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A NOTE ON THE GENERALIZED ENERGY INEQUALITY IN THE NAVIER-STOKES EQUATIONS*

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Abstract. We prove that there exists a suitable weak solution of the Navier-Stokes equation, which satisfies the generalized energy inequality for every nonnegative test function. This improves the famous result on existence of a suitable weak solution which satisfies this inequality for smooth nonnegative test functions with compact support in the space-time.

Keywords: Navier-Stokes equations, suitable weak solution, generalized energy inequality

MSC 2000: 35Q35, 35Q30

Suppose that we solve the Navier-Stokes equations for sufficiently smooth data. It is proved in [1] that then there exists a suitable weak solution of these equations, i.e. the generalized energy inequality holds for the smooth test functions with compact support in the space-time domain. The goal of this paper is to describe briefly the construction of a suitable weak solution which satisfies the above mentioned generalized energy inequality for *every* smooth test function. A similar result is also mentioned in [6], but it is not proved there.

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $\mathscr{C}^{2+\mu}$ boundary $\partial\Omega$ ($\mu > 0$), $T > 0$, $Q_T = \Omega \times (0,T)$. The classical formulation of the Navier-Stokes initial-boundary value problem for a viscous incompressible fluid can be written as

$$\frac{\partial u}{\partial t} - \nu \cdot \Delta u + (u \cdot \nabla)u + \nabla \mathscr{P} = f \quad \text{in } Q_T, \tag{1}$$

$$\nabla \cdot u = 0 \quad \text{in } Q_T, \tag{2}$$

$$u(\cdot,0) = u_0, \tag{3}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{4}$$

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where $u = (u_1, u_2, u_3)$ and $\mathscr{P}$ denote the velocity and the pressure, $\nu > 0$ is the viscosity coefficient and f is an external body force. Throughout the paper we suppose that $\nu = 1$.

Let $\mathscr{V} = \mathscr{C}_0^\infty(\Omega, \mathbb{R}^3) \cap \{v; \nabla \cdot v = 0\}$. As is usual in mathematical literature, H and V, respectively, denote the closures of $\mathscr{V}$ in the norms of $[L^2(\Omega)]^3$ and $[W_0^{1,2}(\Omega)]^3$. Denote further $A = -P_H \Delta$, where P_H is the Helmholtz projection from $[L^2(\Omega)]^3$ onto H. Then $\mathscr{D}(A) = [W^{2,2}(\Omega)]^3 \cap V$ is a Banach space with the norm $\|\cdot\|_{\mathscr{D}(A)}$ which is equivalent to the norm $\|\cdot\|_{[W^{2,2}(\Omega)]^3}$.

We write $L^{\alpha,\beta}$ instead of $L^\alpha(0,T; L^\beta(\Omega))$ and $\|\cdot\|_{L^{\infty,2} \cap L^{2,6}}$ is the sum

$$\|\cdot\|_{L^\infty(0,T;L^2(\Omega))} + \|\cdot\|_{L^2(0,T;L^6(\Omega))}.$$

Throughout the paper we suppose that the following conditions are satisfied:

$$f \in L^2(0,T;H) \tag{5}$$

and for simplicity,

$$u_0 \in \mathscr{D}(A). \tag{6}$$

The suitable weak solution in [1] is defined as follows.

Definition. Let

$$f \in L^q(Q_T), \quad q > 5/2, \tag{7}$$

$$\nabla \cdot f = 0, \tag{8}$$

$$u_0 \in H. \tag{9}$$

The pair $(u, \mathscr{P})$ is called a suitable weak solution of (1)–(4) if

$$u \in L^2(0,T;V) \cap L^\infty(0,T;H), \tag{10}$$

$$u(t) \to u_0 \ \text{ weakly in } \ H \ \text{ for } \ t \to 0, \tag{11}$$

$$\mathscr{P} \in L^{\frac{5}{4}}(Q_T), \tag{12}$$

$$\text{equation (1) holds in the sense of distributions in } Q_T \tag{13}$$

and

$$\begin{aligned} \int_{\Omega\times\{t\}} |u|^2\varphi + 2\int_0^t\int_\Omega |\nabla u|^2\varphi &\leqslant \int_{\Omega\times\{0\}} |u_0|^2\varphi + \int_0^t\int_\Omega |u|^2\Big(\frac{\partial\varphi}{\partial t} + \Delta\varphi\Big) \\ &\quad + \int_0^t\int_\Omega (|u|^2u + 2\mathscr{P}u)\cdot\nabla\varphi + 2\int_0^t\int_\Omega fu\varphi \end{aligned} \tag{14}$$

for every $\varphi \in \mathscr{C}^\infty(\overline{Q_T})$, $\varphi \geqslant 0$, $\varphi = 0$ in a neighbourhood of $\partial\Omega \times (0,T)$ and for every $t \in (0,T)$.

The following result is proved in [1].

Theorem 1. *Let* (7) *and* (8) *hold and* $u_0 \in H \cap W_0^{\frac{2}{5},\frac{5}{4}}(\Omega)$*. Then there exists a suitable weak solution of* (1)–(4).

The following theorem is the main result of this paper.

Theorem 2. *Let* (5)–(6) *hold. Then there exists a suitable weak solution* $(u, \mathscr{P})$ *of* (1)–(4)*. Furthermore,*

$$\text{the function } u\colon [0,T] \to H \text{ is weakly continuous,} \tag{15}$$

$$\mathscr{P} \in L^{r,s^*}, \tag{16}$$

where

$$\frac{2}{r} + \frac{3}{s^*} = 3, \quad 1 < r < 2, \quad \frac{3}{2} < s^* < 3 \tag{17}$$

and

$$\begin{aligned} \int_{\Omega\times\{t_2\}} |u|^2\varphi + 2\int_{t_1}^{t_2}\int_\Omega |\nabla u|^2\varphi &\leqslant \int_{\Omega\times\{t_1\}} |u|^2\varphi + \int_{t_1}^{t_2}\int_\Omega |u|^2\Big(\frac{\partial\varphi}{\partial t} + \Delta\varphi\Big) \\ &\quad + \int_{t_1}^{t_2}\int_\Omega (|u|^2u + 2\mathscr{P}u)\cdot\nabla\varphi + 2\int_{t_1}^{t_2}\int_\Omega fu\varphi \end{aligned} \tag{18}$$

for every $\varphi \in \mathscr{C}^\infty(\overline{Q_T})$, $\varphi \geqslant 0$, *for almost every* $t_1 \in [0,T]$ *and every* $t_2 \in [0,T]$, $t_1 < t_2$*. Moreover,* (18) *holds for* $t_1 = 0$.

First, we present a few lemmas. The first lemma is proved in [5].

Lemma 1. *If* $g \in L^{\infty,2} \cap L^{2,6}$ *and* $\alpha \in [2,\infty]$, $\beta \in [2,6]$, $\frac{2}{\alpha} + \frac{3}{\beta} \geqslant \frac{3}{2}$*, then*

$$\|g\|_{L^{\alpha,\beta}} \leqslant c\|g\|_{L^{2,2}}^{\frac{2}{\alpha}+\frac{3}{\beta}-\frac{3}{2}}\|g\|_{L^{\infty,2}\cap L^{2,6}}^{\frac{5}{2}-(\frac{2}{\alpha}+\frac{3}{\beta})},$$

where $c = c(\Omega)$.

The following lemma is an immediate consequence of Lemma 1.

Lemma 2. *If $g \in L^{\infty,2} \cap L^{2,6}$, $p \in [2,\infty]$, $q \in [2,6]$ and*

$$\frac{2}{p} + \frac{3}{q} = \frac{3}{2}, \tag{19}$$

then

$$\|g\|_{L^{p,q}} \leqslant c\|g\|_{L^{\infty,2} \cap L^{2,6}}, \tag{20}$$

where $c = c(\Omega)$.

Lemma 3. *Let $\psi \in L^2(0,T;\mathscr{D}(A))$, $\psi' \in L^2(0,T;H)$, (5) hold and $w_0 \in V$. Then there exists a unique solution $(w,\mathscr{Q})$ of the problem*

$$w' - \Delta w + \psi \nabla w + \nabla \mathscr{Q} = f, \tag{21}$$

$$w(0) = w_0, \tag{22}$$

where $w \in L^2(0,T;\mathscr{D}(A)) \cap L^\infty(0,T;V)$, $w' \in L^2(0,T;H)$, $\nabla\mathscr{Q} \in L^2(0,T;L^2(\Omega)^3)$, $\mathscr{Q} \in L^{2,6}$, $\int_\Omega \mathscr{Q} = 0$ for almost every t and

$$\|w\|_{L^2(0,T;V)} + \|w\|_{L^\infty(0,T;H)} \leqslant c_1 \cdot (\|f\|_{L^2(0,T;H)} + \|w_0\|_H). \tag{23}$$

Moreover, if $w_0 \in \mathscr{D}(A)$, then

$$\|\mathscr{Q}\|_{L^{r,s^s}}, \|\nabla\mathscr{Q}\|_{L^{r,s}} \leqslant c_2 \cdot (\|f\|_{L^2(0,T;H)} + \|w_0\|_{\mathscr{D}(A)})(\|\psi\|_{L^{\infty,2}\cap L^{2.6}} + 1), \tag{24}$$

where r, s^* satisfy (17), $c_2 = c_2(\Omega)$,

$$\frac{2}{r} + \frac{3}{s} = 4, \quad 1 < s < \frac{3}{2} \tag{25}$$

and c_1, c_2 do not depend on ψ.

P r o o f. Using ([7], Proposition 2) we get that there exists a unique w, $w \in L^2(0,T;\mathscr{D}(A)) \cap L^\infty(0,T;V)$, $w' \in L^2(0,T;H)$, which is a weak solution of (21), (22). Consequently, $\nabla\mathscr{Q} \in L^2(0,T;L^2(\Omega))$, $\mathscr{Q} \in L^2(0,T;L^6(\Omega))$ and w, $\mathscr{Q}$ solve (21), (22). Multiplying (21) by w, we get

$$\|w(t)\|_H^2 + 2\int_s^t \|\nabla w\|_{L^2(\Omega)}^2 \leqslant \|w(s)\|_H^2 + 2\int_s^t \|f\|_H \,\|w\|_H$$

for every $0 \leqslant s < t \leqslant T$. Set $s = 0$. Then

$$\|w(t)\|_H^2 + 2\int_0^t \|\nabla w\|_{L^2(\Omega)}^2 \leqslant \|w(0)\|_H^2 + 2\int_0^t \|f\|_H \,\|w\|_H$$

for every $t \in [0,T]$. The last inequality implies

$$\|w(t)\|_H^2 + \int_0^T \|\nabla w\|^2_{L^2(\Omega)} \leqslant c_1 \cdot (\|w_0\|_H^2 + \int_0^T \|f\|_H^2)$$

and (23) follows immediately.

Using the inequality

$$\|\psi \nabla w\|_{L^{r,s}} \leqslant c\|\nabla w\|_{L^{2,2}}\|\psi\|_{L^{p,q}} \leqslant c\|\nabla w\|_{L^{2,2}}\|\psi\|_{L^{\infty,2}\cap L^{2,6}},$$

where $p = \frac{2r}{2-r}$, $q = \frac{2s}{2-s}$ and r, s satisfy (17) and (25), it is possible to see that also $f - \psi\nabla w \in L^{r,s}$ and

$$\|f - \psi\nabla w\|_{L^{r,s}} \leqslant c(\|f\|_{L^2(0,T;H)} + c\|\nabla w\|_{L^{2,2}}\|\varphi\|_{L^{\infty,2}\cap L^{2,6}}).$$

By virtue of (23), inequality (24) for $\|\nabla \mathscr{Q}\|_{L^{r,s}}$ now follows from the famous $L^r - L^s$ estimates for the Stokes equations (see [2]). Using now the fact that $\int_\Omega \mathscr{Q} = 0$, we get (24) for $\|\mathscr{Q}\|_{L^{r,s^*}}$, where r, s^* satisfy (17). $\square$

Lemma 4. *The unique solution $(w, \mathscr{Q})$ obtained in Lemma 3 satisfies the generalized energy equality for every $\varphi \in \mathscr{C}^\infty(\overline{Q_T})$, $\varphi \geqslant 0$ and every $t_1, t_2 \in [0,T]$, $t_1 < t_2$:*

$$\begin{aligned} \int_{\Omega\times\{t_2\}} |w|^2\varphi + 2\int_{t_1}^{t_2}\int_\Omega |\nabla w|^2\varphi &= \int_{\Omega\times\{t_1\}} |w|^2\varphi + \int_{t_1}^{t_2}\int_\Omega |w|^2\Big(\frac{\partial\varphi}{\partial t} + \Delta\varphi\Big) \\ &\quad + \int_{t_1}^{t_2}\int_\Omega (|w|^2\cdot\psi + 2\mathscr{Q}w)\cdot\nabla\varphi + 2\int_{t_1}^{t_2}\int_\Omega fw\varphi. \end{aligned} \tag{26}$$

P r o o f. Multiplying (21) by $2w\varphi$ and integrating by parts, we get (26). $\square$

Definition. Let $n \in \mathbb{N}$, $\delta_n = T/n$ and $\varphi \in \mathscr{C}(0,T;H)$. Define

$$\Psi_n(\varphi)(t) = \begin{cases} \varphi(0) & \text{for } t \in (0,\delta_n), \\ \varphi(t-\delta_n) & \text{for } t \in [\delta_n, T). \end{cases} \tag{27}$$

Clearly, $\Psi_n(\varphi) \in \mathscr{C}(0,T;H)$.

Let $(w_n, \mathscr{Q}_n)$ be the solution of the problem

$$w_n' - \Delta w_n + \Psi_n(w_n)\nabla w_n + \nabla \mathscr{Q}_n = f, \tag{28}$$

$$w_n(0) = u_0. \tag{29}$$

It is possible to see that $w_n, \Psi_n(w_n) \in L^2(0, \delta_n; \mathscr{D}(A))$ and by applying Lemma 3 inductively on each time interval $(k\delta_n, (k+1)\delta_n)$, $k = 1, \ldots, n-1$, we get that $w_n, \Psi_n(w_n) \in L^2(0, T; \mathscr{D}(A))$. Using (23) and (27), we get that for sufficiently big K

$$\|w_n\|_{L^2(0,T;V)}, \|w_n\|_{L^\infty(0,T;H)}, \|\Psi_n(w_n)\|_{L^2(0,T;V)} \leqslant K, \tag{30}$$

and using (24) and (30), we obtain

$$\|\mathscr{Q}/\mathscr{R}\|_{L^{r,s^*}}, \|\nabla \mathscr{Q}_n\|_{L^{r,s}} \leqslant K. \tag{31}$$

Further, using Lemma 3, we get from (27) and (28) that

$$\|w_n'\|_{L^{\frac{4}{3}}(0,T;V^*)}, \|\Psi_n(w_n)'\|_{L^{\frac{4}{3}}(0,T;V^*)} \leqslant K. \tag{32}$$

Note that K does not depend on n and r, s, s^* satisfy (17) and (25). Using (30), (32) and ([8], Theorem 2.1 in Chapter III), we come to the conclusion that

$$w_n \text{ stay in a compact subset of } L^2(0, T; H). \tag{33}$$

Therefore, there exist $u, u^* \in L^2(0, T; V) \cap L^\infty(0, T; H)$ and $\{w_{n_k}\} \subset \{w_n\}$, $\{\Psi_{n_k}(w_{n_k})\} \subset \{\Psi_n(w_n)\}$ (for simplicity we will use $\{w_{n_k}\} = \{w_n\}$, $\{\Psi_{n_k}(w_{n_k})\} = \{\Psi_n(w_n)\}$) such that

$$w_n \to u \quad \text{weakly in } L^2(0, T; V), \tag{34}$$

$$w_n \to u \quad \text{strongly in } L^2(0, T; H), \tag{35}$$

$$\Psi_n(w_n) \to u^* \quad \text{weakly in } L^2(0, T; V), \tag{36}$$

$$\Psi_n(w_n) \to u^* \quad \text{strongly in } L^2(0, T; H). \tag{37}$$

Lemma 5. *The following equality holds:*

$$u = u^*. \tag{38}$$

Proof. We know (see (35), (37)) that $\{w_n\}$, $\{\Psi_n(w_n)\}$ are relatively compact sets in $L^2(0, T; H)$. Using ([4], Theorem 2.13.1, condition (ii)), we get that w_n^k are

2-mean equicontinuous (w_n^k is the k-th component of w_n, $k = 1, 2, 3$). It means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $h \in \mathbb{R}$, $|h| < \delta$,

$$\int_{Q_T} |w_n^k(x, t+h) - w_n^k(x, t)|^2 < \varepsilon^2.$$

(If necessary, w_n are defined by zero outside Q_T.) It follows from the last inequality and from (27) that $u = u^*$.

It follows immediately from Lemma 1, (34) and (35) that

$$w_n \to w \text{ strongly in } L^{\alpha,\beta}, \text{ where } \frac{2}{\alpha} + \frac{3}{\beta} > \frac{3}{2}, \ \alpha \in (2, \infty), \ \beta \in (2, 6). \tag{39}$$

By virtue of (31), there exists $\mathscr{P} \in L^{r,s^*}$, $\nabla \mathscr{P} \in L^{r,s}$ such that

$$\mathscr{Q}_n \to \mathscr{P} \text{ weakly in } L^{r,s^*} \tag{40}$$

and

$$\nabla \mathscr{Q}_n \to \nabla \mathscr{P} \text{ weakly in } L^{r,s}. \tag{41}$$

Applying Lemma 4 to (28), (29), we obtain the generalized energy equality for $(w_n, \mathscr{Q}_n)$:

$$\begin{aligned}
&\int_{\Omega \times \{t_2\}} |w_n|^2 \varphi + 2 \int_{t_1}^{t_2} \int_\Omega |\nabla w_n|^2 \varphi \\
&\qquad = \int_{\Omega \times \{t_1\}} |w_n|^2 \varphi + \int_{t_1}^{t_2} \int_\Omega |w_n|^2 \Big(\frac{\partial \varphi}{\partial t} + \Delta \varphi \Big) \\
&\qquad\quad + \int_{t_1}^{t_2} \int_\Omega (|w_n|^2 \Psi(w_n) + 2 \mathscr{Q}_n w_n) \cdot \nabla \varphi + 2 \int_{t_1}^{t_2} \int_\Omega f w_n \varphi.
\end{aligned} \tag{42}$$

Proof of Theorem 2. To prove that $u = \lim w_n$ satisfies (10), (13) and (15) we proceed similarly as in ([8], Theorem 3.1 in Chapter III). Further, $\mathscr{P} \in L^{r,s^*}$ is the associated pressure of the weak solution u. Using (35) we get that

$$\|w_n(t)\|_H \to \|u(t)\|_H \tag{43}$$

for almost every $t \in [0, T]$. The relation (34) implies that

$$\int_{t_1}^{t_2} \int_\Omega |\nabla w_n|^2 \geqslant \int_{t_1}^{t_2} \int_\Omega |\nabla u|^2$$

for every $t_1, t_2 \in [0,T], t_1 < t_2$. It is possible to prove (note that $\varphi \geqslant 0$ on Q_T) that also

$$\liminf \int_{t_1}^{t_2} \int_\Omega |\nabla w_n|^2 \varphi \geqslant \int_{t_1}^{t_2} \int_\Omega |\nabla u|^2 \varphi \tag{44}$$

for every t_1, t_2, $0 \leqslant t_1 \leqslant t_2 \leqslant T$. Applying (34)–(40), (43) and (44) to (42), we get also

$$\begin{aligned} \int_{\Omega\times\{t_2\}} |u|^2\varphi + 2\int_{t_1}^{t_2}\int_\Omega |\nabla u|^2\varphi \\ \leqslant \int_{\Omega\times\{t_1\}} |u|^2\varphi + \int_{t_1}^{t_2}\int_\Omega |u|^2\Big(\frac{\partial\varphi}{\partial t} + \Delta\varphi\Big) \\ + \int_{t_1}^{t_2}\int_\Omega (|u|^2 u + 2\mathscr{P}u)\cdot\nabla\varphi + 2\int_{t_1}^{t_2}\int_\Omega fu\varphi \end{aligned} \tag{45}$$

for almost every $t_1, t_2 \in [0,T]$, $t_1 < t_2$. Using the fact that the function

$$t \in [0,T] \to \int_{\Omega\times\{t\}} |u|^2\varphi$$

is semi-lower continuous we get (45) for almost every $t_1 \in [0,T]$ and for every $t_2 \in [0,T]$, $t_1 < t_2$. Moreover, as (43) holds for $t = 0$, (45) is also satisfied for $t_1 = 0$. Theorem 2 is proved. □

R e m a r k 1. Estimate (24) is important for the proof of Theorem 2. We prove (24) using $L^r - L^s$ estimates published in [2]. To fulfil all assumptions necessary to apply the results of [2] and for the sake of simplicity, we suppose a smooth initial condition (6). Note that it is possible to suppose a more general initial condition (see [2], relation (2.5)).

R e m a r k 2. The general energy inequality proved in [1] was used for the study of the set of internal singular points of a suitable weak solution. Analogously, using the general energy inequality proved in this paper may be useful in the study of boundary singular points.

R e m a r k 3. Notice that we get the classical energy inequality

$$\int_{\Omega\times\{t_2\}} |u|^2 + 2\int_{t_1}^{t_2}\int_\Omega |\nabla u|^2 \leqslant \int_{\Omega\times\{t_1\}} |u|^2 + 2\int_{t_1}^{t_2}\int_\Omega fu \tag{46}$$

if we suppose $\varphi \equiv 1$ in (45).

R e m a r k 4. A draft version of this paper was published in [3]. The present main theorem is stronger than that of [3].

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