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TWO SIMPLE DERIVATIONS OF UNIVERSAL BOUNDS FOR THE C.B.S. INEQUALITY CONSTANT*

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Abstract. Universal bounds for the constant in the strengthened Cauchy-Bunyakowski-Schwarz inequality for piecewise linear-linear and piecewise quadratic-linear finite element spaces in 2 space dimensions are derived. The bounds hold for arbitrary shaped triangles, or equivalently, arbitrary matrix coefficients for both the scalar diffusion problems and the elasticity theory equations.

Keywords: finite element method, h- and p-refinement, strengthened Cauchy-Bunyakowski-Schwarz inequality

MSC 2000: 65N30, 65N22, 65F10

1. Introduction

This paper deals with estimates of the constant γ, which appears in the strengthened Cauchy-Bunyakowski-Schwarz (C.B.S.) inequality

$$|a(u,v)| \leqslant \gamma\sqrt{a(u,u)}\sqrt{a(v,v)} \quad \forall u \in U, \ v \in V,$$

where U, V are two (finite dimensional) linear spaces, $U \cap V = \{0\}$ and $a(\cdot,\cdot)$ is a symmetric positive definite or semidefinite bilinear form on a space W containing U and V as subspaces. It follows that γ equals the cosine of the angle between the subspaces U, V in a metric defined by $\sqrt{a(u,u)}$.

More precisely, we are interested in the cases where a comes from the variational formulation of an elliptic boundary value problem in a bounded domain $\Omega \subset \mathbb{R}^2$, U is a finite element space of piecewise linear functions and V is the complement of U in another finite element space $U \oplus V$, which arises by h- or p- refinement of U.

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In such cases the value of γ can be estimated locally. Namely, if $\mathcal{T}$ is a triangulation of Ω which is used for the definition of U, then

$$\gamma = \max_{E \in \mathcal{T}} \gamma_E,$$

$$|a_E(u,v)| \leqslant \gamma_E \sqrt{a_E(u,u)} \sqrt{a_E(v,v)} \quad \forall u \in U_E, \ \ v \in V_E,$$

where U_E and V_E are linear spaces of functions which are restrictions of functions from U and V to $E \in \mathcal{T}$, and where a_E denotes the restriction of the bilinear form a to E.

The estimates of γ can be used for the convergence analysis of many numerical procedures connected with the application of the finite element method; let us mention two-level and multi-level iterative methods and preconditioners, local refinement composite grid methods, a posteriori error estimates etc. Therefore, much effort has been devoted to the estimation of γ via local estimates; we mention here the papers by Axelsson [3], Axelsson and Gustafsson [6], Maitre and Musy [11], Margenov [13], Achchab and Maitre [2], Jung and Maitre [10], Axelsson [4], Achchab et al. [1] and the references therein.

From the estimates presented in literature, it is seen that γ generally depends on the bilinear form a, which includes the dependence on the problem coefficients, and on the type and shape of the finite elements used. We can also see that in some cases it is possible to have *universal bounds*, which do not depend on the problem coefficients and the shape of the finite elements.

This paper will be devoted to the derivation of such universal bounds of the C.B.S. constant. We will present results concerning the bilinear forms corresponding to the anisotropic Laplacian and to anisotropic elasticity operators. Moreover, we will show two ways of simple derivations of the bounds. The resulting bounds generalize and extend the results obtained previously in literature. Especially, a new result is obtained for the case of general anisotropic elasticity. This paper deals with 2D problems, a similar approach for 3D problems can be found in [9].

The paper is organized as follows. In Section 2, we describe more precisely the bilinear forms corresponding to the anisotropic Laplacian and the elasticity operator and formulate the P_1–P_1 and P_1–P_2 strengthened C.B.S. constant estimation problems. Then, in Section 3, we prove a universal bound for the P_1–P_1 problem in 2D. Another way of obtaining the universal bounds, using a relation between P_1–P_1 and P_1–P_2 problems, will be shown in Section 4. The paper ends with some concluding remarks.

2. Formulation of P_1–P_1 and P_1–P_2 problems for the anisotropic Laplacian and anisotropic elasticity operators

We shall consider two general types of bilinear forms. The first can be written in the form

$$a(u,v) = \int_\Omega \sum_{i,j=1}^{2} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \,\mathrm{d}x = \int_\Omega \langle Dd(u), d(v)\rangle \,\mathrm{d}x. \tag{2.1}$$

Here $u, v \in H^1(\Omega)$, $D = [a_{ij}]$ is a matrix of problem coefficients, which is assumed to be symmetric positive definite, $d(u) = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\right)^T$ is the gradient of u and $\langle x, y\rangle = x^T y$ for $x, y \in \mathbb{R}^2$. This bilinear form corresponds to an anisotropic Laplacian in $\mathbb{R}^2$.

The second bilinear form corresponds to a general anisotropic elasticity operator. It has the form

$$a(u,v) = \int_\Omega \sum_{i,j,k,l=1}^{2} c_{ijkl}\varepsilon_{ij}(u)\varepsilon_{kl}(v) \,\mathrm{d}x \tag{2.2}$$

where $u, v \in [H^1(\Omega)]^2$, c_{ijkl} are elasticity moduli, which are nonnegative and possess the symmetries

$$c_{ijkl} = c_{jikl} = c_{klij}, \tag{2.3}$$

see e.g. [12] for further details. The quantities

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2.4}$$

are the small strain tensor components. From (2.3) and (2.4), it follows that the bilinear form can be rewritten in the form

$$a(u,v) = \int_\Omega \langle Cd(u), d(v)\rangle \,\mathrm{d}x \tag{2.5}$$

where

$$d(u) = \left(\frac{\partial u_1}{\partial x_1}, \frac{\partial u_1}{\partial x_2}, \frac{\partial u_2}{\partial x_1}, \frac{\partial u_2}{\partial x_2}\right)^T \tag{2.6}$$

for $u \in [H^1(\Omega)]^2$, $u = (u_1, u_2)$, and C is the matrix consisting of the elasticity moduli,

$$C = \begin{bmatrix} c_{1111} & c_{1112} & c_{1121} & c_{1122} \\ c_{1211} & c_{1212} & c_{1221} & c_{1222} \\ c_{2111} & c_{2112} & c_{2121} & c_{2122} \\ c_{2211} & c_{2212} & c_{2221} & c_{2222} \end{bmatrix}. \tag{2.7}$$

As follows from (2.3), the matrix C is symmetric. This matrix should be also positive semidefinite and positive definite for vectors $w = (w_{11}, w_{12}, w_{21}, w_{22})^T$ which fulfil the symmetry relation $w_{12} = w_{21}$, see e.g. [12]. As an example, the matrix C corresponding to the case of plane strain with isotropic material has the form

$$C = \begin{bmatrix} \lambda + 2\mu & 0 & 0 & \lambda \\ 0 & \mu & \mu & 0 \\ 0 & \mu & \mu & 0 \\ \lambda & 0 & 0 & \lambda + 2\mu \end{bmatrix}, \tag{2.8}$$

where λ, μ, are the Lamé moduli, which are positive numbers.

Further, we shall assume that $\mathcal{T}_H$ is a finite element triangulation of Ω and that the coefficients a_{ij} and c_{ijkl} are constant on each element in $\mathcal{T}_H$. Now we can formulate two strengthened C.B.S. constant estimation problems.

P_1–P_1 **problem:** The h-refinement gives a new division $\mathcal{T}_h$ of Ω, which arises by dividing each $E \in \mathcal{T}_H$ into smaller triangles. How this is done will be outlined in the next section. For each element $E \in \mathcal{T}_H$ we consider spaces

$$U_E = \{v \in C(E)\colon\ v \in P_1\}, \tag{2.9}$$

$$U_E^h = \{v \in C(E)\colon\ v|_e \in P_1\ \ \forall e \in \mathcal{T}_h,\ \ e \subset E\}, \tag{2.10}$$

$$V_E = \{v \in U_E^h\colon\ v(x) = 0 \text{ for all vertices } x \text{ of } E\}, \tag{2.11}$$

$$U_E^h = U_E \oplus V_E. \tag{2.12}$$

Here $C(E)$ denotes the set of continuous functions on E.

Then the P_1–P_1 problem for the anisotropic Laplacian is to find a nontrivial bound for the C.B.S. constant $\gamma_{E,1}$ such that for all $u \in U_E$, $v \in V_E$

$$|a_E(u,v)| \leqslant \gamma_{E,1}\sqrt{a_E(u,u)}\sqrt{a_E(v,v)}. \tag{2.13}$$

Here,

$$a_E(u,v) = \int_E \langle Dd(u), d(v)\rangle. \tag{2.14}$$

For the corresponding P_1–P_1 problem for the anisotropic elasticity problem, (2.13) holds with

$$a_E(u,v) = \int_E \langle Cd(u), d(v)\rangle \,\mathrm{d}x \tag{2.15}$$

and $u \in \hat{U}_E$, $v \in \hat{V}_E$, where

$$\hat{U}_E = \{v = (v_1, v_2):\ v_i \in U_E \ \text{for}\ i = 1, 2\},$$
$$\hat{V}_E = \{v = (v_1, v_2):\ v_i \in V_E \ \text{for}\ i = 1, 2\}.$$

P_1–P_2 **problem:** Consider now a p-refinement, i.e., piecewise linear and quadratic functions over the elements of $\mathcal{T}_H$. For each element $E \in \mathcal{T}_H$ we consider the spaces

$$\begin{aligned}
U_E &= \{v \in C(E):\ v \in P_1\},\\
U_E^p &= \{v \in C(E):\ v \in P_2\},\\
V_E &= \{v \in U_E^p:\ v(x) = 0 \ \text{for all vertices}\ x\ \text{of}\ E\},\\
U_E^p &= U_E \oplus V_E.
\end{aligned}$$

The P_1–P_2 problem for the anisotropic Laplacian is again to find a nontrivial bound for the C.B.S. constant $\gamma_{E,2}$ such that

$$|a_E(u, v)| \leqslant \gamma_{E,2}\sqrt{a_E(u, u)}\sqrt{a_E(v, v)}$$

for all $u \in U_E$, $v \in V_E$ and a_E defined by (2.14). The corresponding P_1–P_2 problem for the elasticity can be defined in the same way as before.

3. Universal estimates for the P_1–P_1 problem

For 2D problems, we will consider triangular elements and the h-refinement of the type which is illustrated in Fig. 1. It means that now each coarse triangle is divided into m^2 smaller congruent triangles with edges which are m times shorter then the edges of the original coarse triangle. Moreover, each edge of the small triangle is parallel to some side of the original triangle. By the described division, we get from the original coarse triangulation with the mesh parameter H a new triangulation with the mesh parameter $h = H/m$.

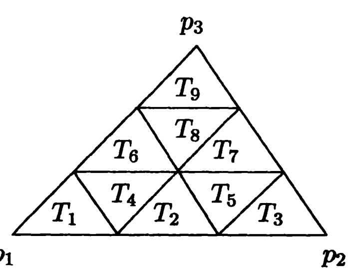

Figure 1. Division of a triangle into m^2 congruent ones, $h = H/m$, $m = 3$.

The aim of this section is to prove the following theorem:

Theorem 3.1. *Consider the bilinear forms* (2.14) *and* (2.15) *corresponding respectively to a general* 2D *anisotropic Laplacian or a general* 2D *anisotropic elasticity operator on* Ω. *Further, let* $\mathcal{T}_H$ *be a triangulation of* Ω *and assume that the problem coefficients are constant on the coarse elements* $E \in \mathcal{T}_H$. *Assume also that each element* $E \in \mathcal{T}_H$ *is divided into* m^2 *smaller congruent triangles in the described way. Then*

$$\gamma_{E,1} \leqslant \sqrt{\frac{m^2-1}{m^2}}. \tag{3.1}$$

We will prove Theorem 1 in several sub-steps in the following subsections. First we restrict our attention to a right angle isosceles reference triangle. For illustrative purposes, we first prove a universal estimate in the case of the anisotropic Laplacian and a division of the coarse triangles into four smaller ones, i.e., for $m = 2$. This simple case is subsequently extended to $m \geqslant 2$ and both anisotropic Laplacian and anisotropic elasticity. Finally, we show how to extend the estimates to the case of general triangles by an affine mapping of these triangles to the reference one.

3.1. Universal estimate of γ for a reference triangle and $m = 2$.

As an illustration of our approach, we will start with the simplest case of a reference triangle E with two axiparallel sides, see Fig. 2. The triangles are ordered as indicated in Fig. 1. We denote

$$\delta_i = \frac{\partial u}{\partial x_i}, \quad i = 1,2, \quad d(u) = \delta = (\delta_1, \delta_2)^T \text{ for } u \in U_E, \tag{3.2}$$

$$d_i^{(k)} = \frac{\partial v}{\partial x_i}\bigg|_{T_k}, \quad i = 1,2, \quad d(v)|_{T_k} = d^{(k)} = \left(d_1^{(k)}, d_2^{(k)}\right)^T \text{ for } v \in V_E. \tag{3.3}$$

Note that all quantities δ_i, $d_i^{(k)}$ are constants. We shall exploit certain relations between $d_i^{(k)}$. These relations are induced by the fact that v is zero at the vertices (which gives $d_1^{(2)} = -d_1^{(1)}$ and $d_2^{(4)} = -d_2^{(1)}$) and the fact that some triangles share an axiparallel side (which gives $d_1^{(3)} = d_1^{(4)}$ and $d_2^{(2)} = d_2^{(3)}$). These relations are illustrated in Fig. 2.

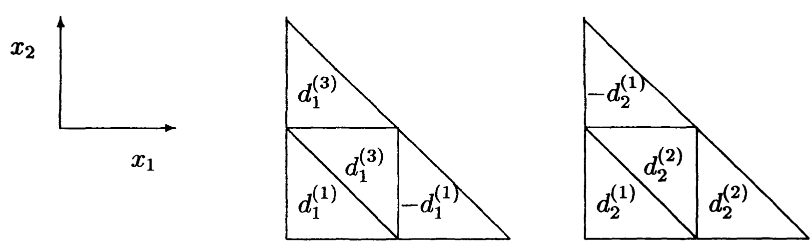

Figure 2. The reference triangle and values of $\frac{\partial v}{\partial x_i}$.

Now we can write

$$
\begin{aligned}
(3.4) \qquad a_E(u,v) &= \sum_{k=1}^{4} \int_{T_k} \langle D\delta, d^{(k)} \rangle \, \mathrm{d}x = \sum_{k=1}^{4} \langle \hat{\delta}, d^{(k)} \rangle \Delta \\
&= \big[\hat{\delta}_1 \big(d_1^{(1)} - d_1^{(1)} + 2d_1^{(3)}\big) + \hat{\delta}_2 \big(d_2^{(1)} - d_2^{(1)} + 2d_2^{(2)}\big)\big] \Delta \\
&= 2\Delta \langle \hat{\delta}, \hat{d} \rangle = 2\Delta \langle D\delta, \hat{d} \rangle \leqslant 2\Delta \|\delta\|_D \, \|\hat{d}\|_D,
\end{aligned}
$$

where $\hat{\delta} = D\delta = \big(\hat{\delta}_1, \hat{\delta}_2\big)^T$. Here we have used the fact that $\hat{\delta}_i$ are constant on E due to δ and D being constant on E. Above, we introduced $\|w\|_D = \sqrt{\langle Dw, w \rangle}$ for $w \in \mathbb{R}^2$ and Δ denotes the area of the smaller triangles, i.e., $|E| = 4\Delta$.

In (3.4) we have also introduced an auxiliary vector $\hat{d} = \big(d_1^{(3)}, d_2^{(2)}\big)^T$. Thus $\hat{d} = d^{(3)}$ and moreover

$$
(3.5) \qquad \hat{d} = d^{(1)} + d^{(2)} + d^{(4)}, \quad \|\hat{d}\|_D^2 \leqslant 3 \sum_{k \neq 3} \|d^{(k)}\|_D^2.
$$

Thus

$$
(3.6) \qquad a_E(v,v) = \sum_{k=1}^{4} \|d^{(k)}\|_D^2 \Delta \geqslant \Big(1 + \frac{1}{3}\Big) \|\hat{d}\|_D^2 \Delta,
$$

$$
(3.7) \qquad a_E(u,u) = \|\delta\|_D^2 4\Delta.
$$

From (3.4), (3.6), and (3.7) we therefore get

$$
(3.8) \qquad a_E(u,v) \leqslant \sqrt{\frac{3}{4}} \sqrt{a_E(u,u)} \sqrt{a_E(v,v)} \quad \forall u \in U_E, \ v \in V_E.
$$

This means that (2.13) holds with $\gamma_{E,1} = \sqrt{\frac{3}{4}}$.
This estimate is in accordance with the estimates in [11] and [4].

Note 3.1. The estimate (3.8) is valid for all coefficient matrices D. For a specific D, we can of course get a better estimate. For example, for $D = I$, it follows that

$$
(3.9) \qquad a_E(v,v) = 2\big[\big(d_1^{(1)}\big)^2 + \big(d_1^{(3)}\big)^2 + \big(d_2^{(1)}\big)^2 + \big(d_2^{(2)}\big)^2\big] \Delta \geqslant 2\Delta \|\hat{d}\|_D^2.
$$

From (3.4), (3.7) and (3.9) we now get

$$
(3.10) \qquad |a_E(u,v)| \leqslant \sqrt{\frac{1}{2}} \sqrt{a_E(u,u)} \sqrt{a_E(v,v)} \quad \forall u \in U_E, \ v \in V_E.
$$

This estimate (3.10) is in accordance with the results in [3] and [11].

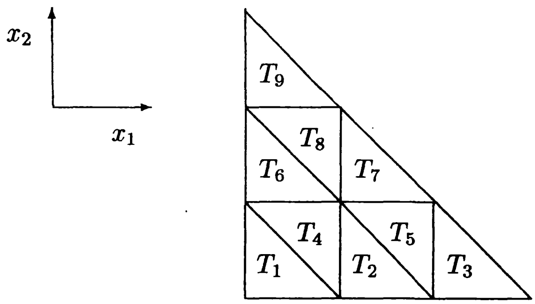

Figure 3. The reference triangle and $m = 3$. $I_1^0 = \{1,2,3\}$, $I_2^0 = \{1,6,9\}$, $I^* = \{4,5,8\}$.

3.2. Universal estimates of γ for a reference triangle and $m \geqslant 2$.

The estimate obtained in Subsection 3.1 can be simply generalized to denser grid refinements. Let us consider the reference triangle and its division into m^2 triangles in the way which is illustrated in Fig. 3.

We shall again consider the derivatives of the functions $u \in U_E$ and $v \in V_E$. By considering the x_1-parallel sides of the small triangles, we can see that

- there are m triangles T_k, $k \in I_1^0$ which do not share this side with other triangles. Because $v \in V_H$ is zero at the vertices, we get $\sum\limits_{k \in I_1^0} d_1^{(k)} = 0$;
- the remaining triangles can be divided into pairs which share one x_1-parallel side. Let I^* be the set of indices of the lower triangles from each pair.

For the x_2-parallel sides of the small triangles, we observe a similar structure:

- there are m triangles T_k, $k \in I_2^0$ which do not share this side with other triangles. Because $v \in V_H$ is zero at the vertices, we get $\sum\limits_{k \in I_2^0} d_2^{(k)} = 0$;
- the remaining triangles can be divided into pairs which share one x_2-parallel side. The set I^* introduced above now gives the indices of the left triangles from each pair.

For $u \in U_E$ and $v \in V_E$ we can now write

$$
\begin{aligned}
(3.11) \qquad a_E(u,v) &= \sum_{k=1}^{m^2} \int_{T_k} \langle D\delta, d^{(k)} \rangle \, \mathrm{d}x = \sum_{k=1}^{m^2} \langle \hat{\delta}, d^{(k)} \rangle \Delta \\
&= 2\Delta \langle \hat{\delta}, \hat{d} \rangle = 2\Delta \langle D\delta, \hat{d} \rangle \leqslant 2\Delta \|\delta\|_D \, \|\hat{d}\|_D
\end{aligned}
$$

where the meaning of $\hat{\delta}$, Δ is the same as in Subsection 3.1, $\hat{\delta}$ is constant on E and

$$
(3.12) \qquad \hat{d} = \left(\sum_{k \in I^*} d_1^{(k)}, \sum_{k \in I^*} d_2^{(k)} \right)^T .
$$

Thus,

$$\hat{d} = \sum_{k \in I^*} d^{(k)} \quad \text{and also} \quad \hat{d} = \sum_{k \in (I^*)^c} d^{(k)}, \tag{3.13}$$

where $(I^*)^c = \{1, \ldots, m^2\} \setminus I^*$ is the complementary set to I^*.

Noting that the sets I^* and $(I^*)^c$ contain $(m^2 - m)/2$ and $(m^2 + m)/2$ indices, respectively, we obtain the estimates

$$\|\hat{d}\|_D^2 \leqslant \frac{m(m-1)}{2} \sum_{k \in I^*} \|d^{(k)}\|_D^2, \tag{3.14}$$

$$\|\hat{d}\|_D^2 \leqslant \frac{m(m+1)}{2} \sum_{k \in (I^*)^c} \|d^{(k)}\|_D^2. \tag{3.15}$$

Thus,

$$a_E(v,v) = \sum_{k=1}^{m^2} \|d^{(k)}\|_D^2 \Delta \geqslant \left[\frac{2}{m(m-1)} + \frac{2}{m(m+1)}\right] \|\hat{d}\|_D^2 \Delta, \tag{3.16}$$

$$a_E(u,u) = \|\delta\|_D^2 m^2 \Delta. \tag{3.17}$$

From (3.11), (3.16), and (3.17), we get

$$a_E(u,v) \leqslant \sqrt{\frac{m^2-1}{m^2}} \sqrt{a_E(u,u)} \sqrt{a_E(v,v)}, \tag{3.18}$$

which implies the general result $\gamma_{E,1} \leqslant \sqrt{(m^2-1)/m^2}$.

3.3. Universal estimate of γ for the elasticity operator.

In this subsection the reference triangle and its division will be the same as in the previous subsection but we will consider the bilinear form and spaces corresponding to the elasticity operator. The expression of the elasticity bilinear form in formula (2.5) allows us to readily extend the previous results.

Let us denote

$$\delta = d(u) = (\delta_{11}, \delta_{12}, \delta_{21}, \delta_{22})^T, \quad \delta_{ij} = \frac{\partial u_i}{\partial x_j} \text{ for } u \in \hat{U}_E, \tag{3.19}$$

$$\begin{aligned} d^{(k)} &= d(v)|_{T_k} = \left(d_{11}^{(k)}, d_{12}^{(k)}, d_{21}^{(k)}, d_{22}^{(k)}\right)^T, \\ d_{ij}^{(k)} &= \left.\frac{\partial v_i}{\partial x_j}\right|_{T_k} \quad \text{for } v \in \hat{V}_E. \end{aligned} \tag{3.20}$$

Then all quantities δ_{ij}, $d_{ij}^{(k)}$ are constants and for $i, j = 1, 2$ the same relations hold between the derivatives $d_{ij}^{(k)}$ as in Subsection 3.2. Thus

$$\begin{aligned} a_E(u,v) &= \sum_{k=1}^{m^2} \int_{T_k} \langle C\delta, d^{(k)} \rangle \,\mathrm{d}x = \sum_{k=1}^{m^2} \langle \hat{\delta}, d^{(k)} \rangle \Delta \\ &= \sum_{i,j=1}^{2} \hat{\delta}_{ij} 2 \sum_{k \in I^*} d_{ij}^{(k)} \cdot \Delta \\ &= 2\Delta \langle \hat{\delta}, \hat{d} \rangle = 2\Delta \langle C\delta, \hat{d} \rangle \leqslant 2\Delta \|\delta\|_C \, \|\hat{d}\|_C, \end{aligned} \tag{3.21}$$

where $\hat{\delta} = C\delta$ is again constant on E due to δ and C being constant, $\|z\|_C = \sqrt{\langle Cz, z \rangle}$ is the seminorm induced by C and

$$\hat{d} = \sum_{k \in I^*} d^{(k)} \quad \text{and also} \quad \hat{d} = \sum_{k \in (I^*)^c} d^{(k)}. \tag{3.22}$$

The above expressions for $\hat{d}$ lead to the estimates

$$\|\hat{d}\|_C^2 \leqslant \frac{m(m-1)}{2} \sum_{k \in I^*} \|d^{(k)}\|_C^2, \tag{3.23}$$

$$\|\hat{d}\|_C^2 \leqslant \frac{m(m+1)}{2} \sum_{k \in (I^*)^c} \|d^{(k)}\|_C^2. \tag{3.24}$$

Thus,

$$a_E(v,v) = \sum_{k=1}^{m^2} \|d^{(k)}\|_C^2 \Delta \geqslant \left[\frac{2}{m(m+1)} + \frac{2}{m(m-1)} \right] \|\hat{d}\|_C^2 \Delta, \tag{3.25}$$

$$a_E(u,u) = \|\delta\|_C^2 m^2 \Delta. \tag{3.26}$$

From (3.21), (3.25) and (3.26), we conclude again that $\gamma_{E,1} \leqslant \sqrt{(m^2-1)/m^2}$.

3.4. Universal estimate of γ for a general triangle.

Now, we shall consider a general triangle E and its division to m^2 congruent smaller triangles with each side parallel to some side of the original triangle, as well as the reference triangle $\tilde{E}$ with vertices (0,0), (1,0), (0,1) and its division in the same way. It is important to note that it is possible to find an affine mapping $F\colon\ \tilde{E} \to E$. This mapping will also map each smaller triangle from the division of $\tilde{E}$ into the corresponding smaller triangle from the division of E, see Fig. 4.

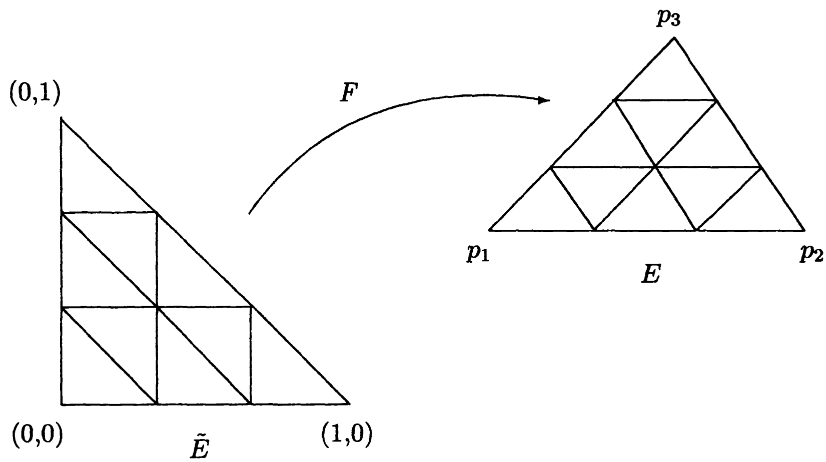

Figure 4. Mapping the reference triangle $\tilde{E}$ to a general triangle E.

Let $p_i = (p_{i1}, p_{i2})$ be the vertices of E. Then the mapping $F\colon \tilde{x} \to x$ can be described analytically by the relations

$$x_1 = p_{11} + (p_{21} - p_{11})\tilde{x}_1 + (p_{31} - p_{11})\tilde{x}_2,$$
$$x_2 = p_{12} + (p_{22} - p_{12})\tilde{x}_1 + (p_{32} - p_{12})\tilde{x}_2.$$

Now, let us consider $u \in U_E$, $v \in V_E$. Then $\tilde{u} = u \circ F \in U_{\tilde{E}}$ and $\tilde{v} = v \circ F \in V_{\tilde{E}}$. Moreover, $a_E(u,u)$, $a_E(v,v)$ and $a_E(u,v)$ can be transformed to $\tilde{a}_{\tilde{E}}(\tilde{u},\tilde{u})$, $\tilde{a}_{\tilde{E}}(\tilde{v},\tilde{v})$ and $\tilde{a}_{\tilde{E}}(\tilde{u},\tilde{v})$, see also [4]. The transformation $a \to \tilde{a}$ starts with the transformation of the derivatives. If $\tilde{u} = u \circ F$ then

$$\frac{\partial \tilde{u}}{\partial \tilde{x}_j}(\tilde{x}) = \frac{\partial u}{\partial x_1}(x)(p_{j+1,1} - p_{1,1}) + \frac{\partial u}{\partial x_2}(x)(p_{j+1,2} - p_{1,2}).$$

This yields $d(\tilde{u}) = G^T d(u)$, where G is the Jacobian matrix of F,

$$G = \begin{bmatrix} p_{21} - p_{11} & p_{31} - p_{11} \\ p_{22} - p_{12} & p_{32} - p_{12} \end{bmatrix}.$$

Now, we can write

$$\begin{aligned} a_E(u,u) &= \int_E \langle Dd(u), d(u)\rangle \,\mathrm{d}x = \int_{\tilde{E}} \langle DG^{-T}d(\tilde{u}), G^{-T}d(\tilde{u})\rangle |\det(G)| \,\mathrm{d}\tilde{x} \\ &= \int_{\tilde{E}} \langle \tilde{D}d(\tilde{u}), d(\tilde{u})\rangle \,\mathrm{d}\tilde{x} = \tilde{a}_{\tilde{E}}(\tilde{u},\tilde{u}), \quad \tilde{u} = u \circ F, \end{aligned}$$

where $\tilde{D} = |\det(G)| G^{-1} D G^{-T}$ is the transformed coefficient matrix, which is symmetric and positive definite, and

$$\tilde{a}_{\tilde{E}}(\tilde{u},\tilde{v}) = \int_{\tilde{E}} \langle \tilde{D}d(\tilde{u}), d(\tilde{v})\rangle \,\mathrm{d}\tilde{x} \quad \forall \tilde{u} \in U_{\tilde{E}},\ \tilde{v} \in V_{\tilde{E}}. \tag{3.27}$$

Similarly, we can show that

$$a_E(v,v) = \tilde{a}_{\tilde{E}}(\tilde{v},\tilde{v}) \quad \text{and} \quad a_E(u,v) = \tilde{a}_{\tilde{E}}(\tilde{u},\tilde{v}).$$

For $u \in U_E$, $v \in V_E$ we get $\tilde{u} \in U_{\tilde{E}}$ and $\tilde{v} \in V_{\tilde{E}}$. Using now the universal estimate of γ, which has been proved for the case of the reference triangle and an arbitrary symmetric positive definite coefficient matrix, we get

$$\begin{aligned}
|a_E(u,v)| &= |\tilde{a}_{\tilde{E}}(\tilde{u},\tilde{v})| \\
&\leqslant \sqrt{\frac{m^2-1}{m^2}}\sqrt{\tilde{a}_{\tilde{E}}(\tilde{u},\tilde{u})}\sqrt{\tilde{a}_{\tilde{E}}(\tilde{v},\tilde{v})} \\
&= \sqrt{\frac{m^2-1}{m^2}}\sqrt{a_E(u,u)}\sqrt{a_E(v,v)}.
\end{aligned}$$

The extension of this transformation technique to the case of general elasticity with the bilinear form (2.5) is straightforward. The transformed matrix $\tilde{C}$ will be in the form $\tilde{C} = |\det(G)| G_2^{-1} C G_2^{-T}$, where G_2 is a block diagonal 4×4 matrix with the diagonal blocks equal to G.

Remark 3.1. The universal estimates of the C.B.S. constant have been proven for a special grid refinement. For other types of division of triangles, it may be impossible to derive a nontrivial universal estimate of the C.B.S. inequality. For example, for the division of triangles illustrated in Fig. 5 and 2D elasticity (plane strain), we do not get a nontrivial estimate, as the C.B.S. constant γ depends on the Poisson ratio ν and γ tends to unity as ν goes to 1/2, see [8], [10].

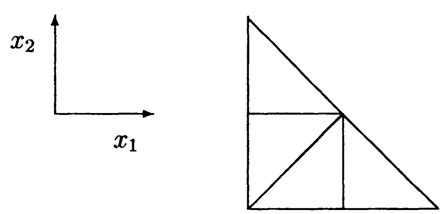

Figure 5. A different type of division of the triangle.

The technique of derivation of universal estimates of the C.B.S. constant exploited in this section can be also used for analysis of the 3D anisotropic Laplacian and 3D elasticity assuming that the linear tetrahedral finite elements are used, see [9].

4. An algebraic approach to the derivation the P_1–P_2 C.B.S. constant

Given a triangular element E with angles α, β, γ which has been regularly refined in four subtriangles, the local assembled P_1 and P_2 matrices have the form (see e.g. [5])

$$A_{H/2}^{(1)} = \frac{1}{2}\begin{bmatrix} 2d & -2c & -2b & 0 & -a & -a \\ -2c & 2d & -2a & -b & 0 & -b \\ -2b & -2a & 2d & -c & -c & 0 \\ 0 & -b & -c & b+c & 0 & 0 \\ -a & 0 & -c & 0 & a+c & 0 \\ -a & -b & 0 & 0 & 0 & a+b \end{bmatrix},$$

which is the matrix assembled from the element matrix for the four subtriangles corresponding to piecewise linear basis functions, and

$$A_H^{(2)} = \frac{1}{6}\begin{bmatrix} 8d & -8c & -8b & 0 & -4a & -4a \\ -8c & 8d & -8a & -4b & 0 & -4b \\ -8b & -8a & 8d & -4c & -4c & 0 \\ 0 & -4b & -4c & 3(b+c) & c & b \\ -4a & 0 & -4c & c & 3(a+c) & a \\ -4a & -4b & 0 & b & a & 3(a+b) \end{bmatrix},$$

which is the local finite element matrix corresponding to quadratic basis functions on E. Here $a = \cot\alpha$, $b = \cot\beta$, $c = \cot\gamma$ and $d = a + b + c$. These matrices yield the relation

$$A_H^{(2)} = \frac{4}{3}A_{H/2}^{(1)} - N, \tag{4.28}$$

where

$$N = \frac{1}{3}\begin{bmatrix} 0 & 0 \\ 0 & A_H^{(1)} \end{bmatrix} \tag{4.29}$$

and

$$A_H^{(1)} = \frac{1}{2}\begin{bmatrix} (b+c) & -c & -b \\ -c & (a+c) & -a \\ -b & -a & (a+b) \end{bmatrix},$$

the latter being the local finite element matrix for the vertex nodes of E, corresponding to linear basis functions.

R e m a r k 4.1. The relation

$$A_H^{(2)} = \frac{4}{3}A_{H/2}^{(1)} - \frac{1}{3}\begin{bmatrix} 0 & 0 \\ 0 & A_H^{(1)} \end{bmatrix}$$

could have been derived in an alternate way, using one step of the classical Richardson H-extrapolation to derive a locally third order accurate scheme $\left(A_H^{(2)}\right)$ from the two locally second order accurate schemes $\left(A_{H/2}^{(1)}\right.$ and $\left.A_H^{(1)}\right)$, by elimination of the $O(H^2)$ error components.

Now relation (4.28) can be used to derive a relation between the C.B.S. constants for the P_1–P_2 and P_1–P_1 hierarchical finite element spaces combinations.

Theorem 4.1. *For any finite element mesh regularly refined into congruent elements, for which* (4.28) *holds, one has*

$$\gamma_2^2 = \frac{4}{3}\gamma_1^2 \tag{4.30}$$

where γ_1, γ_2 are the C.B.S. constants for piecewise linear and piecewise quadratic finite elements, respectively.

Proof. Since all block parts of the first two matrices in (4.28) except the lower right block are equal we can take Schur complements, and (4.29) shows that

$$S^{(2)} = \frac{4}{3}S^{(1)} - \frac{1}{3}A_H^{(1)}.$$

Hence

$$\frac{x_2^T S^{(2)} x_2}{x_2^T A_H^{(1)} x_2} = \frac{4}{3}\frac{x_2^T S^{(1)} x_2}{x_2^T A_H^{(1)} x_2} - \frac{1}{3},$$

i.e.,

$$\begin{aligned} 1 - \gamma_2^2 &= \min_{x_2} \frac{x_2^T S^{(2)} x_2}{x_2^T A_H^{(1)} x_2} = \frac{4}{3}\min_{x_2} \frac{x_2^T S^{(1)} x_2}{x_2^T A_H^{(1)} x_2} - \frac{1}{3} \\ &= \frac{4}{3}(1 - \gamma_1^2) - \frac{1}{3} = 1 - \frac{4}{3}\gamma_1^2. \end{aligned}$$

□

Remark 4.2. The relation (4.30) was shown previously in [11], [4] using a more involved derivation. Using the already derived expressions for γ_1 in Section 3, we have then also a general expression for $\gamma_2 = \frac{2}{\sqrt{3}}\gamma_1$. We note that γ_2 can take values arbitrarily close to one for degenerate triangles, or equivalently, for certain anisotropies of the coefficients in the differential operator. On the other hand, (4.30) implies again the universal bound for γ_1.

5. Concluding remarks

The matrices of triangular finite elements derived from the two-level methods can be partitioned into a two-by-two block matrix form $[A_{ij}]_{i,j=1}^{2}$, where A_{11} corresponds to the node points added in the refinement process, A_{22} corresponds to the original vertex nodes and A_{12} (A_{21}) corresponds to the coupling between the two finite element subspaces used.

The following matrix relation holds:

$$u^T A_{12} v \leqslant \gamma \{u^T A_{11} u v^T A_{22} v\}^{\frac{1}{2}},$$

where $u = (\alpha_1, \alpha_2, \alpha_3, 0, 0, 0)^T$, $v = (0, 0, 0, \beta_1, \beta_2, \beta_3)^T$ are the corresponding orthogonal vectors.

Using the hierarchical form of the matrices one can precondition the block matrix by its block-diagonal part, $\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$, in which case the condition number becomes $(1+\gamma)/(1-\gamma)$. Alternatively, one can precondition the reduced (Schur complement) system, $A_{22} - A_{21}A_{11}^{-1}A_{12}$ with A_{22}, in which case the condition number becomes $1/(1-\gamma^2)$. Here it should be noted that the Schur complements will be the same as for the hierarchical basis even if the reduction takes place from the standard basis function matrix (i.e., from $A_H^{(2)}$ or $A_{H/m}^{(1)}$). For further details, see e.g. [5]. Systems with the matrix A_{22} which occur in this preconditioning can be solved using the same method recursively, unless one finds that the matrix $A_{11}^{(1)}$ is already sufficiently coarse to use a direct solution method or a simpler iterative method.

Matrix A_{11} is frequently well-conditioned and systems with it can be solved efficiently by some simple iteration method. However, for nearly degenerate triangles or, equivalently, strongly anisotropic coefficients it becomes very ill-conditioned and requires some special element by element preconditioner, see e.g. [7] for details.

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