Ahmet Tekcan

Proper cycles of indefinite quadratic forms and their right neighbors

Applications of Mathematics, Vol. 52 (2007), No. 5, 407–415

Persistent URL: http://dml.cz/dmlcz/134685

PROPER CYCLES OF INDEFINITE QUADRATIC FORMS AND THEIR RIGHT NEIGHBORS

Ahmet Tekcan, Bursa

(Received February 7, 2006, in revised version August 16, 2006)

Abstract. In this paper we consider proper cycles of indefinite integral quadratic forms $F=(a,b,c)$ with discriminant Δ. We prove that the proper cycles of F can be obtained using their consecutive right neighbors $R^i(F)$ for $i\geqslant 0$. We also derive explicit relations in the cycle and proper cycle of F when the length l of the cycle of F is odd, using the transformations $\tau(F)=(-a,b,-c)$ and $\chi(F)=(-c,b,-a)$.

Keywords: quadratic form, indefinite form, cycle, proper cycle, right neighbor

MSC 2000: 11E04, 11E12, 11E16

1. Introduction

A real binary quadratic form (or just a form) F is a polynomial in two variables x and y of the type

$$F=F(x,y)=ax^2+bxy+cy^2$$

with real coefficients a, b, c. We denote F briefly by $F=(a,b,c)$. The discriminant of F is defined by the formula b^2-4ac and is denoted by $\Delta=\Delta(F)$. F is an integral form if and only if $a,b,c\in\mathbb{Z}$, and is indefinite if and only if $\Delta(F)>0$. An indefinite quadratic form $F=(a,b,c)$ with discriminant Δ is said to be reduced if

$$\bigl|\sqrt{\Delta}-2|a|\bigr|<b<\sqrt{\Delta}. \tag{1.1}$$

Let $\mathrm{GL}(2,\mathbb{Z})$ be the multiplicative group of 2×2 matrices $g=\begin{pmatrix} r & s\\ t & u\end{pmatrix}$ such that $r,s,t,u\in\mathbb{Z}$ and $\det g=\pm1$. Gauss (1777–1855) defined the group action of $\mathrm{GL}(2,\mathbb{Z})$ on the set of forms by the following formula: Let $F=(a,b,c)$ be a form and let $g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z})$. Then the form gF is defined by

$$gF(x, y) = a(rx + ty)^2 + b(rx + ty)(sx + uy) + c(sx + uy)^2.$$

That is, gF is obtained from F by making the substitution $x \to rx+tu$, $y \to sx+uy$. Moreover, $\Delta(F) = \Delta(gF)$ for all $g \in \mathrm{GL}(2, \mathbb{Z})$. That is, the action of $\mathrm{GL}(2, \mathbb{Z})$ on forms leaves the discriminant invariant. If F is indefinite or integral, then so is gF for all $g \in \mathrm{GL}(2, \mathbb{Z})$.

Let F and G be two forms. If there exists a $g \in \mathrm{GL}(2, \mathbb{Z})$ such that $gF = G$, then F and G are called equivalent. If $\det g = 1$, then F and G are called properly equivalent. If $\det g = -1$, then F and G are called improperly equivalent.

Let $\varrho(F)$ denote the normalization (i.e., replacing F by its normalization, for further details see [1, p. 88]) of $(c, -b, a)$. To be more explicit, we set

$$\varrho(F) = (c, -b + 2cs, cs^2 - bs + a), \tag{1.2}$$

where

$$s = s(F) = \begin{cases} \mathrm{sign}(c)\left\lfloor \dfrac{b}{2|c|} \right\rfloor & \text{for } |c| \geqslant \sqrt{\Delta}, \\ \mathrm{sign}(c)\left\lfloor \dfrac{b+\sqrt{\Delta}}{2|c|} \right\rfloor & \text{for } |c| < \sqrt{\Delta}. \end{cases}$$

Note that, if F is reduced, then $\varrho(F)$ is also reduced due to (1.1). In fact, ϱ is a permutation of the set of all reduced indefinite forms.

Now consider the transformations

$$\chi(F) = \chi(a, b, c) = (-c, b, -a) \tag{1.3}$$

and

$$\tau(F) = \tau(a, b, c) = (-a, b, -c). \tag{1.4}$$

If

$$\chi(F) = F,$$

that is, $F = (a, b, -a)$ for the transformation χ defined in (1.3), then F is called symmetric. We assume that $F = (a, b, c)$ is indefinite and integral throughout the paper.

The cycle of F is the sequence $((\tau\varrho)^i(G))$ for $i \in \mathbb{Z}$, where $G = (k, l, m)$ is a reduced form with $k > 0$ which is equivalent to F. Similarly, the proper cycle of F is the sequence $(\varrho^i(G))$ for $i \in \mathbb{Z}$, where G is a reduced form which is properly equivalent to F. The cycle and the proper cycle of F are invariants of the equivalence class of F. We represent the cycle or proper cycle of F by its period

$$F_0 \sim F_1 \sim \ldots \sim F_{l-1}$$

of length l. We explain how to compute the cycle and proper cycle of F by the following lemma.

Lemma 1.1. *Let $F = (a, b, c)$ be a reduced quadratic form of discriminant Δ. Then the cycle of F is $F_0 \sim F_1 \sim F_2 \sim \ldots \sim F_{l-1}$ of length l, where $F_0 = F = (a_0, b_0, c_0)$,*

$$s_i = |s(F_i)| = \left\lfloor \frac{b_i + \lfloor\sqrt{\Delta}\rfloor}{2|c_i|} \right\rfloor \tag{1.5}$$

and

$$F_{i+1} = (a_{i+1}, b_{i+1}, c_{i+1}) = \bigl(|c_i|, -b_i + 2s_i|c_i|, -(a_i + b_i s_i + c_i s_i^2)\bigr) \tag{1.6}$$

for $1 \leqslant i \leqslant l-2$. The proper cycle of F is

$$\begin{aligned} F_0 &\sim \tau(F_1) \sim F_2 \sim \tau(F_3) \sim \ldots \sim \tau(F_{l-2}) \sim F_{l-1} \\ &\sim \tau(F_0) \sim F_1 \sim \tau(F_2) \sim \ldots \sim F_{l-2} \sim \tau(F_{l-1}) \end{aligned} \tag{1.7}$$

of length $2l$ if l is odd, and is

$$F_0 \sim \tau(F_1) \sim F_2 \sim \tau(F_3) \sim \ldots \sim F_{l-2} \sim \tau(F_{l-1}) \tag{1.8}$$

of length l if l is even. In this case the equivalence class of F is the disjoint union of the proper equivalence class of F and the proper equivalence class of $\tau(F)$ ([1]).

The right neighbor of $F = (a, b, c)$, denoted by $R(F)$, is the form (A, B, C) determined by the conditions

(i) $A = c$,
(ii) $b + B \equiv 0 \pmod{2A}$ and $\sqrt{\Delta} - 2|A| < B < \sqrt{\Delta}$,
(iii) $B^2 - 4AC = \Delta$.

It is clear from the definition that

$$R(F) = (A, B, C) = \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (a, b, c) = \begin{pmatrix} 0 & -1 \\ 1 & -\delta \end{pmatrix} (a, b, c), \tag{1.9}$$

where

$$b + B = 2c\delta. \tag{1.10}$$

Therefore F is properly equivalent to its right neighbor $R(F)$ (see [2]).

2. Proper cycles of indefinite quadratic forms and their right neighbors

In this section we will consider the proper cycles of indefinite reduced quadratic forms $F = (a, b, c)$. We will show that the proper cycle of F can be given by using its consecutive right neighbors $R^i(F)$ for $i \geqslant 0$. We also derive some relations in the cycle and proper cycle of F.

Theorem 2.1. *Let $F_0 \sim F_1 \sim \ldots \sim F_{l-1}$ be the cycle of F of length l, and let $R^i(F_0)$ be the consecutive right neighbors of F_0 for $i \geqslant 0$. Then*
(1) *If l is odd, then the proper cycle of F is*

$$F_0 \sim R^1(F_0) \sim R^2(F_0) \sim \ldots \sim R^{2l-2}(F_0) \sim R^{2l-1}(F_0)$$

of length $2l$.
(2) *If l is even, then the proper cycle of F is*

$$F_0 \sim R^1(F_0) \sim R^2(F_0) \sim \ldots \sim R^{l-2}(F_0) \sim R^{l-1}(F_0)$$

of length l.

Proof. (1) Let l be odd. Then we know from (1.7) that the proper cycle of F is

$$F_0 \sim \tau(F_1) \sim F_2 \sim \tau(F_3) \sim \ldots \sim F_{l-2} \sim \tau(F_{l-1})$$

of length $2l$. Let F_i and F_{i+1} be two forms in the cycle of F. Then

$$s_i = \left\lfloor \frac{b_i + \lfloor\sqrt{\Delta}\rfloor}{2|c_i|} \right\rfloor$$

by (1.5). For the right neighbor $R(F_i)$ of F_i we have

$$\delta_i = \frac{b_i + B_i}{2c_i}$$

by (1.10). On the other hand, $b_i + B_i \equiv 0 \pmod{2A_i}$ and $\sqrt{\Delta} - 2|A_i| < B_i < \sqrt{\Delta}$ by the definition. Hence it is easily seen that

$$s_i = \begin{cases} -\delta_i & \text{if } i \text{ is even,} \\ \delta_i & \text{if } i \text{ is odd.} \end{cases}$$

Hence we can write $s_i = |\delta_i|$. Therefore $|\delta_i|$ coincides with s_i. For the quadratic form $F_0 = (a_0, b_0, c_0)$ we have from (1.5) and (1.6)

$$\begin{aligned} F_1 &= (|c_0|, -b_0 + 2s_0|c_0|, -a_0 - b_0 s_0 - c_0 s_0^2) \\ &= (-c_0, -b_0 - 2s_0 c_0, -a_0 - b_0 s_0 - c_0 s_0^2). \end{aligned} \tag{2.1}$$

The first right neighbor of $F_0 = (a_0, b_0, c_0)$ is

$$\begin{aligned} R^1(F_0) &= \begin{pmatrix} 0 & -1 \\ 1 & -\delta_0 \end{pmatrix} (a_0, b_0, c_0) \\ &= (c_0, -b_0 + 2\delta_0 c_0,\ a_0 - b_0\delta_0 + c_0\delta_0^2) \end{aligned} \tag{2.2}$$

due to (1.9). Replacing δ_0 by $-s_0$ in (2.2) we get

$$R^1(F_0) = (c_0, -b_0 - 2s_0 c_0, a_0 + b_0 s_0 + c_0 s_0^2) \tag{2.3}$$

since $s_0 = -\delta_0$. Applying (1.4) we get from (2.1)

$$\tau(F_1) = (c_0, -b_0 - 2s_0 c_0, a_0 + b_0 s_0 + c_0 s_0^2). \tag{2.4}$$

Consequently, (2.3) and (2.4) yield that

$$R^1(F_0) = (c_0, -b_0 - 2s_0 c_0, a_0 + b_0 s_0 + c_0 s_0^2) = \tau(F_1).$$

Similarly it can be shown that

$$\begin{aligned} R^2(F_0) &= F_2, \\ R^3(F_0) &= \tau(F_3), \\ &\dots \\ R^{l-1}(F_0) &= F_{l-1}, \\ R^l(F_0) &= \tau(F_0), \\ R^{l+1}(F_0) &= F_1, \\ &\dots \\ R^{2l-2}(F_0) &= F_{l-2}, \\ R^{2l-1}(F_0) &= \tau(F_{l-1}). \end{aligned}$$

Therefore

$$F_0 \sim R^1(F_0) \sim R^2(F_0) \sim \dots \sim R^{2l-2}(F_0) \sim R^{2l-1}(F_0)$$

is the proper cycle of F of length $2l$.

The second assertion can be proved in the same way. □

E x a m p l e 2.1. The cycle of $F = (1, 5, -4)$ is

$$F_0 = (1, 5, -4) \sim F_1 = (4, 3, -2) \sim F_2 = (2, 5, -2) \\ \sim F_3 = (2, 3, -4) \sim F_4 = (4, 5, -1)$$

of length 5 which is an odd number. Therefore the proper cycle of F is

$$F_0 \sim R^1(F_0) \sim R^2(F_0) \sim R^3(F_0) \sim R^4(F_0) \sim R^5(F_0) \\ \sim R^6(F_0) \sim R^7(F_0) \sim R^8(F_0) \sim R^9(F_0)$$

of length 10 since

$$\begin{aligned} R^1(F_0) &= (-4, 3, 2) = \tau(F_1), \\ R^2(F_0) &= (2, 5, -2) = F_2, \\ R^3(F_0) &= (-2, 3, 4) = \tau(F_3), \\ R^4(F_0) &= (4, 5, -1) = F_4, \\ R^5(F_0) &= (-1, 5, 4) = \tau(F_0), \\ R^6(F_0) &= (4, 3, -2) = F_1, \\ R^7(F_0) &= (-2, 5, 2) = \tau(F_2), \\ R^8(F_0) &= (2, 3, -4) = F_3, \\ R^9(F_0) &= (-4, 5, 1) = \tau(F_4). \end{aligned}$$

E x a m p l e 2.2. The cycle of $F = (1, 8, -5)$ is

$$F_0 = (1, 8, -5) \sim F_1 = (5, 2, -4) \sim F_2 = (4, 6, -3) \\ \sim F_3 = (3, 6, -4) \sim F_4 = (4, 2, -5) \sim F_5 = (5, 8, -1)$$

of length 6 which is an even number. Therefore the proper cycle of F is

$$F_0 \sim R^1(F_0) \sim R^2(F_0) \sim R^3(F_0) \sim R^4(F_0) \sim R^5(F_0)$$

of length 6 since

$$\begin{aligned} R^1(F_0) &= (-5, 2, 4) = \tau(F_1), \\ R^2(F_0) &= (4, 6, -3) = F_2, \\ R^3(F_0) &= (-3, 6, 4) = \tau(F_3), \\ R^4(F_0) &= (4, 2, -5) = F_4, \\ R^5(F_0) &= (-5, 8, 1) = \tau(F_5). \end{aligned}$$

From Theorem 2.1 we can deduce the following corollary.

Corollary 2.2. *Let $F_0 \sim F_1 \sim \ldots \sim F_{l-1}$ be the cycle of F of length l.*

(1) *If l is odd, then*

$$R^i(F_0) = \begin{cases} F_i & \text{if } i \text{ is even,} \\ \tau(F_i) & \text{if } i \text{ is odd} \end{cases}$$

for $1 \leqslant i \leqslant l-1$, and

$$R^i(F_0) = \begin{cases} F_{i-l} & \text{if } i \text{ is even,} \\ \tau(F_{i-l}) & \text{if } i \text{ is odd} \end{cases}$$

for $l \leqslant i \leqslant 2l-1$.

(2) *If l is even, then*

$$R^i(F_0) = \begin{cases} F_i & \text{if } i \text{ is even,} \\ \tau(F_i) & \text{if } i \text{ is odd} \end{cases}$$

for $1 \leqslant i \leqslant l-1$.

Theorem 2.3. *If l is odd, then in the cycle $F_0 \sim F_1 \sim \ldots \sim F_{l-1}$ of F,*

$$\chi(F_i) = F_{l-1-i}$$

for $0 \leqslant i \leqslant l-1$ and $F_{(l-1)/2}$ is a symmetric form.

Proof. Let $F = (a, b, c)$ be a quadratic form. Then applying (1.5) and (1.6) we get

$$\begin{aligned} F_0 &= (a_0, b_0, c_0), \\ F_1 &= (a_1, b_1, c_1), \\ F_2 &= (a_2, b_2, c_2), \\ F_3 &= (a_3, b_3, c_3), \\ &\ldots \\ F_{(l-3)/2} &= \big(a_{(l-3)/2}, b_{(l-3)/2}, c_{(l-3)/2}\big), \\ F_{(l-1)/2} &= \big(a_{(l-1)/2}, b_{(l-1)/2}, c_{(l-3)/2}\big), \\ F_{(l+1)/2} &= \big(-c_{(l-3)/2}, b_{(l-3)/2}, -a_{(l-3)/2}\big), \\ &\ldots \\ F_{l-3} &= (-c_2, b_2, -a_2), \\ F_{l-2} &= (-c_1, b_1, -a_1), \\ F_{l-1} &= (-c_0, b_0, -a_0). \end{aligned} \tag{2.5}$$

It is clear from (2.5) that

$$\begin{aligned}
\chi(F_0) &= (-c_0, b_0, -a_0) = F_{l-1},\\
\chi(F_1) &= (-c_1, b_1, -a_1) = F_{l-2},\\
\chi(F_2) &= (-c_2, b_2, -a_2) = F_{l-3},\\
&\dots\\
\chi\big(F_{(l-3)/2}\big) &= \big(-c_{(l-3)/2}, b_{(l-3)/2}, -a_{(l-3)/2}\big) = F_{(l+1)/2},\\
\chi\big(F_{(l-1)/2}\big) &= \big(a_{(l-1)/2}, b_{(l-1)/2}, c_{(l-3)/2}\big) = F_{(l-1)/2},\\
\chi\big(F_{(l+1)/2}\big) &= \big(a_{(l-3)/2}, b_{(l-3)/2}, c_{(l-3)/2}\big) = F_{(l-3)/2},\\
&\dots\\
\chi(F_{l-3}) &= (a_2, b_2, c_2) = F_2,\\
\chi(F_{l-2}) &= (a_1, b_1, c_1) = F_1,\\
\chi(F_{l-1}) &= (a_0, b_0, c_0) = F_0.
\end{aligned}$$

So $\chi(F_i) = F_{l-1-i}$ for $0 \leqslant i \leqslant l-1$ and $F_{(l-1)/2}$ is a symmetric form since $\chi(F_{(l-1)/2}) = F_{(l-1)/2}$ by (1.3). □

From Theorem 2.3, we can obtain the following result.

Corollary 2.4. *The cycle of F is*

$$F_0 \sim F_1 \sim F_2 \sim \dots \sim F_{(l-3)/2} \sim F_{(l-1)/2} \sim \chi(F_{(l-3)/2}) \sim \dots \\ \sim \chi(F_2) \sim \chi(F_1) \sim \chi(F_0).$$

Now we can give the cycle of $\chi(F)$ by the following theorem.

Theorem 2.5. *If l is odd, then the cycle of $\chi(F)$ is*

$$\chi(F_l) \sim \chi(F_{l-1}) \sim \chi(F_{l-2}) \sim \dots \sim \chi(F_1)$$

of length l.

P r o o f. Let $F_0 \sim F_1 \sim F_2 \sim \dots \sim F_{l-1}$ be the cycle of F. Then $F_l = F_0, F_{l+1} = F_1, \dots, F_{2l} = F_{l-1}$. We know from Theorem 2.3 that

$$\chi(F_i) = F_{l-1-i}$$

for $0 \leqslant i \leqslant l-1$. So $\chi(F_{l-1}) = F_0$. In particular, $\chi(F_{l-2}) = F_1, \chi(F_{l-3}) = F_2, \dots, \chi(F_0) = F_{l-1}$. Consequently, the cycle of $\chi(F)$ is

$$\chi(F_l) \sim \chi(F_{l-1}) \sim \chi(F_{l-2}) \sim \dots \sim \chi(F_1)$$

of length l. □

E x a m p l e 2.3. The cycle of $F = (1, 7, -6)$ is

$$\begin{aligned} F_0 = (1,7,-6) \sim F_1 = (6,5,-2) \sim F_2 = (2,7,-3) \sim F_3 = (3,5,-4) \\ \sim F_4 = (4,3,-4) \sim F_5 = (4,5,-3) \sim F_6 = (3,7,-2) \\ \sim F_7 = (2,5,-6) \sim F_8 = (6,7,-1) \end{aligned}$$

of length 9. Note that

$$\begin{aligned} \chi(F_0) &= (6,7,-1), \\ \chi(F_8) &= (1,7,-6), \\ \chi(F_7) &= (6,5,-2), \\ \chi(F_6) &= (2,7,-3), \\ \chi(F_5) &= (3,5,-4), \\ \chi(F_4) &= (4,3,-4), \\ \chi(F_3) &= (4,5,-3), \\ \chi(F_2) &= (3,7,-2), \\ \chi(F_1) &= (2,5,-6). \end{aligned}$$

Therefore the cycle of $\chi(F) = (6, 7, -1)$ is

$$\begin{aligned} \chi(F_9) = (6,7,-1) \sim \chi(F_8) = (1,7,-6) \sim \chi(F_7) = (6,5,-2) \\ \sim \chi(F_6) = (2,7,-3) \sim \chi(F_5) = (3,5,-4) \sim \chi(F_4) = (4,3,-4) \\ \sim \chi(F_3) = (4,5,-3) \sim \chi(F_2) = (3,7,-2) \sim \chi(F_1) = (2,5,-6) \end{aligned}$$

of length 9.

References

[1] *J. Buchmann*: Algorithms for Binary Quadratic Forms. Springer-Verlag, accepted.

[2] *D. E. Flath*: Introduction to Number Theory. Wiley, New York, 1989. zbl

Author's address: *A. Tekcan*, Uludag University, Faculty of Science, Department of Mathematics, Görükle 16059, Bursa, Turkey, e-mail: `tekcan@uludag.edu.tr`.