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ADDITIONAL NOTE ON PARTIAL REGULARITY OF WEAK SOLUTIONS OF THE NAVIER-STOKES EQUATIONS IN THE CLASS $L^\infty(0,T,L^3(\Omega)^3)$*

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Abstract. We present a simplified proof of a theorem proved recently concerning the number of singular points of weak solutions to the Navier-Stokes equations. If a weak solution $\mathbf{u}$ belongs to $L^\infty(0,T,L^3(\Omega)^3)$, then the set of all possible singular points of $\mathbf{u}$ in Ω is at most finite at every time $t_0 \in (0,T)$.

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Let Ω be a bounded domain in $\mathbb{R}^3$ with $C^{2+\mu}$ ($\mu > 0$) boundary $\partial\Omega$, $T > 0$ and $Q_T = \Omega \times (0,T)$. Consider the Navier-Stokes equations describing the evolution of the velocity $\mathbf{u}$ and the pressure p in Q_T:

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f}, \tag{1}$$

$$\operatorname{div}\mathbf{u} = 0, \tag{2}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega\times(0,T), \tag{3}$$

$$\mathbf{u}|_{t=0} = \mathbf{u}_0, \tag{4}$$

where $\mathbf{f}$ is the external body force and $\nu > 0$ is the viscosity coefficient. The existence and properties of weak solutions to (1)–(4) are discussed for example in [6].

The attention of many authors in the last decades has been directed to the question whether a smooth solution to (1)–(4) can at a certain instant of time lose its smoothness and develop a singularity. One of the basic papers concerning the problem was

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written by L. Caffarelli, R. Kohn and L. Nirenberg (see [1]). They established the concept of a suitable weak solution to (1)–(4) and proved that if $\mathbf{u}$ is such a weak solution then the one-dimensional Hausdorff measure of the set S of all singular points of $\mathbf{u}$ is equal to zero. A point $(\mathbf{x},t)\in Q_T$ is called regular if $\mathbf{u}$ is essentially bounded on some space-time neighbourhood of $(\mathbf{x},t)$. Otherwise, the point is singular.

We will concentrate on an interesting result proved by J. Neustupa in [4]:

Theorem 1. *Let* $\mathbf{u}\in L^\infty(0,T,L^3(\Omega)^3)$ *be a weak solution to* (1)–(4), *where* $\mathbf{f}\in L^2(Q_T)^3\cup L^q_{\mathrm{loc}}(Q_T)^3$ *for some* $q>5/2$ *and* $\operatorname{div}\mathbf{f}=0$. *Then the set* $S_{t_0}=S\cap\{(\mathbf{x},t)\in Q_T;\ t=t_0\}$ *contains no more than* K^3/ε_5^3 *points for every* $t_0\in(0,T)$, *where*

$$K=\sup_{t\in(0,T)}\left(\int_\Omega|\mathbf{u}(\mathbf{x},t)|^3\,\mathrm{d}\mathbf{x}\right)^{1/3}$$

and ε_5 *is the number given by Lemma* 3.

In other words, the solution $\mathbf{u}$ can develop only a finite number of singularities at every particular time. The goal of this paper is to present a simplified proof of Theorem 1. Our simplification is due to the following two facts. Firstly, it is not necessary to use the concept of a separated subset of S_{t_0} (as was done in [4]) and so we can avoid the proofs of some technical lemmas. Secondly, we do not use the cut-off function technique. The boundary integrals developing as a result of this approach can be handled easily and do not represent any major problem. When reading the paper it is helpful to have [4] at hand.

Lemma 1 was proved in [2] for weak solutions to (1)–(4) and $\mathbf{f}=0$. It is not difficult to generalize it to the case $\mathbf{f}\neq\mathbf{0}$.

Lemma 1. *There exists an absolute constant* $\varepsilon_0>0$ *such that if*

$$\sup_{t\in(t_0-\sigma,\,t_0+\sigma)}\left(\int_{B_r(\mathbf{x}_0)}|\mathbf{u}(\mathbf{x},t)|^3\,\mathrm{d}\mathbf{x}\right)^{1/3}<\varepsilon_0$$

for some $r>0$, $\sigma>0$, *then* $(\mathbf{x}_0,t_0)$ *is a regular point.*

The following lemma is an easy consequence of Lemma 1.

Lemma 2. *There exists an absolute constant* ε_0 *such that for every singular point* $(\mathbf{x}_0,t_0)\in S$ *we have*

$$\lim_{r\to0_+}\limsup_{t\to t_0^-}\left(\int_{B_r(\mathbf{x}_0)}|\mathbf{u}(\mathbf{x},t)|^3\,\mathrm{d}\mathbf{x}\right)^{1/3}>\varepsilon_0.$$

Suppose throughout the paper that $\mathbf{f}$ satisfies the assumptions of Theorem 1. Further, let $\mathbf{u} \in L^\infty(0,T,L^3(\Omega)^3)$ be a weak solution to (1)–(4). Finally, let $(\mathbf{x}_0,t_0) \in Q_T$ be an arbitrary singular point of $\mathbf{u}$, $r \in (0,1)$, $\sigma > 0$ and $\overline{B_r(\mathbf{x}_0)} \times \langle t_0-\sigma, t_0+\sigma\rangle \subset Q_T$, where $B_r(\mathbf{x}_0) = \{\mathbf{x} \in \mathbb{R}^3;\ |\mathbf{x}-\mathbf{x_0}| < r\}$. Using the uniqueness theorem for the weak solutions to (1)–(4) from $L^\infty(0,T,L^3(\Omega)^3)$ proved by H. Kozono and H. Sohr in [3] and the theorem on the existence of suitable weak solutions from [1] we obtain that $\mathbf{u}$ is suitable on (δ,T) for every δ. Using [1] once more we come to the conclusion that the one-dimensional Hausdorff measure of the set S of all singular points of $\mathbf{u}$ is equal to zero. Therefore, we may suppose without loss of generality that $S \cap (\partial B_r(\mathbf{x}_0) \times \langle t_0-\sigma, t_0+\sigma) = \emptyset$. Denote $D = B_r(\mathbf{x}_0)$ and $\Gamma = \partial D$. Under the assumptions of this paragraph we can state the following lemma.

Lemma 3. *Under the assumptions of the preceding paragraph there exists an absolute constant $\varepsilon_5 > 0$ independent of $(\mathbf{x}_0,t_0)$ and r such that*

$$\liminf_{t\to t_0^-}\left(\int_D |\mathbf{u}(\mathbf{x},t)|^3\,\mathrm{d}\mathbf{x}\right)^{1/3} \geqslant \varepsilon_5.$$

P r o o f. We can write $(t_0-\sigma,t_0+\sigma) = G \cup \bigcup_{\gamma\in N}(a_\gamma,b_\gamma)$, where G is a countable set and $\mathbf{u} \in L^2_{\mathrm{loc}}(a_\gamma,b_\gamma,W^{2,2}(\Omega)\cap W^{1,2}_{0,\sigma}(\Omega))$, $\mathrm{d}\mathbf{u}/\mathrm{d}t \in L^2_{\mathrm{loc}}(a_\gamma,b_\gamma,L^2_\sigma(\Omega))$ and $p \in W^{1,2}(\Omega)$ for almost every $t \in (a_\gamma,b_\gamma)$. According to [6] we have $L^2(D)^3 = H_0 \oplus H_1 \oplus H_2$, where

$$\begin{aligned} H_0 &= \{\mathbf{u} \in L^2(D)^3;\ \operatorname{div}\mathbf{u} = 0,\ \ (\mathbf{u}\cdot\mathbf{n})|_\Gamma = 0\},\\ H_1 &= \{\mathbf{u} \in L^2(D)^3;\ \mathbf{u} = \nabla q,\ \ q \in W^{1,2}(D),\ \ \Delta q = 0\},\\ H_2 &= \{\mathbf{u} \in L^2(D)^3;\ \mathbf{u} = \nabla P,\ \ P \in W^{1,2}_0(D)\}. \end{aligned}$$

Let π_i denote the projector operator from $L^2(D)^3$ on H_i, $i = 1,2,3$ and put $\pi_{01} = \pi_0 + \pi_1$. Suppose now that $t \in (a_\gamma,b_\gamma)$ for some $\gamma \in N$. Multiplying equation (1) by $\pi_{01}(\mathbf{u}|\mathbf{u}|)$ and integrating over D we get

$$(5)\qquad \frac{1}{3}\frac{\mathrm{d}}{\mathrm{dt}}\int_D |\mathbf{u}|^3\,\mathrm{d}\mathbf{x} - \nu\int_D (\Delta\mathbf{u})\cdot\pi_{01}(\mathbf{u}|\mathbf{u}|)\,\mathrm{d}\mathbf{x} + \int_D (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\pi_{01}(\mathbf{u}|\mathbf{u}|)\,\mathrm{d}\mathbf{x} + \int_D (\nabla p)\cdot\pi_{01}(\mathbf{u}|\mathbf{u}|)\,\mathrm{d}\mathbf{x} = \int_D \mathbf{f}\cdot\pi_{01}(\mathbf{u}|\mathbf{u}|)\,\mathrm{d}\mathbf{x}.$$

Let us now estimate the last four terms in (5). We denote by c a generic constant independent of $(\mathbf{x}_0,t_0), r$ and σ, C will denote a generic L^1-function on $(t_0-\sigma,t_0+\sigma)$

and $\delta > 0$ will be specified later. We have

$$(6)\quad \nu\int_D (\Delta\mathbf{u})\cdot\pi_{01}(\mathbf{u}|\mathbf{u}|)\,\mathrm{d}\mathbf{x} = \nu\int_D (\Delta\mathbf{u})\cdot(\mathbf{u}|\mathbf{u}|)\,\mathrm{d}\mathbf{x} = \nu\int_\Gamma \frac{\partial u_i}{\partial x_j} n_j u_i |\mathbf{u}|\,\mathrm{d}\Gamma - \nu\int_D \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}|\mathbf{u}|\,\mathrm{d}\mathbf{x} - \nu\int_D \frac{\partial u_i}{\partial x_j} u_i \frac{u_k}{|\mathbf{u}|}\frac{\partial u_k}{\partial x_j}\,\mathrm{d}\mathbf{x}.$$

Since $\mathbf{u}$ is bounded on $\Gamma\times(t_0-\sigma, t_0+\sigma)$ and $\nabla\mathbf{u}\in L^2(Q_T)$, the first integral on the right-hand side of (6) can be viewed as a function from $L^1(t_0-\sigma, t_0+\sigma)$ and we get

$$(7)\qquad \nu\int_D (\Delta\mathbf{u})\cdot\pi_{01}(\mathbf{u}|\mathbf{u}|)\,\mathrm{d}\mathbf{x} = C(t) - \nu\int_D |\mathbf{u}||\nabla\mathbf{u}|^2\,\mathrm{d}\mathbf{x} - \frac{4}{9}\nu\int_D |\nabla|\mathbf{u}|^{3/2}|^2\,\mathrm{d}\mathbf{x}.$$

Further,

$$(8)\qquad \left|\int_D (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\pi_{01}(\mathbf{u}|\mathbf{u}|)\,\mathrm{d}\mathbf{x}\right|$$
$$\leqslant \left(\int_D |\mathbf{u}|^{3/2}\cdot|\nabla\mathbf{u}|^{3/2}\,\mathrm{d}\mathbf{x}\right)^{2/3}\cdot\left(\int_D |\pi_{01}(\mathbf{u}|\mathbf{u}|)|^3\,\mathrm{d}\mathbf{x}\right)^{1/3}$$
$$\leqslant c\left(\int_D |\mathbf{u}|^3\,\mathrm{d}\mathbf{x}\right)^{1/6}\left(\int_D |\mathbf{u}|\cdot|\nabla\mathbf{u}|^2\,\mathrm{d}\mathbf{x}\right)^{1/2}\left(\int_D |\mathbf{u}|^6\,\mathrm{d}\mathbf{x}\right)^{1/3}$$
$$\leqslant \delta\int_D |\mathbf{u}|\cdot|\nabla\mathbf{u}|^2\,\mathrm{d}\mathbf{x} + \frac{c}{\delta}\left(\int_D |\mathbf{u}|^3\,\mathrm{d}\mathbf{x}\right)^{1/3}\left(\int_D |\mathbf{u}|^6\,\mathrm{d}\mathbf{x}\right)^{2/3}$$
$$\leqslant \delta\int_D |\mathbf{u}|\cdot|\nabla\mathbf{u}|^2\,\mathrm{d}\mathbf{x} + \frac{c}{\delta}\left(\int_D |\mathbf{u}|^3\,\mathrm{d}\mathbf{x}\right)^{2/3}\left(\int_D |\mathbf{u}|^9\,\mathrm{d}\mathbf{x}\right)^{1/3}$$
$$\leqslant \delta\int_D |\mathbf{u}|\cdot|\nabla\mathbf{u}|^2\,\mathrm{d}\mathbf{x} + \left(\delta + \frac{c}{\delta^2}\int_D |\mathbf{u}|^3\,\mathrm{d}\mathbf{x}\right)\big||\mathbf{u}|^{3/2}\big|_6^2$$
$$\leqslant \delta\int_D |\mathbf{u}|\cdot|\nabla\mathbf{u}|^2\,\mathrm{d}\mathbf{x} + c\left(\delta + \frac{c}{\delta^2}\int_D |\mathbf{u}|^3\,\mathrm{d}\mathbf{x}\right)$$
$$\times\left(\frac{1}{r^2}\int_D |\mathbf{u}|^3\,\mathrm{d}\mathbf{x} + \int_D \left|\nabla|\mathbf{u}|^{3/2}\right|^2\,\mathrm{d}\mathbf{x}\right).$$

Since $\mathbf{u}\in L^\infty(0,T,L^3(\Omega)^3)$ it follows from (8) that

$$(9)\quad \left|\int_D (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\pi_{01}(\mathbf{u}|\mathbf{u}|)\,\mathrm{d}\mathbf{x}\right|$$
$$\leqslant \delta\int_D |\mathbf{u}|\cdot|\nabla\mathbf{u}|^2\,\mathrm{d}\mathbf{x} + c\left(\delta + \frac{c}{\delta^2}\int_D |\mathbf{u}|^3\,\mathrm{d}\mathbf{x}\right)\int_D \left|\nabla|\mathbf{u}|^{3/2}\right|^2\,\mathrm{d}\mathbf{x} + C(t).$$

Let us estimate the last term on the right-hand side of (5):

$$\int_D (\nabla p)\cdot\pi_{01}(\mathbf{u}|\mathbf{u}|)\,\mathrm{d}\mathbf{x} = \int_D (\nabla p)\cdot\pi_1(\mathbf{u}|\mathbf{u}|)\,\mathrm{d}\mathbf{x} = \int_D \nabla p\cdot\nabla q\,\mathrm{d}\mathbf{x},$$

where $\Delta q = 0$, $\frac{\partial q}{\partial n}|_\Gamma = (\mathbf{u}|\mathbf{u}| - \nabla P)\cdot n|_\Gamma$, $\Delta P = \operatorname{div}(\mathbf{u}|\mathbf{u}|)$ and $P \in W_0^{1,2}(D) \cap W^{2,2}(D)$ (see [6] Chapter 1). Moreover, $q \in W^{2,2}(D)$. Therefore,

$$\left|\int_D (\nabla p)\cdot \pi_{01}(\mathbf{u}|\mathbf{u}|)\,\mathrm{d}\mathbf{x}\right| = \left|\int_\Gamma p\frac{\partial q}{\partial n}\,\mathrm{d}\Gamma\right| \leqslant \delta\int_\Gamma \left|\frac{\partial q}{\partial n}\right|^2 \mathrm{d}\Gamma + \frac{1}{\delta}\int_\Gamma |p|^2\,\mathrm{d}\Gamma. \tag{10}$$

It was proved in [5] that $p \in L^2(\varepsilon, T, L^2(\Omega))$ for every $\varepsilon > 0$ and therefore the last term from (10) can be viewed as a function from $L^1(t_0-\sigma, t_0+\sigma)$. Further, we have

$$\begin{aligned}\delta\int_\Gamma \left|\frac{\partial q}{\partial n}\right|^2 \mathrm{d}\Gamma &= \delta\int_\Gamma \big|(\mathbf{u}|\mathbf{u}| - \nabla P)\cdot n\big|^2\,\mathrm{d}\Gamma \\ &\leqslant \delta\int_\Gamma |\mathbf{u}|^4\,\mathrm{d}\Gamma + \delta\int_\Gamma\left|\frac{\partial P}{\partial n}\right|^2\mathrm{d}\Gamma\end{aligned} \tag{11}$$

and $t \mapsto \delta\int_\Gamma |\mathbf{u}|^4\,\mathrm{d}\Gamma$ is a bounded function on $(t_0-\sigma, t_0+\sigma)$. It follows further that

$$\begin{aligned}\int_\Gamma \left|\frac{\partial P}{\partial n}\right|^2 \mathrm{d}\Gamma &\leqslant c\Big(\frac{1}{r^{3/2}}|\nabla P|_{3/2}^{3/2} + |\nabla^2 P|_{3/2}^{3/2}\Big)^{\frac{4}{3}} \leqslant c\big|\operatorname{div}(\mathbf{u}|\mathbf{u}|)\big|_{3/2}^2 \\ &\leqslant c\Big(\int_D |\mathbf{u}|^{3/2}|\nabla\mathbf{u}|^{3/2}\,\mathrm{d}\mathbf{x}\Big)^{4/3} \\ &\leqslant c\Big(\int_D |\mathbf{u}|^3\,\mathrm{d}\mathbf{x}\Big)^{1/3}\Big(\int_D |\mathbf{u}||\nabla\mathbf{u}|^2\,\mathrm{d}\mathbf{x}\Big).\end{aligned} \tag{12}$$

Summing up (10), (11) and (12) we obtain

$$\left|\int_D (\nabla p)\cdot\pi_{01}(\mathbf{u}|\mathbf{u}|)\,\mathrm{d}\mathbf{x}\right| = c\delta\int_D |\mathbf{u}||\nabla\mathbf{u}|^2\,\mathrm{d}\mathbf{x} + C(t). \tag{13}$$

Finally, the right-hand side of (5) can be viewed as an L^1-function on $(t_0-\sigma, t_0+\sigma)$ and we can conclude from (5), (7), (9) and (13) that

$$\begin{aligned}&\frac{1}{3}\frac{\mathrm{d}}{\mathrm{dt}}\int_D |\mathbf{u}|^3\,\mathrm{d}\mathbf{x} + (\nu - c\delta)\int_D |\mathbf{u}||\nabla\mathbf{u}|^2\,\mathrm{d}\mathbf{x} \\ &\qquad + \Big(\frac{4\nu}{9} - c\delta - \frac{c}{\delta^2}\int_D |\mathbf{u}|^3\,\mathrm{d}\mathbf{x}\Big)\int_D \big|\nabla|\mathbf{u}|^{3/2}\big|^2\,\mathrm{d}\mathbf{x} \leqslant C(t).\end{aligned} \tag{14}$$

Supposing that from now on c and C are fixed and choosing $\delta = (2\nu)/(9c)$ we obtain

$$\begin{aligned}&\frac{1}{3}\frac{\mathrm{d}}{\mathrm{dt}}\int_D |\mathbf{u}|^3\,\mathrm{d}\mathbf{x} + \frac{7\nu}{9}\int_D |\mathbf{u}||\nabla\mathbf{u}|^2\,\mathrm{d}\mathbf{x} \\ &\qquad + \Big(\frac{2\nu}{9} - \frac{c}{\delta^2}\int_D |\mathbf{u}|^3\,\mathrm{d}\mathbf{x}\Big)\int_D \big|\nabla|\mathbf{u}|^{3/2}\big|^2\,\mathrm{d}\mathbf{x} \leqslant C(t).\end{aligned} \tag{15}$$

Let now

$$(16)\qquad (c/\delta^2)\int_D |\mathbf{u}(\mathbf{x},t^*)|^3\,\mathrm{d}\mathbf{x} < \nu/9$$

for some $t^* \in (t_0-\sigma, t_0+\sigma)$. Then

$$(17)\qquad (c/\delta^2)\int_D |\mathbf{u}(\mathbf{x},t)|^3\,\mathrm{d}\mathbf{x} < 2\nu/9$$

for every $t \in \langle t^*, t^*+\tau)\cap(t_0-\sigma, t_0+\sigma)$, where τ is a constant independent of t^* such that $\int_I C(t)\,\mathrm{d}t < (\nu\delta^2)/(27c)$ for every interval $I\subset(t_0-\sigma,t_0+\sigma)$ of the length τ. Indeed, define $A=\sup M$, where $M=\{\eta\in(0,\tau\rangle;\ (c/\delta^2)\int_D|\mathbf{u}(\mathbf{x},t)|^3\,\mathrm{d}\mathbf{x} < 2\nu/9,\ \forall\, t\in\langle t^*,t^*+\eta)\cap(t_0-\sigma,t_0+\sigma)\}$. Obviously, it follows from the $L^3(\Omega)^3$-right continuity of $\mathbf{u}$ on $(t_0-\sigma,t_0+\sigma)$ that $M\neq\emptyset$ and further $A\in M$. It suffices to show that $A=\tau$. Assuming that $A<t_0+\sigma-t^*$ and integrating (15) over (t^*,t^*+A) we obtain

$$(18)\qquad \begin{aligned}\int_D |\mathbf{u}(\mathbf{x},t^*+A)|^3\,\mathrm{d}\mathbf{x} &\leqslant \int_D|\mathbf{u}(\mathbf{x},t^*)|^3\,\mathrm{d}\mathbf{x} + 3\int_{t^*}^{t^*+A} C(t)\,\mathrm{d}t\\ &< \frac{\nu\delta^2}{9c}+\frac{3\nu\delta^2}{27c}=\frac{2\nu\delta^2}{9c},\end{aligned}$$

that is

$$(19)\qquad (c/\delta^2)\int_D |\mathbf{u}(\mathbf{x},t^*+A)|^3\,\mathrm{d}\mathbf{x} < 2\nu/9.$$

The equality $A=\tau$ now follows from (19), the definition of A and the $L^3(\Omega)^3$-right continuity of $\mathbf{u}$ on $(t_0-\sigma,t_0+\sigma)$.

Put further $\delta_0=(\nu\delta^2)/(9c)$ and choose $\varepsilon_5>0$ such that $(2\varepsilon_5)^3<\delta_0$. If $\liminf_{t\to t_0^-}(\int_D|\mathbf{u}(\mathbf{x},t)|^3\,\mathrm{d}\mathbf{x})^{1/3}<\varepsilon_5$ then there would exist a sequence $t_n\to t_0^-$ such that $t_0-t_n<\tau$ and $(\int_D|\mathbf{u}(\mathbf{x},t_n)|^3\,\mathrm{d}\mathbf{x})^{1/3}<2\varepsilon_5$, $\forall\, n\in\mathbb{N}$, i.e. $\int_D|\mathbf{u}(\mathbf{x},t_n)|^3\,\mathrm{d}\mathbf{x}<(2\varepsilon_5)^3<\delta_0$. Integrating (15) over (t_n,t), where $t\in\langle t_n,t_0\rangle$, we obtain

$$(20)\qquad \int_D|\mathbf{u}(\mathbf{x},t)|^3\,\mathrm{d}\mathbf{x}\leqslant\int_D|\mathbf{u}(\mathbf{x},t_n)|^3\,\mathrm{d}\mathbf{x}+3\int_{t_n}^{t}C(\xi)\,\mathrm{d}\xi\leqslant(2\varepsilon_5)^3+\varepsilon_4.$$

Since ε_4 can be made arbitrarily small when considering sufficiently big n, it follows that

$$(21)\qquad \limsup_{t\to t_0^-}\Big(\int_D|\mathbf{u}(\mathbf{x},t)|^3\,\mathrm{d}\mathbf{x}\Big)^{1/3}\leqslant((2\varepsilon_5)^3+\varepsilon_4)^{1/3}.$$

Choosing now ε_5 and ε_4 sufficiently small, the right-hand side of (21) can be made arbitrarily small, which contradicts Lemma 2. It means that in fact

$$\liminf_{t\to t_0^-}\left(\int_D |\mathbf{u}(\mathbf{x},t)|^3\,\mathrm{d}\mathbf{x}\right)^{1/3} \geqslant \varepsilon_5$$

and ε_5 is an absolute constant independent of $(\mathbf{x}_0,t_0)$ and r. Lemma 3 is proved. □

P r o o f of Theorem 1. Now it is easy to prove Theorem 1. Let $\{(\mathbf{x}_{01},t_0),\ldots,(\mathbf{x}_{0n},t_0)\}$ be a finite set of singular points of $\mathbf{u}$. Then there exists $r>0$ and $\sigma>0$ such that $B_r(\mathbf{x}_{0i})\cap B_r(\mathbf{x}_{0j})=\emptyset$ for $i\neq j$ and

$$K^3 \geqslant \int_\Omega |\mathbf{u}(\mathbf{x},t)|^3\,\mathrm{d}\mathbf{x} \geqslant \sum_{i=1}^n \int_{B_r(\mathbf{x}_{0i})} |\mathbf{u}(\mathbf{x},t)|^3\,\mathrm{d}\mathbf{x} \geqslant n\varepsilon_5^3 \tag{22}$$

for every $t\in(t_0-\sigma,t_0)$. This implies that the number of singular points developing at the time t_0 cannot exceed K^3/ε_5^3. □

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