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ON GENERAL TWO-SCALE CONVERGENCE AND ITS APPLICATION TO THE CHARACTERIZATION OF G-LIMITS

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Abstract. We characterize some G-limits using two-scale techniques and investigate a method to detect deviations from the arithmetic mean in the obtained G-limit provided no periodicity assumptions are involved. We also prove some results on the properties of generalized two-scale convergence.

Keywords: homogenization, G-convergence, two-scale convergence

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1. Introduction

Convergence of linear differential operators is a matter of interest concerning a wide variety of applications, and in this process, it is usually a defining sequence $\{A_h\}$ of matrices that plays the leading role. Proving existence of a limit operator with appropriate characteristics has been extensively studied for quite general cases. The problem of finding procedures for computing such limits, however, is usually restricted to certain special cases, the most developed one being periodic homogenization. There is no problem if we demand quite strong convergence of $\{A_h\}$. With weaker convergence assumptions on the sequence $\{A_h\}$, though, we encounter difficulties. We are going to characterize limits of sequences of operators for non trivial cases without any periodicity demands.

A well studied and very illuminating example is the convergence of sequences of linear elliptic operators. We study sequences $\{u_h\}$ of weak solutions to sequences of equations of the type

$$\begin{aligned} -\nabla\cdot(A_h(x)\nabla u_h(x)) &= f(x) \quad \text{in } \Omega, \\ u_h(x) &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

Assume that matrix functions

$$A_h \to A \quad \text{in } L^2(\Omega)^{N\times N} \text{ (strongly)}, \tag{2}$$

where A_h and $A \in L^\infty(\Omega)^{N\times N}$, satisfy the structure conditions

$$A_h(x)\xi \cdot \xi \geqslant C_0|\xi|^2 \tag{3}$$

and

$$|A_h(x)\xi| \leqslant C_1|\xi|, \tag{4}$$

where $0 < C_0 \leqslant C_1 < \infty$. Then

$$u_h \rightharpoonup u \quad \text{in } W_0^{1,2}(\Omega),$$

where u solves

$$\begin{aligned} -\nabla\cdot(A(x)\nabla u(x)) &= f(x) \quad \text{in } \Omega,\\ u(x) &= 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{5}$$

(see [3], Lemma 1.2.22). This is, however, in general not the case if the convergence of $\{A_h\}$ is somewhat less strong. In periodic homogenization (see [5] and [8]) we study (1) for

$$A_h(x) = A\Big(\frac{x}{\varepsilon_h}\Big),$$

where A is periodic with respect to the unit cube Y and $\varepsilon_h \to 0$ when $h \to \infty$. We have

$$A_h(x) = A\Big(\frac{x}{\varepsilon_h}\Big) \rightharpoonup \int_Y A(y)\,\mathrm{d}y \quad \text{in } L^2(\Omega)^{N\times N}, \tag{6}$$

but the matrix representing the limit problem does not coincide with the limit in (6). We have weak convergence in $L^2(\Omega)^{N\times N}$, and this is not strong enough. A correction term appears in the limit matrix.

The convergence of (1) to (5) is an example of G-convergence (see [8]). We say that the sequence $\{\mathcal{A}_h\}$ of differential operators, defined by the sequence $\{A_h\}$ of matrices through the left-hand side of (1), G-converges to an operator $\mathcal{B}$, represented by a matrix B, if for each f the solutions u_h to (1) converge weakly in $W_0^{1,2}(\Omega)$ to the solution u to

$$\begin{aligned} -\nabla\cdot(B(x)\nabla u(x)) &= f(x) \quad \text{in } \Omega,\\ u(x) &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{7}$$

However, it is not obvious how to determine B from the limit behavior of $\{A_h\}$. The question is if there is some kind of convergence weaker than (2), yet not identical with (6), where we do not have to bother about a correction term? How much can we disturb the behavior of $\{A_h\}$ until such a term appears?

In Section 2 we introduce a generalization of the two-scale convergence to the case without any periodicity assumption. In Section 3 we study G-convergence for (1), where $\{A_h\}$ is obtained by certain two-scale techniques based on the ideas in [12]. It turns out that it is possible to find an exact criterion for when we can compute the G-limit directly.

Notation 1. In general we use the Einstein tensor summation convention where indices are repeated. In a few places, however, we want to avoid summation. Let A_{ij} and B_{ij} be elements in the matrices A and B, respectively. For the product of the elements A_{ij} and B_{ij} without summation we write

$$[A_{ij}B_{ij}]_{i,j},$$

while

$$[A_{ij}B_{ij}]_i = \sum_{j=1}^{N} A_{ij}B_{ij}$$

means that there is no summation over i, just over j.

2. Two-scale convergence and two-scale compatible operators

If $\{u_h\}$ and $\{\nu_h\}$ are bounded sequences in $L^2(\Omega)^N$, we cannot be sure that

$$\int_\Omega u_h(x)\cdot\nu_h(x)\varphi(x)\,\mathrm{d}x \to \int_\Omega u(x)\cdot\nu(x)\varphi(x)\,\mathrm{d}x \tag{8}$$

holds with u and ν the weak limits of $\{u_h\}$ and $\{\nu_h\}$, respectively. We need to strengthen the assumptions on the sequences. One way to do this is to make further assumption that $\{\nu_h\}$ converges strongly in $L^2(\Omega)^N$, and thus obtain "weak-strong" convergence. Another option is to extend the strengthening of the convergence to both $\{u_h\}$ and $\{\nu_h\}$, like in compensated compactness, where we have certain demands on the derivatives of the sequences.

The following example illustrates the effect of the choice of sequences provided we only require boundedness in $L^2(\Omega)$.

E x a m p l e 2. Let $\{u_h\}$ and $\{\nu_h\}$ be two weakly convergent sequences in $L^2(\Omega)$, and let u and ν denote their weak limits. In general, it is not true that $u_h\nu_h$ will approach $u\nu$ in any traditional sense. If we choose $u_h(x) = \cos(x/\varepsilon_h)$, $\nu_h(x) = \sin(x/\varepsilon_h)$ and $\Omega = (a,b)$, we have

$$u(x)\nu(x) = 0 \tag{9}$$

and hence

$$u_h\nu_h \rightharpoonup u\nu \quad \text{in } D'(\Omega). \tag{10}$$

If we let $u_h(x) = \sin(x/\varepsilon_h)$ instead, we obtain

$$u_h\nu_h \rightharpoonup 1/2 \neq u\nu \quad \text{in } D'(\Omega),$$

so we do not have (10).

The notion of two-scale convergence, however, offers us a third alternative.

2.1. Periodic two-scale convergence

In two-scale convergence we have pairing of a sequence $\{u_h\}$ of functions bounded in $L^2(\Omega)$ and a sequence $\{\nu_h\}$ of test functions defined by

$$\nu_h(x) = v\Big(x, \frac{x}{\varepsilon_h}\Big), \tag{11}$$

where $v \in L^2(\Omega \times Y)$ is sufficiently smooth and periodic in the second argument with period being the unit cube Y in $\mathbb{R}^N$. The original result by Nguetseng reads that, up to a subsequence, for some $u_0 \in L^2(\Omega \times Y)$, we have

$$\lim_{h\to\infty} \int_\Omega u_h(x) v\Big(x, \frac{x}{\varepsilon_h}\Big)\,\mathrm{d}x = \int_\Omega \int_Y u_0(x,y) v(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

For this kind of test function v we also know that

$$\nu_h(x) = v\Big(x, \frac{x}{\varepsilon_h}\Big) \rightharpoonup \int_Y v(x,y)\,\mathrm{d}y \quad \text{in } L^2(\Omega), \tag{12}$$

without additional differential constraints.

Periodic two-scale convergence is well established and frequently used, but can also be viewed as a special case of the general two-scale convergence, which will be investigated in the next section. Here we will examine in particular the relation between the two-scale limit u_0 and the weak limit in $L^2(\Omega)$ in the periodic case. Later, the corresponding investigations will be done for the general case.

Let us start with the definition of the periodic case for two-scale convergence.

Definition 3. A sequence $\{u_h\}$ in $L^2(\Omega)$ is said to two-scale converge to a limit $u_0 \in L^2(\Omega \times Y)$ if

$$\lim_{h\to\infty} \int_\Omega u_h(x) v\Big(x, \frac{x}{\varepsilon_h}\Big)\,\mathrm{d}x = \int_\Omega \int_Y u_0(x,y) v(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{13}$$

for every $v \in L^2(\Omega; C_\#(Y))$. We denote this fact by

$$u_h \rightharpoonup\rightharpoonup u_0.$$

Theorem 4. *Let $\{u_h\}$ be a sequence in $L^2(\Omega)$ two-scale converging to $u_0 \in L^2(\Omega \times Y)$. Then $\{u_h\}$ is bounded in $L^2(\Omega)$ and*

$$u_h \rightharpoonup \int_Y u_0(x,y)\,\mathrm{d}y \quad \textit{in } L^2(\Omega). \tag{14}$$

Proof. See [2] and [19]. □

Theorem 4 follows if (13) is true for all $v \in L^2(\Omega; C_\#(Y))$, but not necessarily if this holds for a smaller class X, even if X is dense in $L^2(\Omega; C_\#(Y))$; see the discussion in Example 5.

In other words, if a sequence $\{u_h\}$ in $L^2(\Omega)$ two-scale converges to a function u_0, it also converges weakly to its integral mean value over Y in $L^2(\Omega)$. One can think that this is true if we assume only (13) for a smaller subset of $L^2(\Omega; C_\#(Y))$, like $C_0^\infty(\Omega; C_\#^\infty(Y))$ or $C(\Omega; C_\#^\infty(Y))$, but that is not the case. The following example from [13] shows that even though we obtain a limit as in (13), we cannot apply Theorem 4 since $\{u_h\}$ is not bounded in $L^2(\Omega)$.

Example 5. Let $\Omega = (0,2)$ and define $\{u_h\}$ by

$$u_h(x) = \begin{cases} \varepsilon_h^{-1}, & 0 < x < \varepsilon_h, \\ 0, & \varepsilon_h < x < 2. \end{cases}$$

Then (13) is satisfied with $u_0 = 0$ for all $v \in C_0^\infty(\Omega; C_\#^\infty(Y))$ while e.g. for $v \equiv 1$

$$\int_\Omega u_h(x) v(x)\,\mathrm{d}x \to 1 \neq 0.$$

This is due to the fact that $\{u_h\}$ is unbounded in $L^2(\Omega)$.

If we add the condition that $\{u_h\}$ is bounded in $L^2(\Omega)$ we may use $X = C_0^\infty(\Omega; C_\#^\infty(Y))$ in Definition 3.

Proposition 6. *Let $\{u_h\}$ be a bounded sequence in $L^2(\Omega)$ such that*

$$\lim_{h\to\infty}\int_\Omega u_h(x)v\Big(x,\frac{x}{\varepsilon_h}\Big)\,\mathrm{d}x=\int_\Omega\int_Y u_0(x,y)v(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

for every $v\in C_0^\infty(\Omega;C_\#^\infty(Y))$. Then $\{u_h\}$ two-scale converges to u_0.

By density arguments, it can also be proved that (13) holds for test functions in other spaces if $\{u_h\}$ two-scale converges, such as $L^2_\#(Y;C(\overline{\Omega}))$ for example. This is proved in [13], where a proof of Proposition 6 can also be found.

Remark 7 (historical). In the late 80's, Nguetseng (see [19]) presented a radically new approach to the homogenization of partial differential equations. The name of the method, two-scale convergence, is introduced in [2], where Allaire uses and develops it in various ways, and also applies it to a variety of problems. Moreover, in [1] by Allaire and Briane, two-scale convergence is extended to the linear stationary multi-scale case. The extension to the almost periodic case is found in [6] by Casado-Diaz and Gayte. See also the careful investigation of the theoretical fundaments and properties of periodic two-scale convergence in [13] by Lukkassen et al. Cioranescu et al. investigate a new approach, unfolding, closely related to two-scale convergence, in [7], see also [18] by Nechvátal. In [14], Mascarenhas and Toader introduce a concept called "scale-convergence" for Young measures. Further refinements to the method of two-scale convergence are made by Nguetseng in [20].

2.2. General two-scale convergence

In the definition of periodic two-scale convergence the key role was played by mapping test functions $v(x,y)$ of two variables to functions $v\big(x,x/\varepsilon_h\big)$ of one variable. The generalization is based on replacing this mapping with a sequence $\{\tau_h\}$ of transformations. We will introduce a certain kind of linear operators τ_h, investigate sequences of integral expressions of the type

$$\int_\Omega u_h(x)\nu_h(x)\,\mathrm{d}x=\int_\Omega u_h(x)\tau_h v(x)\,\mathrm{d}x,$$

and see that results of the same kind as (12) hold, see [12]. In the sequel we let Ω and Y be open and bounded subsets of $\mathbb{R}^N$ and $\mathbb{R}^M$, respectively.

We define general two-scale convergence.

Definition 8 (General two-scale convergence). Let X be a subspace of $L^2(\Omega\times Y)$ and let $\{\tau_h\}$ be a sequence of operators

$$\tau_h\colon\ X\to L^2(\Omega).$$

A sequence $\{u_h\}$ in $L^2(\Omega)$ is said to two-scale converge to $u_0 \in L^2(\Omega \times Y)$ with respect to $\{\tau_h\}$ if

$$\lim_{h\to\infty} \int_\Omega u_h(x)\tau_h v(x)\,\mathrm{d}x = \int_\Omega \int_Y u_0(x,y)v(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

for all $v \in X$.

Below we specify the necessary properties of $\{\tau_h\}$ and X to obtain compactness of a kind similar to the case traditional periodic two-scale convergence.

Definition 9 (Two-scale compatible operators). Let X be a normed linear space $X \subset L^2(\Omega \times Y)$. A sequence of operators $\{\tau_h\}$ is said to be two-scale compatible with respect to X if the operators

$$\tau_h \colon X \to L^2(\Omega)$$

are linear and have the properties

$$\lim_{h\to\infty} \|\tau_h v\|_{L^2(\Omega)} \leqslant C\|v\|_{L^2(\Omega\times Y)} \tag{15}$$

and

$$\|\tau_h v\|_{L^2(\Omega)} \leqslant C\|v\|_X \tag{16}$$

for all $v \in X$. The space X will be called the space of admissible test functions with respect to $\{\tau_h\}$.

The conditions in Definition 9 are not sufficient to create a connection between the limits of two-scale type that will appear, and the corresponding traditional weak limits. We strengthen the assumptions on $\{\tau_h\}$ in the next definition.

Definition 10 (Strongly two-scale compatible operator). Let a sequence of operators $\{\tau_h\}$ be two-scale compatible with respect to X. It is called strongly two-scale compatible if, in addition, the following two conditions are satisfied:

(i) For all v in X, we have

$$\tau_h v \rightharpoonup \int_Y v(x,y)\,\mathrm{d}y \quad \text{in } L^2(\Omega).$$

(ii) For all sequences $\{u_h\}$ two-scale converging to u_0 with respect to $\{\tau_h\}$, we have

$$u_h \rightharpoonup \int_Y u_0(x,y)\,\mathrm{d}y \quad \text{in } L^2(\Omega).$$

R e m a r k 11. The sequence $\{\tau_h\}$ of operators defined by

$$\tau_h v(x) = v\Big(x, \frac{x}{\varepsilon_h}\Big),$$

where $\varepsilon_h \to 0$ for $h \to \infty$, is strongly two-scale compatible with respect to $X = L^2(\Omega; C_{\#}(Y))$.

The next theorem means that compactness results of two-scale convergence type can be applied also to cases which do not involve any periodicity assumptions. Note that the admissible space X depends of the choice of $\{\tau_h\}$.

Theorem 12. *Let $\{u_h\}$ be a bounded sequence in $L^2(\Omega)$ and $X \subset L^2(\Omega \times Y)$ a separable Banach space. Then there exists $u_0 \in L^2(\Omega \times Y)$ such that, up to a subsequence,*

$$\lim_{h\to\infty} \int_\Omega u_h(x)\tau_h v(x)\,\mathrm{d}x = \int_\Omega \int_Y u_0(x,y)v(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

for all $v \in X$ if $\{\tau_h\}$ is two-scale compatible with respect to X.

P r o o f. See [12] and [21]. □

2.2.1. Properties of general two-scale convergence. The following propositions show how a relation between the two-scale limit u_0 and the weak $L^2(\Omega)$-limit of $\{u_h\}$ can be established. We state sufficient conditions for operators τ_h to be strongly two-scale compatible, and hence obtain the wanted connection between the limits.

Proposition 13. *Assume that $\{\tau_h\}$ is two-scale compatible with respect to X, where $C_0^\infty(\Omega) \subset X \subset L^2(\Omega \times Y)$, and that τ_h is working as an identity operator while acting on any $v \in X$ that is independent of the variable y. Assume further that the sequence $\{u_h\}$ is bounded in $L^2(\Omega)$ and two-scale converging with respect to $\{\tau_h\}$ to some $u_0 \in L^2(\Omega \times Y)$. Then*

$$u_h \rightharpoonup \int_Y u_0(x,y)\,\mathrm{d}y \quad \text{in } L^2(\Omega). \tag{17}$$

P r o o f. Suppose $\{u_h\}$ is bounded in $L^2(\Omega)$ and two-scale converging to $u_0 \in L^2(\Omega \times Y)$ with respect to $\{\tau_h\}$. Then

$$\int_\Omega u_h(x)v(x)\,\mathrm{d}x = \int_\Omega u_h(x)\tau_h v(x)\,\mathrm{d}x$$
$$\to \int_\Omega \int_Y u_0(x,y)v(x)\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega \Big(\int_Y u_0(x,y)\,\mathrm{d}y\Big)v(x)\,\mathrm{d}x$$

for any $v \in C_0^\infty(\Omega)$. Since $\{u_h\}$ is bounded in $L^2(\Omega)$, this means that

$$u_h \rightharpoonup \int_Y u_0(x,y)\,\mathrm{d}y \quad \text{in } L^2(\Omega)$$

and the proposition is proved. □

If we assume that $\{u_h\}$ converges strongly in $L^2(\Omega)$ then the second scale in the two-scale limit is lost, as proved in the proposition below.

Proposition 14. *Let $\{\tau_h\}$ be two-scale compatible with the additional condition*

$$\nu_h = \tau_h v \rightharpoonup \nu = \int_Y v(x,y)\,\mathrm{d}y \quad \textit{in } L^2(\Omega).$$

Assume further that

$$u_h \to u \quad \textit{in } L^2(\Omega).$$

Then

$$\lim_{h\to\infty} \int_\Omega u_h(x)\tau_h v(x)\,\mathrm{d}x = \int_\Omega \int_Y u(x)v(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega u(x)\nu(x)\,\mathrm{d}x$$

for all admissible v.

P r o o f. The "weak-strong" convergence immediately yields

$$\int_\Omega u_h(x)\tau_h v(x)\,\mathrm{d}x \to \int_\Omega u(x) \int_Y v(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_\Omega u(x)\nu(x)\,\mathrm{d}x$$

and the proof is complete. □

The following example from [12] introduces a construction of sequences $\{\tau_h\}$ of two-scale compatible operators without involving any periodicity demands.

E x a m p l e 15. Let $\{w_h\}$ be a bounded sequence in $L^4(\Omega \times Y)$ such that the conditions

$$\int_Y w_h(x,y)\,\mathrm{d}y = 1, \tag{18}$$

$$w_h \rightharpoonup w \quad \text{in } L^4(\Omega \times Y) \tag{19}$$

and

$$w_h^2 \rightharpoonup W \quad \text{in } L^2(\Omega \times Y) \tag{20}$$

hold with W in $L^\infty(\Omega \times Y)$. Then the operators $\{\tau_h\}$ defined by

$$\nu_h(x) = \tau_h v(x) = \int_Y w_h(x,y)v(x,y)\,\mathrm{d}y \tag{21}$$

form a two-scale compatible sequence of operators with respect to $L^4(\Omega \times Y)$. If in addition $w = 1$ and $|Y| = 1$, we have

$$\nu_h(x) = \tau_h v(x) \rightharpoonup \int_Y v(x,y)\,\mathrm{d}y \quad \text{in } L^2(\Omega)$$

and hence $\{\tau_h\}$ is strongly two-scale compatible.

Our next example introduces a particular choice of w_h to define a certain kind of two-scale convergence.

Example 16. Let

$$s_n = \sum_{k=1}^{n} \frac{2}{k}$$

with

$$s_0 = 0;$$

clearly $s_n \to \infty$. Let us introduce an auxiliary function φ defined on $[0,\infty)$,

$$\varphi(y) = \begin{cases} 2, & y \in \left[s_n, s_n + \dfrac{1}{n+1}\right), \\ 0, & \text{otherwhise.} \end{cases}$$

We shall study a sequence $\{\tau_h\}$ of two-scale operators defined by (21) for

$$w_h(x,y) = \varphi(y + p_h(x)), \tag{22}$$

where $y \in Y = (0,1)$, $x \in \Omega = (a,b)$ and p_h is piecewise smooth. The sequence $\{w_h\}$ is bounded in $L^\infty(\Omega \times Y)$ and picking a suitable subsequence, it is not difficult to see that $\{\tau_h\}$ defined by (22) yields two-scale convergence with respect to $\{\tau_h\}$. It is also easy to find sequences $\{p_h\}$ such that the key property (17) holds. Further, if we choose e.g.

$$p_h(x) = h$$

we get

$$w_h(x,y) = w_h(y) = w(y+h) \rightharpoonup 1 \quad \text{in } L^2(Y)$$

and hence (see Section 3.1)

$$\nu_h(x) = \int_Y w_h(y)v(x,y)\,\mathrm{d}y \to \nu(x) = \int_Y v(x,y)\,\mathrm{d}y \quad \text{in } L^2(\Omega).$$

Thus we obtain the conventional limit and arrive at

$$\lim_{h\to\infty}\int_\Omega u_h(x)\nu_h(x)\,\mathrm{d}x = \int_\Omega u(x)\nu(x)\,\mathrm{d}x$$

if

$$u_h \rightharpoonup u \quad \text{in } L^2(\Omega).$$

In the next section we study how the convergence of differential operators is affected when the strong convergence of the defining matrices is disturbed by adding dependence on the variable x in a similar way as above.

3. G-convergence and two-scale compatible operators

This chapter is devoted to the concept of G-convergence, a type of convergence for differential operators. We will characterize some G-limits for sequences $\{\mathcal{A}_h\}$ of differential operators obtained from sequences of symmetric matrices $\{A_h\}$ created by operators similar to those discussed in Example 15. We will in particular investigate whether and in what way the G-limits differ from the corresponding weak limits of $\{A_h\}$.

For the special case of linear elliptic operators we have the following definition.

Definition 17 (G-convergence)**.** Let $\{A_h\}$ be a sequence of symmetric matrices in $L^\infty(\Omega)^{N\times N}$ satisfying the structural conditions (3) and (4). Furthermore, let B be a symmetric matrix in the same space as $\{A_h\}$, satisfying the same conditions. If, for every $f \in W^{-1,2}(\Omega)$, the sequence $\{u_h\}$ of solutions to

$$\begin{aligned} -\nabla\cdot(A_h(x)\nabla u_h(x)) &= f(x) \quad \text{in } \Omega,\\ u_h(x) &= 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{23}$$

converges in the sense that

$$u_h \rightharpoonup u \quad \text{in } W_0^{1,2}(\Omega)$$

as $h\to\infty$, where u is the unique solution to

$$\begin{aligned} -\nabla\cdot(B(x)\nabla u(x)) &= f(x) \quad \text{in } \Omega,\\ u(x) &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{24}$$

we say that the sequence $\{\mathcal{A}_h\}$ of elliptic operators defined by $\{A_h\}$ G-converges to the elliptic operator $\mathcal{B}$ defined by B.

The following theorem states the sequential compactness of operators $\{\mathcal{A}_h\}$ introduced in Definition 17 with respect to G-convergence.

Theorem 18. *Given $\{A_h\}$, a sequence of symmetric matrices in $L^\infty(\Omega)^{N\times N}$ satisfying* (3) *and* (4), *then there exists a subsequence such that the corresponding sequence $\{\mathcal{A}_h\}$ of operators G-converges.*

Proof. See [22]. □

In periodic homogenization (see [5] and [8]) we study (23) for

$$A_h(x) = A\Big(\frac{x}{\varepsilon_h}\Big), \tag{25}$$

where A is periodic with respect to the unit cube Y and $\varepsilon_h \to 0$ when $h \to \infty$. Then we have

$$A_h(x) = A\Big(\frac{x}{\varepsilon_h}\Big) \rightharpoonup \int_Y A(y)\,\mathrm{d}y \quad \text{in } L^2(\Omega)^{N\times N}, \tag{26}$$

but the matrix representing the effective properties does not coincide with the limit in (26). We have weak convergence in $L^2(\Omega)$, and this is not strong enough. Here, the effective matrix B can be computed as

$$B_{ij} = \int_Y A_{ij}(y) + A(y)\nabla_y z_i \cdot e_j\,\mathrm{d}y, \tag{27}$$

where z_i is Y-periodic and solves

$$-\nabla_y \cdot (A(y)(e_i + \nabla_y z_i(y))) = 0 \quad \text{in } Y. \tag{28}$$

We know what the matrix B looks like when $\{A_h\}$ is periodic in the sense of (25), but what if it is not? In the next section we will first introduce sequences $\{A_h\}$ of matrices created by means of a certain kind of compact operators, and investigate what happens when the operators are modified in a way that relaxes compactness.

Remark 19. G-convergence was first introduced by Spagnolo ([22], [23] and [24]). Later, Murat and Tartar generalized the concept under the name of H-convergence, see [15], [16], [17], [25], [26]. Chiadò Piat and Defranceschi made generalizations to non-linear monotone cases in [9], which was also done in [10] by Chiadò Piat et al. Tartar studied the properties of H-convergence for a type of non-linear monotone problems in [25]. See also the monograph [11] on weak convergence by Evans.

3.1. Hilbert-Schmidt operators

Hilbert-Schmidt operators are compact and thus have the ability to transform weakly convergent sequences in reflexive Banach spaces into strongly convergent ones (see [4], 8.9). In

$$\nu_h(x) = \int_Y w_h(y)v(x,y)\,\mathrm{d}y, \tag{29}$$

such an operator is acting on the sequence $\{w_h\}$. The compactness of this operator means that it transforms weakly convergent (in an appropriate space) sequences $\{w_h\}$ to strongly convergent ones (under some additional assumptions).

The matrix version of (29) should not cause any problem. If $\{A_h\}$ is obtained as

$$(A_h)_{ij}(x) = \int_Y [(w_h)_{ij}(y)A_{ij}(x,y)]_{i,j}\,\mathrm{d}y, \tag{30}$$

where A is regular enough and

$$w_h \rightharpoonup 1 \quad \text{in } L^2(Y)^{N\times N}, \tag{31}$$

then

$$A_h \to A \quad \text{in } L^2(\Omega)^{N\times N}. \tag{32}$$

This means that the G-limit is represented by the strong $L^2(\Omega)$-limit of $\{A_h\}$ (see [3, Lemma 1.2.22]), and the solutions to (23) for $\{A_h\}$ as in (30) will then converge to the solution to (24) with

$$B(x) = \int_Y A(x,y)\,\mathrm{d}y. \tag{33}$$

Here the matrix B coincides with the weak limit of $\{A_h\}$ thanks to the strong convergence (32). This is, however, in general not the case if we let w_h depend on an additional variable. We will investigate this case in the next section.

3.2. Characterization of some G-limits

The purpose of this section is to compute the limit matrix B using information available about the sequence $\{A_h\}$. In periodic homogenization this B deviates from the weak limit of $\{A_h\}$, and the correction of the limit in (26) is contained in the term

$$\int_Y A(y)\nabla_y z_i \cdot e_j\,\mathrm{d}y$$

in (27). In [27] it is proved that this correction vanishes when

$$\partial_{y_i} A_{ij}(y) = 0. \tag{34}$$

Could we expect something similar to (34), when dealing with non periodic structures? To find out, we study (23) for

$$(A_h)_{ij}(x) = \int_Y [(w_h)_{ij}(x,y)A_{ij}(x,y)]_{i,j}\,\mathrm{d}y. \tag{35}$$

Since w_h does now depend on the variable x as well, we cannot be sure that $\{A_h\}$ converges strongly and clearly it is not sure that (33) holds. It turns out that the switch between when (33) holds, and when a correction term appears, is exactly when

$$(r_h)_j(x) = \int_Y [(\partial_{x_i}(w_h)_{ij}(x,y))A_{ij}(x,y)]_j\,\mathrm{d}y \rightharpoonup 0 \quad \text{in } L^2(\Omega). \tag{36}$$

The following proposition deals with this situation.

Proposition 20 (Two-scale G-convergence). *Let $\{u_h\}$ be a sequence of solutions to*

$$\begin{aligned} -\nabla\cdot(A_h(x)\nabla u_h(x)) &= f(x) \quad \text{in } \Omega,\\ u_h(x) &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{37}$$

The matrices A_h are defined by

$$A^h_{ij}(x) = \int_Y [(w_h)_{ij}(x,y)A_{ij}(x,y)]_{i,j}\,\mathrm{d}y \tag{38}$$

for $A, w_h \in C^1(\bar{\Omega}\times\bar{A})^{N\times N}$, where A and $\{w_h\}$ are chosen such that $\{A_h\}$ satisfy structural conditions of the kind introduced in (3) *and* (4)*. Assume further that*

$$w_h \rightharpoonup 1 \quad \text{in } L^2(\Omega\times Y)^{N\times N} \tag{39}$$

and

$$\int_Y [\partial_{x_i}(w_h)_{ij}(x,y)A_{ij}(x,y)]_j\,\mathrm{d}y \rightharpoonup 0 \quad \text{in } L^2(\Omega). \tag{40}$$

Then the G-limit is represented by

$$B(x) = \int_Y A(x,y)\,\mathrm{d}y$$

in the sense that

$$u_h \rightharpoonup u \quad \text{in } W^{1,2}_0(\Omega),$$

where u solves the homogenized problem

$$\begin{aligned} -\nabla\cdot(B(x)\nabla u(x)) &= f(x) \quad \textit{in } \Omega, \\ u(x) &= 0 \qquad \textit{on } \partial\Omega. \end{aligned}$$

Proof. The weak form of the left-hand side of (37) for $v \in C_0^\infty(\Omega)$ can be written as

$$\int_\Omega \partial_{x_i} u_h(x)((A_h)_{ij}(x)\partial_{x_j} v(x))\,\mathrm{d}x$$

and will after partial integration appear as

$$-\int_\Omega u_h(x)(\partial_{x_i}(A_h)_{ij}(x)\partial_{x_j} v(x) + (A_h)_{ij}(x)\partial^2_{x_ix_j} v(x))\,\mathrm{d}x. \tag{41}$$

Using (38), we arrive at

$$\begin{aligned} -\int_\Omega u_h(x)\bigg(\partial_{x_i}\bigg(\int_Y [(w_h)_{ij}(x,y)A_{ij}(x,y)]_{i,j}\,\mathrm{d}y\bigg)\partial_{x_j} v(x) \\ + \int_Y [(w_h)_{ij}(x,y)A_{ij}(x,y)]_{i,j}\,\mathrm{d}y\,\partial^2_{x_ix_j} v(x)\bigg)\,\mathrm{d}x, \end{aligned}$$

and further expansion gives

$$\begin{aligned} -\int_\Omega u_h(x)\bigg(\int_Y [\partial_{x_i}(w_h)_{ij}(x,y)A_{ij}(x,y)]_j\partial_{x_j} v(x)\,\mathrm{d}y \\ + \int_Y [(w_h)_{ij}(x,y)\partial_{x_i}A_{ij}(x,y)]_j\partial_{x_j} v(x)\,\mathrm{d}y \\ + \int_Y [(w_h)_{ij}(x,y)A_{ij}(x,y)]_{i,j}\partial^2_{x_ix_j} v(x)\,\mathrm{d}y\bigg)\,\mathrm{d}x. \end{aligned}$$

Letting $h \to \infty$ and using (39), (40) and the fact that $u_h \to u$ strongly in $L^2(\Omega)$, we obtain

$$\begin{aligned} -\int_\Omega u(x)\bigg[\bigg(\int_Y \partial_{x_i}A_{ij}(x,y)\,\mathrm{d}y\bigg)\partial_{x_j} v(x) + \bigg(\int_Y A_{ij}(x,y)\,\mathrm{d}y\bigg)\partial^2_{x_ix_j} v(x)\bigg]\,\mathrm{d}x \\ = -\int_\Omega u(x)\partial_{x_i}\bigg(\int_Y A_{ij}(x,y)\partial_{x_j} v(x)\,\mathrm{d}y\bigg)\,\mathrm{d}x. \end{aligned}$$

Integration by parts results in

$$\int_\Omega \partial_{x_i} u(x)\bigg(\int_Y A_{ij}(x,y)\partial_{x_j} v(x)\,\mathrm{d}y\bigg)\,\mathrm{d}x,$$

which we recognize as

$$\int_\Omega \left(\int_Y A_{ij}(x,y)\,\mathrm{d}y \right) \partial_{x_i} u(x) \partial_{x_j} v(x)\,\mathrm{d}x.$$

We have now obtained the homogenized matrix B given by

$$B(x) = \int_Y A(x,y)\,\mathrm{d}y$$

directly without using any local problem. We have shown that the weak limit u solves the homogenized equation

$$\begin{aligned} -\nabla \cdot (B(x) \nabla u(x)) = f(x) \quad &\text{in } \Omega, \\ u(x) = 0 \qquad &\text{on } \partial\Omega \end{aligned}$$

and hence that G-convergence is proved. □

Remark 21. It is the behavior of $[\partial_{x_i}(w_h)_{ij}]_{i,j}$ as $h \to \infty$ that decides whether the matrix representing the G-limit coincides with the mean value over y in the matrix A or not. If (40) holds, we obtain

$$B(x) = \int_Y A(x,y)\,\mathrm{d}y.$$

Otherwise, we get a deflection. The easiest way of obtaining (40) is to assume

$$[\partial_{x_i}(w_h)_{ij}(x,y)]_{i,j} = 0.$$

This means that w_h is independent of x, and hence we have returned to the situation in (30). If we assume that

$$A(x,y) = A(y),$$

then the conditions (40) turn into

$$\partial_{x_i} \int_Y [(w_h)_{ij}(x,y) A_{ij}(y)]_{i,j}\,\mathrm{d}y = \partial_{x_i}(A_h)_{ij} \rightharpoonup 0 \quad \text{in } L^2(\Omega). \tag{42}$$

In ordinary periodic homogenization, we have

$$(A_h)_{ij}(x) = A_{ij}\left(\frac{x}{\varepsilon_h}\right),$$

where A_{ij} is Y-periodic. The condition (42) is then satisfied if

$$\partial_{x_i}(A_h)_{ij}(x) = \partial_{x_i} A_{ij}\left(\frac{x}{\varepsilon_h}\right) = \frac{1}{\varepsilon_h} \partial_{y_i} A_{ij}\left(\frac{x}{\varepsilon_h}\right) = 0,$$

which holds true if

$$\partial_{y_i} A_{ij}(y) = 0.$$

This coincides with the classical condition (see [27, p. 39]) to obtain

$$B = \int_Y A(y)\,\mathrm{d}y$$

in periodic homogenization.

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