\widetilde{M}_{\infty+1\frac{1}{2}}(l^R)}$. Then, define $\widetilde{h}_2$ analogously to $\widetilde{h}_1$. Then, $\widetilde{h}_2$ provides a control law for u_2 at $j = 0, 2, 4, \ldots$ Note that since u_2 is only updated every two time steps, $\widetilde{h}_2$ also will be the control law used for u_2 at time steps $j = 1, 3, 5, \ldots$

Utilizing each of the above steps, a control law which makes CINV(OBJECT$^0_\gamma$) multirate controlled invariant under F_γ is $\widetilde{g}$, where

$$\widetilde{g}(j) = \begin{pmatrix} \widetilde{h}_2(x(j)) \\ \widetilde{h}_1(x(j)) \end{pmatrix}, \quad j = 0, 2, 4, \ldots$$

$$\widetilde{g}(j) = \begin{pmatrix} \widetilde{h}_2(x(j-1)) \\ \widetilde{g}_1(x(j), x(j-1), \widetilde{g}(j-1)) \end{pmatrix}, \quad j = 1, 3, 5, \ldots$$

Note that the control input value for u_2 changes only every two steps, as required by the multirate structure of the system.

For the case in which $\dim(u_1) > 1$ or $\dim(u_2) > 1$, an appropriate controller may be formulated by extending the logic of the above algorithm. To illustrate the technique, we will focus on the construction of $\widetilde{g}_1$, with the construction of the other elements of $\widetilde{g}$ being analogous. For clarity, consider a situation in which both u_1 and u_2 have a dimension of two. Then, let

$$(\overline{MR}_\infty(l^R), \overline{mr}_\infty(l^R)) \in Rack\left[MR_\infty(l^R), mr_\infty(l^R)\right].$$

Construct $\phi^+_{MR_\infty(l^R)}(x)$ and $\phi^-_{MR_\infty(l^R)}(x)$ and, likewise $\phi^+_{\overline{MR}_\infty(l^R)}(x)$ and $\phi^-_{\overline{MR}_\infty(l^R)}(x)$ for each *Region*(l^R). Define $\widetilde{g}_{11}(l^R;x) : \mathcal{R}^n \to \mathcal{R}$ for each *Region*(l^R), such that

$$\widetilde{g}_{11}(l^R;x) = \alpha\,\phi^+_{\overline{MR}_\infty(l^R)}(x) + (1-\alpha)\,\phi^-_{\overline{MR}_\infty(l^R)}(x), \qquad \alpha \in [0,1].$$

Then, define $\widetilde{g}_{12}(l^R;x) : \mathcal{R}^n \to \mathcal{R}$ such that

$$\begin{aligned}\widetilde{g}_{12}(l^R;x) &= \beta\,\phi^+_{MR_\infty(l^R)}(x)\begin{pmatrix} x \\ \widetilde{g}_{11}(l^R;x)\end{pmatrix} \\ &\quad + (1-\beta)\,\phi^-_{MR_\infty(l^R)}(x)\begin{pmatrix} x \\ \widetilde{g}_{11}(l^R;x)\end{pmatrix}, \quad \beta \in [0,1].\end{aligned}$$

Then, set $\widetilde{g}_1(l^R;x) : \mathcal{R}^n \to \mathcal{R}$ equal to

$$\widetilde{g}_1(l^R;x) = \begin{pmatrix}\widetilde{g}_{11}(l^R;x) \\ \widetilde{g}_{12}(l^R;x)\end{pmatrix}.$$

To form a control law which holds for any *Region*(l^R), $\widetilde{g}_1(x)$ may be defined as

$$\widetilde{g}_1\left(x^{2j+1}, x^{2j}, u^{2j}\right) = \left\{\widetilde{g}_1(l^R;x),\ \text{where } (x^{2j+1}, x^{2j}, u^{2j}) \in \textit{Region}(l^R)\right\}.$$

The above iterative process is necessary to construct the desired controller when u_1 and u_2 are not scalars due to the structure imposed upon the problem by the *Rack* operator. When constructing CINV(OBJECT$^0_\gamma$), the *Rack* operator is used successively to remove one control input at a time from the matrix which describes the inequality constraints on system and control inputs. Therefore, it is possible to construct the controller for the entire system by constructing a controller from each of the matrices produced by the *Rack* operator in reverse order. In order to produce $\widetilde{h}_1(x)$ and $\widetilde{h}_2(x)$ a process similar to the one described above would be used, beginning with $M_{\infty+1}(l^R)$ and $m_{\infty+1}(l^R)$. The primary difference is that four successive *Rack* operations are used to construct the two elements of $\widetilde{h}_1(x)$, followed by the two elements of $\widetilde{h}_2(x)$, as all components of the control input may be altered at the even time steps. It is clear that the above process may be extended to a system with $\dim(u_1) > 2$ or $\dim(u_2) > 2$ by continuing to work back through the matrices produced by the *Rack* operator such that $\widetilde{g}_{13}(l^R;x)$, for example, would be constructed from $\widetilde{g}_{12}(l^R;x)$ and $\widetilde{g}_{11}(l^R;x)$. And, after constructing $M_{\infty+1\frac{1}{2}}(l^R)$, the control law for time steps $j = 0, 2, 4, \ldots$ can be formulated, such that $\widetilde{h}_{13}(x)$, for example, would be constructed from $\widetilde{h}_{12}(x)$ and $\widetilde{h}_{11}(x)$.

The above process produces a controller which renders CINV(OBJECT$^0_\gamma$) multirate controlled invariant under the action of F_γ. To produce a controller which has a performance of γ, we define

$$\begin{aligned}p_0(x^{2j}) &= \inf\left\{\beta \in \mathcal{R}^+ : x^{2j} \in \beta\,\text{CINV}(\text{OBJECT}^0_\gamma)\right\} \\ p_1(x_1^{2j+1}, x^{2j}, u^{2j}) &= \inf\left\{\beta \in \mathcal{R}^+ : x^{2j+1} \in \beta\,\text{CINV}(\text{OBJECT}^1_\gamma),\right. \\ &\qquad \left.\forall\, x^{2j+1} \in \textit{Est}(x_1^{2j+1}, x^{2j}, u^{2j})\right\}\end{aligned}$$

where $\text{CINV}(\text{OBJECT}^1_\gamma)$ is the closed subset of $\text{CINV}(\text{OBJECT}^0_\gamma)$ to which the state will be confined at the time steps $2j+1,\ j=0,\,1,\ldots$ under the action of the control law $\widetilde{g}$. This set may be constructed by using a similar method to that used to construct set valued estimates. Specifically, the *Rack* operator may be used to map the set $\text{CINV}(\text{OBJECT}^0_\gamma)$ forward in time, utilizing the state equations (3), the known form of the control law $\widetilde{g}$, and the assumption that $\|w\| \leq 1/\gamma$.

Using these scalings with the previously constructed control law $\widetilde{g}$, the new control law g results, where g is defined such that

$$\begin{aligned} g(x^{2j}) &= p_0(x^{2j})\,\widetilde{g}\left(\frac{x^{2j}}{p_0(x^{2j})}\right), \\ & \qquad t = 0,\,2,\,4,\ldots \\ g(x^{2j+1}, x^{2j}) &= p_1(x^{2j+1}, x^{2j}, u^{2j})\,\widetilde{g}\left(\frac{x^{2j+1}}{p_1(x^{2j+1}, x^{2j}, u^{2j})}, \frac{x^{2j}}{p_1(x^{2j+1}, x^{2j}, u^{2j})}\right), \\ & \qquad t = 1,\,3,\,5,\ldots \end{aligned}$$

Using the above definitions, the following proposition may be proven in a manner similar to that used in [12].

Proposition 6.1. Assume that each state x and each u appears only at a rate of T or $2T$ (i.e. $l_1 = \cdots = l_{n_{x_1}} = k_1 = \cdots = k_{m_{u_1}} = 1$ and $l_{n_{x_1}+1} = \cdots = l_n = k_{m_{u_1}+1} = \cdots = k_m = 2$). Also, assume that $\text{CINV}(\text{OBJECT}^0_\gamma)$ exists. Then, the control law g, as defined above, is internally stabilizing with a performance of γ for the multirate system defined by (3).

When using the algorithms in this and previous sections to construct $\text{CINV}(\text{OBJECT}^0_\gamma)$, it is possible that an infinite number of iterations of the appropriate algorithm will be required. To address this problem, it is possible to construct an algorithm which produces an internally stabilizing controller in a finite number of steps. This finite termination algorithm essentially defines an auxiliary set $\widetilde{K}_{R_j}$ for each set K_{R_j}, such that p successive of the K_{R_j}'s will belong to their corresponding $\widetilde{K}_{R_j}$'s *only* when all of the K_{R_j}'s are close to their limit sets in $\text{CINV}(\text{OBJECT}^0_\gamma)$. It can be shown that the final K_{R_j} is controlled invariant, such that an ϵ-suboptimal control law may be constructed, and the algorithm terminates. The formulation of this algorithm follows that of the finite termination algorithm in [13].

In the general multirate case, in which the system being considered has sampling rates different from, or in addition to, T and $2T$. The general technique of constructing a controlled invariant set and then using this set to construct an optimal controller remains largely the same for the general multirate problem, with a few modifications being necessary. The controlled difference inclusion F_γ must now span R time steps. It must be defined such that the unavailable control inputs at each time step are equated to their previous values. In addition, set valued estimators for each of the unmeasured states at each time step are constructed in a manner analogous to that used to construct Est_γ. An analogue to Algorithm 6.1 may then be constructed. Note that the space of the measured variables must again be sectioned as one moves back in time in order to insure that the set valued estimator at each time step has a constant functional form.

7. EXAMPLE

As an example of the implementation of Algorithm 6.1 and of the construction of a memoryless multirate controller, consider the multirate system

$$\begin{aligned} x(t+1) &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u(t) + \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} w(t) \\ z(t) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t) \end{aligned} \tag{24}$$

where

$$l_1 = 1, \quad l_2 = l_3 = 2, \quad k_1 = 1, \quad k_2 = 2$$

such that x_2, x_3, and u_1 appear at a rate of $2T$; and x_1 and u_1 appear at a rate of T. Note that as previously indicated the control inputs are ordered such that $u(t) = \begin{pmatrix} u_2 \\ u_1 \end{pmatrix}$. Since both x_2 and x_3 are unknown at odd time steps, an estimating set must be constructed by using Algorithm 5.1. The resulting estimating set is

$$Est_\gamma\left(x_1^{2j+1}, x^{2j}, u^{2j}\right)$$

$$= \left\{ \begin{pmatrix} x_2^{2j+1} \\ x_3^{2j+1} \end{pmatrix} : \left[\begin{array}{ccc|ccc|cc} 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 & -1 & 1 & -1 \\ 1 & 0 & -1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & -1 & 1 & 0 & -1 & 0 \end{array}\right] \begin{pmatrix} x_1^{2j+1} \\ x_2^{2j+1} \\ x_3^{2j+1} \\ --- \\ x^{2j} \\ --- \\ u^{2j} \end{pmatrix} \le \begin{pmatrix} \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \end{pmatrix} \right\}.$$

As both D_{11} and D_{12} are zero, Algorithm 6.1 may be used to construct the multirate controlled invariance kernel, $\text{CINV}(\text{OBJECT}_\gamma^0)$. Choosing $\gamma = 8$, which implies that

$\|w\|_\infty \leq \frac{1}{8}$, the algorithm converges to

$$\text{CINV}(\text{OBJECT}^0_\gamma) = \left\{ x : \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{4}{3} & 0 \\ 0 & 0 & 1 \\ -\frac{2}{3} & \frac{2}{3} & 0 \\ -1 & 0 & 0 \\ 0 & -\frac{4}{3} & 0 \\ 0 & 0 & -1 \\ \frac{2}{3} & -\frac{2}{3} & 0 \end{bmatrix} x \leq 1 \right\}.$$

Having obtained $\text{CINV}(\text{OBJECT}^0_\gamma)$ for $\gamma = 8$, a controller may be found which insures that the system remains within this controlled invariant set by utilizing the result of Section 6.2. One possible controller, which results from this type of construction and which achieves a performance of $\gamma = 8$ has the form $u(x,j) = \binom{u_2(x)}{u_1(x)}$, where

$$\begin{aligned} u_1(j) &= h_1(x(j)) && \text{: for } j \text{ even} \\ &= g_1(x_1(j),\, x(j-1)) && \text{: for } j \text{ odd} \end{aligned}$$

$$\begin{aligned} u_2(j) &= h_2(x(j)) && \text{: for } j \text{ even} \\ &= h_2(x(j-1)) && \text{: for } j \text{ odd} \end{aligned}$$

and

$$h_1(j) = \frac{1}{2}\max\left\{-p_0\frac{1}{2}-x_3(j)\right\}+\frac{1}{2}\min\left\{p_0\frac{1}{2}-x_3(j)\right\}$$

$$h_2(j) = \frac{1}{2}\max\left\{p_0\frac{7}{8}-x_1(j),\, p_0\frac{5}{8}-x_2(j)\right\}+\frac{1}{2}\min\left\{-p_0\frac{7}{8}-x_1(j),\, -p_0\frac{5}{8}-x_2(j)\right\}$$

$$g_1(j) = \begin{cases} \left.\begin{aligned} &\tfrac{1}{2}\max\Big\{-p_1\tfrac{3}{8}-x_1(j-1)-u_2(j-1), \\ &-p_1\tfrac{9}{8}-x_1(j-1)+x_3(j-1)+u_1(j-1)-u_2(j-1)\Big\} \\ &+\tfrac{1}{2}\min\Big\{p_1\tfrac{3}{8}-x_1(j)-x_1(j-1)+x_2(j-1)-u_2(j-1), \\ &p_1\tfrac{9}{8}-x_1(j-1)+x_3(j-1)+u_1(j-1)-u_2(j-1)\Big\} \end{aligned}\right\} & \text{if } x_1(j) \leq x_2(j-1) \\ \left.\begin{aligned} &\tfrac{1}{2}\max\Big\{-p_1\tfrac{3}{8}-x_1(j)-x_1(j-1)+x_2(j-1)-u_2(j-1), \\ &-p_1\tfrac{9}{8}-x_1(j-1)+x_3(j-1)+u_1(j-1)-u_2(j-1)\Big\} \\ &+\tfrac{1}{2}\min\Big\{p_1\tfrac{3}{8}-x_1(j-1)-u_2(j-1), \\ &p_1\tfrac{9}{8}-x_1(j-1)+x_3(j-1)+u_1(j-1)-u_2(j-1)\Big\} \end{aligned}\right\} & \text{if } x_1(j) \geq x_2(j-1) \end{cases}$$

$$p_0 = \max\left\{ |x_1(j)|,\ \frac{4}{3}|x_2(j)|,\ |x_3(j)|,\ \frac{2}{3}| - x_1(j) + x_2(j)| \right\}$$

$$p_1 = \begin{cases} \left.\begin{array}{l} \max\Big\{ 2(x_1(j) - x_2(j-1) + x_3(j-1) + u_1(j-1)), \\ \frac{8}{7}(x_1(j) + x_1(j-1) - x_2(j-1) + u_2(j-1)), \\ -2(x_3(j-1) + u_1(j-1)), -\frac{8}{7}(x_1(j-1) + u_2(j-1)), \\ |x_1(j)|, \frac{4}{5}| - x_2(j-1) + x_3(j-1) + u_i(j-1)| \Big\} \end{array}\right\} & \text{if } \begin{array}{l} x_1(j) \leq \\ x_2(j-1) \end{array} \\ \left.\begin{array}{l} \max\Big\{ -2(x_1(j) - x_2(j-1) + x_3(j-1) + u_1(j-1)), \\ -\frac{8}{7}(x_1(j) + x_1(j-1) - x_2(j-1) + u_2(j-1)), \\ 2(x_3(j-1) + u_1(j-1)), \frac{8}{7}(x_1(j-1) + u_2(j-1)), \\ |x_1(j)|, \frac{4}{5}| - x_2(j-1) + x_3(j-1) + u_i(j-1)| \Big\} \end{array}\right\} & \text{if } \begin{array}{l} x_1(j) \geq \\ x_2(j-1). \end{array} \end{cases}$$

Note that $p_0(x(j))$ and $p_1(x(j), x(j-1))$ are the Minkowski scaling functions which were mentioned in Section 6.2. The calculation of p_1 requires the construction of $\text{CINV}(\text{OBJECT}^1_\gamma)$. It can be shown that $\text{CINV}(\text{OBJECT}^1_\gamma) = \text{CINV}(\text{OBJECT}^0_\gamma)$.

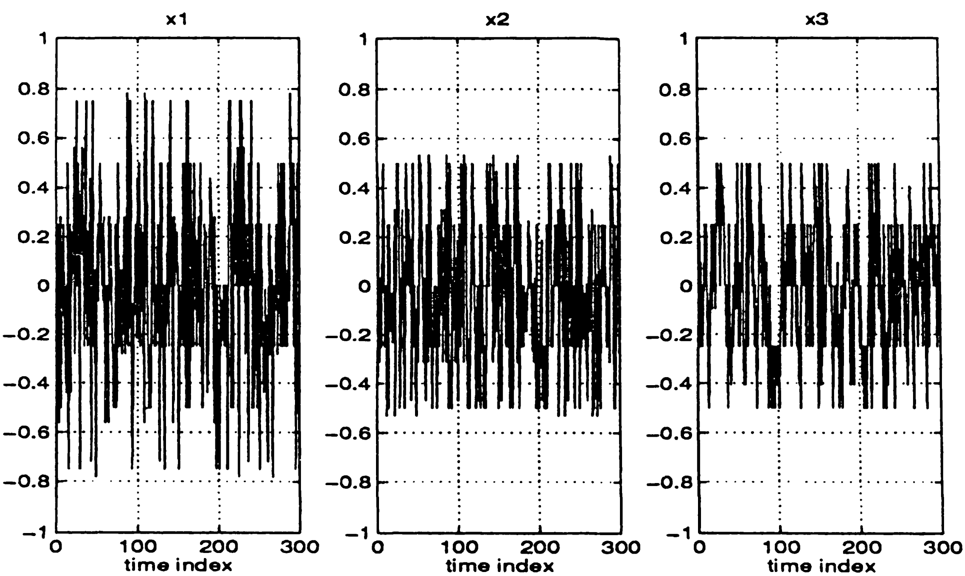

Fig. 1. Time history of state variables with $\|w\|_\infty \leq \frac{1}{8}$.

Using the above controller, a simulation was run with initial conditions of $(x_1, x_2, x_3) = \left(-1, -\frac{3}{4}, 1\right)$. The disturbance $w(t)$ was chosen with a uniform distribution such that $\|w\|_\infty \leq \frac{1}{8}$. The time history of the state variables from this simulation appears in Figure 1. Clearly, as the C_1 system matrix is the identify matrix, $||z||_\infty$ remains less than one in this simulation, thereby confirming the efficacy of the controller.

8. CONCLUSIONS

A state-space approach was taken and the concepts of viability theory and controlled invariance were used to produce a method for the construction of near optimal control laws for multirate systems when full state information is available for feedback. The algorithm which was constructed explicitly in this paper is limited in applicability to two-rate systems, in which all the controls and all the states appear only at rates of T and $2T$. But, as previously discussed, the extension of this algorithm to the general multirate system is straightforward. The resulting optimum control laws are static and contain R different piecewise linear elements, where R is the least common multiple of all the sampling rates, which are sequentially applied to the multirate system. This construction method is attractive due to the desirable static nature of the resulting control laws. Thus, it can potentially serve as an alternative to the well-known input-output synthesis methods.

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