Ricardo M. S. Rosa

Some results on the Navier-Stokes equations in connection with the statistical theory of stationary turbulence

Applications of Mathematics, Vol. 47 (2002), No. 6, 485–516

Persistent URL: http://dml.cz/dmlcz/134510

SOME RESULTS ON THE NAVIER-STOKES EQUATIONS IN CONNECTION WITH THE STATISTICAL THEORY OF STATIONARY TURBULENCE*

Ricardo M. S. Rosa, Rio de Janeiro

Abstract. Some rigorous results connected with the conventional statistical theory of turbulence in both the two- and three-dimensional cases are discussed. Such results are based on the concept of stationary statistical solution, related to the notion of ensemble average for turbulence in statistical equilibrium, and concern, in particular, the mean kinetic energy and enstrophy fluxes and their corresponding cascades. Some of the results are developed here in the case of nonsmooth boundaries and a less regular forcing term and for arbitrary stationary statistical solutions.

Keywords: Navier-Stokes equations, statistical solutions, turbulence, energy cascade, enstrophy cascade

MSC 2000: 35Q30, 76D05, 76D06, 37L40, 76F05, 76F20

Introduction

Major advances in turbulence theory were made from a phenomenological or heuristic point of view. The mathematical rigor, on the other hand, has lagged behind in building a theory of turbulence from the first principles. It is our intention here to present some recent rigorous results obtained in the direction of shortening this gap. While it is not clear how far they can be advanced we are hopeful that the rigorous framework of statistical solutions of the Navier-Stokes equations exploited here can serve as an important backbone for a rigorous approach.

Most of the issues discussed in this note can be found in [15], [16], [13] and in the book by Foias, Manley, Rosa and Temam [14], which is in great part a compilation of previous works by the authors and their collaborators. Some of the results in those works are developed here in the novel case of nonsmooth boundaries and a less regular forcing term. They are also derived for arbitrary stationary statistical

* This work was partially supported by a fellowship from CNPq, Brasília, Brazil.

solutions in the three-dimensional case, instead of only those obtained as generalized limits of time-averages of individual weak solutions.

This note has stemmed from a series of lectures presented at the Seventh Paseky School on Fluid Mechanics, in the Czech Republic. Due to size restrictions only some aspects pertaining to stationary statistical turbulence are included here. We begin with some fundamental results in the conventional theory of turbulence [4], [18], [20], [34], [36], [39], [50]. The following sections build up the concepts and applications connecting the mathematical theory of the Navier-Stokes equations with the conventional theory of turbulence, addressing, in particular, the mean kinetic energy and enstrophy fluxes and their corresponding cascades.

1. The conventional statistical theory of turbulence

For incompressible Newtonian flows with homogeneous density, the *Navier-Stokes equations*, or simply NSE, written in the Eulerian representation, take the form

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} + \boldsymbol{\nabla} p = \mathbf{f}, \quad \boldsymbol{\nabla}\cdot\mathbf{u} = 0.$$

Here, $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ denotes the velocity vector of an idealized fluid particle located at $\mathbf{x} = (x_1, x_2, x_3)$ at time t, $p = p(\mathbf{x}, t)$ is the kinematic pressure, $\mathbf{f} = \mathbf{f}(\mathbf{x}, t)$ represents the mass density of volume forces, and the constant $\nu > 0$ denotes the kinematic viscosity of the fluid. The flow is confined to a region Ω in the physical space $\mathbb{R}^3$ (or $\mathbb{R}^2$), which we assume independent of time.

Reynolds number and turbulence.

In a series of experiments in the late 19th century, O. Reynolds singled out a certain nondimensional parameter as an important quantity of the complexity of the flow. This *Reynolds number* takes the form $\mathrm{Re} = LU/\nu$, for typical large-scale length L and velocity U. In the experiments, for small values of Re, the flow is laminar, becoming more complicate, and eventually turbulent, as the parameter is increased.

A simple phenomenological dimensional analysis illustrates those facts. The viscous and the inertial terms have physical dimensions $\nu U/L^2$ and U^2/L, respectively. Hence the inertial term, which is responsible for the (possibly complicate) dynamic redistribution of energy among the flow structures, dominates precisely when $U^2/L \gg \nu U/L^2$, i.e. when $\mathrm{Re} \gg 1$. A more sophisticated analysis, however, reveals that this balance actually varies among the flow structures according to their length scales, with the dissipation dominating at the smallest scales even for high Reynolds number flows.

For relatively low Reynolds numbers, some "routes to chaos" can be identified and the corresponding flows are usually termed *weakly turbulent*. The situation is different in *fully-developed turbulence*, a somewhat vague term to characterize a highly disorganised motion (although coherent structures may still be present) which is local in nature, and which is active on a wide range of length scales (extending way below those associated with the production of energy). In actual flows this fully-developed turbulent core is usually restricted to certain subregions of the physical space. Within this core, physical quantities are highly unpredictable, but a certain order emerges in a statistical sense.

Ensemble averages and ergodic assumption.

In a later work O. Reynolds proposed representing the values of the physical quantities in a turbulent flow as a sum of a regular mean part and an irregular fluctuating part. For the characterization of the mean values, Reynolds proposed a formal averaging operator with suitable properties. In experiments, local space or time averages were used.

The macroscopic randomness of the motion was regarded in analogy with statistical mechanics. Nowadays, it is accepted that this randomness is a consequence of the chaotic nature of the system, ubiquitous to many other deterministic nonlinear differential equations.

The average processes are also understood now in a different sense. Ideally, one can think of realizing a number of experiments at nearly identical conditions and averaging the flow properties among all the experiments. Because of the random character of the motion, the values of, say, the instantaneous velocity vector $\mathbf{u}^{(n)}(\mathbf{x}, t)$ at a given point $\mathbf{x}$ in space and at a given time t could vary widely among the $n = 1, \ldots, N$ experiments, but the average value

$$\mathbf{U}(\mathbf{x}, t) = \frac{1}{N} \sum_{n=1}^{N} \mathbf{u}^{(n)}(\mathbf{x}, t)$$

is expected to behave in a more predictable manner. Averages, or *mean values*, of physical quantities with respect to several experiments in the sense described above are called *ensemble averages*. They are usually denoted by the symbol $\langle \cdot \rangle$. The *mean velocity field*, for instance, is usually denoted by $\mathbf{U}(\mathbf{x}, t) = \langle \mathbf{u}(\mathbf{x}, t) \rangle$. A more formal definition of the ensemble average is made using the notion of a random variable $\mathbf{u}(\mathbf{x}, t)$ of the probability theory.

Note that the statistics of the flow is allowed to change with time, as illustrated by the dependence on t of $\mathbf{U}(\mathbf{x}, t)$. A particular case is when a *statistical equilibrium* is reached and the ensemble averages are independent of t. Another case of interest is that of *decaying turbulence*, which is exemplified by flows past an array of rods or a honeycomb in a wind tunnel: downstream from the rods or the honeycomb, one may follow the mean flow and observe a (spatially) homogeneous turbulent behavior as it dies out.

In pratice, one does not usually calculate ensemble averages, i.e. one does not usually repeat the same experiment over and over again. An ergodic assumption is usually invoked, inspired by statistical mechanics. In the case of statistical equilibrium, with $\mathbf{U}(\mathbf{x}, t) = \mathbf{U}(\mathbf{x})$ independent of t, the *ergodic assumption* is that for "most" individual flows $\mathbf{u}^{(j)}(\mathbf{x}, t)$, $j = 1, \ldots, N$, the time averages along this flow converge, as the period of the average increases, to the mean value obtained by the ensemble average:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{u}^{(j)}(\mathbf{x}, t+s)\,\mathrm{d}s = \mathbf{U}(\mathbf{x}) = \frac{1}{N}\sum_{n=1}^{N} \mathbf{u}^{(n)}(\mathbf{x}, t).$$

Based upon this assumption, the averages are calculated as time averages of a specific individual flow over a period T sufficiently large.

Taylor's statistical theory.

The intrinsic random nature of turbulent motions was clearly recognized by Taylor [46], [47]. In doing so he used several probability tools, introducing in particular the study of the two-point, second-order spatial correlations of the velocity field, $\langle u_i(\mathbf{x}_1)u_j(\mathbf{x}_2)\rangle$. Backed by experimental observations, Taylor [47] introduced the concept of the homogeneous turbulence, with the assumption that spatial correlations depend only on the relative positions between the points, which is simply $\boldsymbol{\ell} = \mathbf{x}_2 - \mathbf{x}_1$ in the two-point case. This reduced the study of the correlations $\langle u_i(\mathbf{x})u_j(\mathbf{x}+\boldsymbol{\ell})\rangle$ as functions only of $\boldsymbol{\ell}$.

Taylor also introduced the notion of isotropic turbulence, assuming that the mean value of the physical quantities, written in relation to a given set of axes, are "unaltered if the axes of reference are rotated in any manner". In real turbulence, these assumptions are not strictly valid, but in some situations at least locally those assumptions may indeed be a good approximation for the smaller scales. When the isotropy assumption is valid, it simplifies greatly the analysis. This led to several relations among the different moments that could be tested in laboratories for the validity of the assumptions and the study of the nature of turbulence. In particular, it was observed that within the turbulent core of the flow this correlation decays considerably fast as the separation $\mathbf{x}_2 - \mathbf{x}_1$ increases, indicating the local character of turbulence. A microscale length was introduced as representing the average characteristic length of the small eddies affected by the dissipation of energy by viscosity.

Taylor studied in particular the *normalized lateral second-order correlation*

$$g(\ell) = \frac{\langle u_1(\mathbf{x}) u_1(\mathbf{x} + \ell \mathbf{e}_2)\rangle}{\langle u_1(\mathbf{x})^2\rangle}, \quad \ell \geqslant 0$$

which is more natural to measure in experiments. Note that $g(\ell)$ can also be defined for negative ℓ, but theoretically this is not necessary since the homogeneity assumption implies $g(\ell) = g(-\ell)$. In pratice, this can be a first test for homogeneity. Note also that this symmetry in ℓ implies $g'(0) = 0$. Moreover, $g(0) = 1$. Taylor then obtained the expansion

$$g(\ell) = 1 - \left(\frac{\ell}{\ell_T}\right)^2 + \mathcal{O}(\ell^4),$$

where ℓ_T is the *Taylor microlength* defined by

$$\frac{1}{\ell_T^2} = \lim_{\ell \to 0} \frac{1 - g(\ell)}{\ell^2} = -\frac{1}{2} g''(\ell) = \frac{1}{2} \frac{\left\langle \left(\frac{\partial u_1(\mathbf{x})}{\partial x_2}\right)^2 \right\rangle}{\langle u_1(\mathbf{x})^2\rangle}.$$

The quantity ℓ_T^2 measures the radius of curvature of the correlation $g(\ell)$ at $\ell = 0$, and it “may roughly be regarded as a measure of the diameters of the smallest eddies which are responsible for the dissipation of energy”.

Taylor also derived a direct connection between ℓ_T and other physical quantities. More precisely, let $e = \frac{1}{2}\langle |\mathbf{u}(\mathbf{x})|^2\rangle$ be the mean kinetic energy per unit mass of the flow and let $\epsilon = \nu \sum_{i=1}^{3} \langle |\nabla u_i(\mathbf{x})|^2\rangle$ denote the mean rate of energy dissipation by viscosity per unit mass and unit time. Then

$$\ell_T^2 = 30 \frac{\nu e}{\epsilon} = 15 \frac{\langle |\mathbf{u}(\mathbf{x})|^2\rangle}{\sum_{i=1}^3 \langle |\boldsymbol{\nabla} u_i(\mathbf{x})|^2\rangle}.$$

Von Karman and Howarth [25] realized that the Reynolds stresses form indeed a second order tensor, which in the case of homogeneous turbulence is only a function of $\boldsymbol{\ell}$:

$$R(\boldsymbol{\ell}) = \langle \mathbf{u}(\mathbf{x}) : \mathbf{u}(\mathbf{x} + \boldsymbol{\ell})\rangle = (R_{ij}(\boldsymbol{\ell}))_{i,j=1}^3 = (\langle u_i(\mathbf{x}) u_j(\mathbf{x} + \boldsymbol{\ell})\rangle)_{i,j=1}^3.$$

They obtained in this way several simplifications for the relations obtained by Taylor. Taylor [48] then introduced the notion of the energy spectrum, which would later permeate the theory of turbulence. He considered the Fourier transform of the correlation of the velocity field, interpreting the Fourier components as the energy associated with an eddy of the corresponding characteristic length. This *energy spectrum* turned out to be a fundamental tool in analysing the distribution of energy among different eddies of a flow. We follow the consideration of the energy spectrum as done in [4].

Consider the trace of the correlation tensor for a homogeneous flow:

$$\operatorname{Tr} R(\boldsymbol{\ell}) = R_{11}(\boldsymbol{\ell}) + R_{22}(\boldsymbol{\ell}) + R_{33}(\boldsymbol{\ell}), \quad \boldsymbol{\ell} \in \mathbb{R}^3.$$

Note that the mean kinetic energy can be written as $e = \operatorname{Tr} R(0)/2$. We may formally assume that the Fourier transform of $\operatorname{Tr} R(\boldsymbol{\ell})$ exists, and we denote it by $Q(\boldsymbol{\kappa})$, $\boldsymbol{\kappa} \in \mathbb{R}^3$. Then,

$$\operatorname{Tr} R(\boldsymbol{\ell}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} Q(\boldsymbol{\kappa}) e^{\mathrm{i}\boldsymbol{\ell}\cdot\boldsymbol{\kappa}} \, \mathrm{d}\boldsymbol{\kappa}.$$

The *energy spectrum* is defined by

$$\mathcal{S}(\kappa) = \frac{1}{2(2\pi)^{3/2}} \int_{|\boldsymbol{\kappa}|=\kappa} Q(\boldsymbol{\kappa}) \, \mathrm{d}\Sigma(\boldsymbol{\kappa}) \ \ \forall \kappa > 0,$$

where $\mathrm{d}\Sigma(\kappa)$ is the area element of the 2-sphere of radius $\boldsymbol{\kappa}$. With this definition, we can formally write

$$e = \frac{1}{2}\langle |\mathbf{u}(\mathbf{x})|^2 \rangle = \frac{1}{2} \operatorname{Tr} R(0) = \int_0^\infty \mathcal{S}(\kappa) \, \mathrm{d}\kappa.$$

The quantity $\mathcal{S}(\kappa)$ is an energy spectral density (dimension of $(\text{length})^3/(\text{time})^2$), and $\mathcal{S}(\kappa)\,\mathrm{d}\kappa$ is interpreted as the energy of the component of the flow formed by the "eddies" with characteristic length κ, i.e. the Fourier modes $\exp(\boldsymbol{\ell} \cdot \boldsymbol{\kappa})$ with $\kappa \leqslant |\boldsymbol{\kappa}| \leqslant \kappa + \mathrm{d}\kappa$.

Kolmogorov's theory of locally isotropic turbulence.

The next major breakthrough came in a series of short papers by Kolmogorov [26], [27], [28], introducing the "theory of locally isotropic turbulence." It was partly based on Richardson's *energy cascade* process [43], in which eddies of a given length scale break down into smaller eddies, and in turn these smaller eddies break down into still smaller ones, and so on, until small enough scales are reached in which viscosity plays an important role and the kinetic energy is finally dissipated into heat.

Kolmogorov envisaged that due to the turbulent nature of the transfer of energy to smaller eddies, the orienting effects of the shear forces originated in the anisotropic, energy-containing large-scale eddies become the weaker the smaller the eddies are. Then, for sufficiently small eddies, the orienting effect is completely lost and the flow can be regarded as homogeneous and isotropic. In other words, despite the inhomogeneity and anisotropy of the large-scale eddies, the turbulent motion of sufficiently small eddies may eventually become statistically homogeneous, isotropic, and independent of the particular energy-productive mechanisms, provided the Reynolds number is sufficiently large.

The energy injected in the system at the larger scales is transferred to smaller scales by this cascade process until the viscous effects become predominant at the smallest scales and the kinetic energy is dissipated into heat. The flow is also assumed to be in a state of statistical equilibrium with respect to time. He postulated then that for sufficiently large Reynolds numbers the statistical regime of the small-scale components is universal and is determined only by two dimensional parameters: the viscosity ν and the mean rate of energy dissipation per unit time and unit mass, $\epsilon = \nu \sum_{i=1}^{3} \langle |\boldsymbol{\nabla} u_i(\mathbf{x}, t)|^2 \rangle$. The range of scales in which this universality holds is termed the *equilibrium range*. Here, the notion of homogeneity and statistical equilibrium play a role in that ϵ becomes constant, independent of $\mathbf{x}$ and t (but it depends on time in the case of decaying homogeneous turbulence).

Simple dimensional analysis shows that the only length scale of the form $\nu^\alpha \epsilon^\beta$, with α and β real, is $\ell_\epsilon = (\nu^3/\epsilon)^{1/4}$. This length scale is interpreted as that near which most of the energy dissipation takes place. The scale ℓ_ϵ is termed the *Kolmogorov dissipation length.*

In Kolmogorov's theory for fully-developed turbulent flows there is a wide separation between the energy-productive scales and the energy-dissipative ones, so that there exists a range of scales much smaller than the large-scale energy-containing scales and much larger than the dissipative scale ℓ_ϵ where viscosity is still negligible and, hence, the statistical regime depends only on the mean rate of energy dissipation ϵ. This range of scales is termed the *inertial range*. By simple dimensional analysis, one finds that in this range the second order correlations must be proportional to $(\epsilon\ell)^{2/3}$, i.e.,

$$S_2(\ell) \stackrel{\text{def}}{=} \left\langle \left(\mathbf{u}(\mathbf{x} + \boldsymbol{\ell}) \cdot \frac{\boldsymbol{\ell}}{\ell} - \mathbf{u}(\mathbf{x}) \cdot \frac{\boldsymbol{\ell}}{\ell} \right)^2 \right\rangle = \beta_{\mathrm{Ko}} (\epsilon \ell)^{2/3}.$$

The constant β_{Ko} is termed the *Kolmogorov constant in the physical space* (there is a related constant in the spectral space). The particular second order correlation $S_2(\ell)$ is called the second order structure function. The above power law for this structure function is called *Kolmogorov's two-third law.*

Dimensional analysis allows a multiplicative nondimensional factor of the form $g(\nu^{-3/4}\epsilon^{1/4}\ell) = g(\ell/\ell_\epsilon)$. The Kolmogorov constant may be regarded as the limit $g(0)$ as the viscosity ν goes to zero. If this limit is zero, the rate of decay to zero may lead to a multiplicative factor of the order of $(\ell/\ell_\epsilon)^\alpha$, for some power α.

Kolmogorov's derivation of ℓ_ϵ and of the two-third law was not by dimensional analysis. It was indeed by a more convincing universality argument based on scaling invariance. More precisely, it is assumed that there exists a characteristic dissipative length scale ℓ_d which is related to the turbulent eddies which are rapidly annihilated by viscosity, and that ℓ_d, being in the equilibrium range, should be a universal function of ϵ and ν: $\ell_\mathrm{d} = f(\nu, \epsilon)$. In particular, f should be independent of the choice of units for space and time. Thus, if we pass from $\mathbf{x}$ and t to $\mathbf{x}' = \xi \mathbf{x}$ and $t' = \tau t$, we should still have a characteristic length scale in the form of $\ell_d' = f(\nu', \epsilon')$. For dimensional reasons, $\ell_d' = \xi \ell_d$, $\nu' = \xi^2 \tau^{-1} \nu$, and $\epsilon' = \xi^2 \tau^{-3} \epsilon$. Thus, $\xi f(\nu, \epsilon) = f(\xi^2 \nu/\tau, \xi^2 \epsilon/\tau^3)$. Taking $\tau = (\nu\epsilon)^{1/2}$ and $\xi = \epsilon^{1/4}\nu^{-3/4}$, we obtain

$$\ell_\mathrm{d} = \left(\frac{\nu^3}{\epsilon}\right)^{1/4} f(1,1) \sim \ell_\epsilon.$$

As for the two-third law, we should have $S_2(\ell) = g(\epsilon, \ell)$ for $\ell_\epsilon \ll \ell \ll \ell_0$, where ℓ_0 is a typical large-scale length. The universality of g implies that $\xi^2/\tau^2 g(\epsilon, \ell) = g(\xi^2\epsilon/\tau^3, \xi\ell)$. Hence, taking $\xi = \ell^{-1}$ and $\tau = \epsilon^{1/3} r^{-2/3}$ one obtains

$$S_2(\ell) = g(\epsilon, \ell) = \epsilon^{2/3}\ell^{2/3} g(1,1) \sim (\epsilon\ell)^{2/3}.$$

In this case, $\beta_\mathrm{Ko} = g(1,1)$.

Oboukhoff's mechanism and the Kolmogorov spectral five-third law.
A spectral version of the two-third law for the energy spectrum $\mathcal{S}(\kappa)$ was given in [41]. Since $\mathcal{S}(\kappa)$ is an energy spectrum, hence of physical dimension (length)^3/(time)^2, the spectral version of the two-third law takes the form

$$\mathcal{S}(\kappa) = \beta_\mathrm{Ko}' \epsilon^{2/3} \kappa^{-5/3},$$

which is called *Kolmogorov's five-third law.* The constant β_Ko' is called *Kolmogorov's constant in the spectral space.* The dissipation length also has its spectral version, namely *Kolmogorov's wavenumber* $\kappa_\epsilon = (\epsilon/\nu^3)^{1/4}$. Obukhoff's derivation, however, differs from Kolmogorov's in that a mechanism inspired by the energy cascade process is assumed: First, a characteristic time $\tau_\kappa \sim (\kappa^3 \mathcal{S}(\kappa))^{-1/2}$ is introduced for the eddies with length scale of the order of κ^{-1}. The mean kinetic energy of those eddies is of the order of $\kappa\mathcal{S}(\kappa)$. According to the cascade process, this kinetic energy is transferred per unit time to smaller scales, with this energy flux being of the order of the rate ϵ of dissipation of energy per unit time. Thus, $\epsilon \sim \kappa\mathcal{S}(\kappa)/\tau_\kappa \sim \kappa^{5/2}\mathcal{S}(\kappa)^{3/2}$. Hence, according to *Obukhoff's mechanism,*

$$\mathcal{S}(\kappa) \sim \epsilon^{2/3}\kappa^{-5/3}.$$

The Kolmogorov length scale can be related to the Reynolds number as follows. If U is a characteristic velocity of the flow, $e = U^2/2$ is the mean kinetic energy per unit mass of the flow. The characteristic time for the macroscales is $\tau_0 = \ell_0/U$, also called the *circulation time*. Hence, the rate ϵ of energy dissipation per unit time and unit mass is expected to be of the order of e/τ_0, i.e. $\epsilon \sim U^3/\ell_0$, which is called the *energy dissipation law*. Since $\ell_\epsilon = (\nu^3/\epsilon)^{1/4}$, we find $\ell_0/\ell_\epsilon \sim \mathrm{Re}^{3/4}$. For a well-defined separation between the dissipative scale ℓ_ϵ and the macroscopic scale ℓ_0, we need $\ell_0 \gg \ell_\epsilon$, which means $\mathrm{Re}^{3/4} \gg 1$. Following [4], notice that for the existence of a well-defined Kolmogorov inertial range a larger Reynolds number is required. Indeed, we need a range of scales ℓ such that $\ell_0 \gg \ell \gg \ell_\epsilon$. This is equivalent[1] to $(\ell_0/\ell_\epsilon)^{1/2} \gg 1$. Therefore, for a well-defined inertial range, a Reynolds number such that $\mathrm{Re}^{3/8} \gg 1$ is required. In a wind-tunnel experiment with $\ell_0 \sim 3\,\mathrm{m}$ and $\ell_\epsilon \sim 1\,\mathrm{mm}$, we have $\mathrm{Re} \sim 40.000$, but only $\mathrm{Re}^{3/8} \sim 54$, which may be not enough for the existence of a well-defined inertial range. Atmospheric flows, on the other hand, may have ℓ_0 of the order of thousands of kilometers with the dissipation scale of the order of centimeters, yielding $\mathrm{Re} \sim 10^{10}$ and $\mathrm{Re}^{3/8} \sim 10^4$, which is more than enough for an inertial range extending for tens of kilometers.

The energy dissipation law can also be used to relate certain quantities to the Taylor length ℓ_T. In fact, $\ell_T^2 \sim \nu e/\epsilon \sim \nu e \ell_0/U^3 \sim \ell_0^2/\mathrm{Re}$, thus $\ell_T/\ell_0 \sim \mathrm{Re}^{1/2}$. Similarly, $\ell_T^2 \sim \nu e/\epsilon \sim \nu U^2/\epsilon \sim \nu \epsilon^{2/3}\ell_0^{2/3}/\epsilon \sim \ell_0^{2/3}\ell_\epsilon^{4/3}$, hence $\ell_T \sim \ell_0^{1/3}\ell_\epsilon^{2/3}$.

The universality hypothesis was later criticized, the reason being possible deviations of the mean rate of energy dissipation throughout the flow. Such deviations correspond to the phenomenon known as *intermittency*. In three-dimensional flows an important mechanism responsible for intermittent behavior is vortex stretching and thinning, which lead to the formation of high-vorticity/low-dissipative structures moving within a supposedly statistically homogeneous flow. These coherent structures are tube-like vortex filaments with diameter as small as the Kolmogorov scale and with longitudinal length extending from the Taylor scale up to the integral scale. Within these tubes the flow is nearly a rigid motion with nearly no energy dissipation, and this could upset the homogeneity of the mean energy dissipation rate. The issues of intermittency and coherent structures in fully-developed turbulence and whether and how they could affect the deductions of Kolmogorov's universality theory, or at least the power laws for higher-order moments, are currently one of the major issues being addressed in turbulence theory [3], [7], [8], [19], [21], [24], [29], [36], [40], [42], [45].

[1] Here, two (nonnegative) quantities a and b of the same physical dimension are related by $b \gg a$ when $a/b \leqslant \delta$ for a small nondimensional error parameter δ which, for consistency, is held fixed throughout a certain string of relations.

Kraichnan's theory of two-dimensional turbulence.

We close this section by considering two-dimensional turbulence [30], [31], [35], [5]. Such flows play an important role in some situations, particularly in geophysical flows. They are also more amenable to analysis and to numerical simulations, making it an important test bed for turbulence modeling (both analytical and numerical). The dynamics in two dimensions is simpler in the sense that vorticity always points perpendicular to the two-dimensional domain. Nevertheless, two-dimensional flows still exhibit a very complicate behavior.

Due to the conservation property of vorticity particular to two-dimensional flows an important role in two-dimensional turbulence is played by the *enstrophy*, which is defined as one-half the square of the vorticity. Correspondingly, an important role is played by the rate of enstrophy dissipation, $\eta = \nu\langle|\Delta\mathbf{u}(\mathbf{x})|^2\rangle$. A universality argument or a simple dimensional analysis yields the *Kraichnan dissipation length* $\ell_\eta = (\nu^3/\eta)^{1/6}$, with the corresponding *Kraichnan dissipation wavenumber* $\kappa_\eta = (\eta/\nu^3)^{1/6}$, and the *Kraichnan energy spectrum* $\mathcal{S}(\kappa) \sim \beta_{\mathrm{Kr}}\eta^{2/3}\kappa^{-3}$ for κ in the inertial range $\kappa_{\mathbf{f}} \ll \kappa \ll \kappa_\eta$; we will call β_{Kr} the *Kraichnan constant.* The characteristic wavenumber $\kappa_{\mathbf{f}}$ is associated with the energy-productive scales, which are allowed to occur at length scales much smaller than the larger scales characterized by the wavenumber κ_0. Within a range of wavenumbers κ with $\kappa_0 \ll \kappa \ll \kappa_{\mathbf{f}}$, Kraichnan suggested the existence of an *inverse energy cascade*, from the small scales to the larger scales. In this range, the effects of energy transfer are more important than those of enstrophy transfer, so that the five-third energy spectrum $\mathcal{S}(\kappa) \sim \epsilon^{2/3}\kappa^{-5/3}$ is recovered for $\kappa_0 \ll \kappa \ll \kappa_{\mathbf{f}}$.

Kraichnan actually devised a cascade mechanism to explain the form of the spectrum in the enstrophy-cascade range. This mechanim is akin to Obukhoff's, but it is in terms of the enstrophy flux, and is slightly more explicit. The *Kraichnan cascade mechanism* asserts that within the enstrophy-cascade range $\kappa_{\mathbf{f}} \ll \kappa \ll \kappa_\eta$, the viscous dissipation of enstrophy is negligible and the motion is dominated by the transfer of enstrophy to smaller scales through a break-down process in which eddies of a given linear size $\sim 1/\kappa$ break down into eddies of about half their linear size while traveling a distance compared to their linear size.

Let us introduce some concepts and quantities which will help us exploit this mechanism. The flow is decomposed into parts $\mathbf{u}_\kappa$ containing wavenumbers between κ and 2κ. Let η_κ denote the enstrophy flux per unit mass and unit time from the wavenumbers less than 2κ to those greater than or equal to 2κ. By the break-down mechanism described above, η_κ is nearly the enstrophy flux from $\mathbf{u}_\kappa$ to $2\mathbf{u}_\kappa$. The enstrophy contained in the component $\mathbf{u}_\kappa$ is approximately $\kappa^3\mathcal{S}(\kappa)$, and the energy is $\kappa\mathcal{S}(\kappa)$. The *eddy turnaround time* is the time an eddy takes to rotate once around its axis. This is the characteristic time of an eddy whithin the enstrophy cascade region in two-dimensional turbulence, the reason being that in this case enstrophy is supposed to be the most important conserved quantity. Hence, since enstrophy has dimension $(\text{time})^{-3}$, the turnaround time of an eddy with linear size of the order of κ^{-1} and, thus, enstrophy $\kappa^3\mathcal{S}(\kappa)$, is $\tau_\kappa \sim (\kappa^3\mathcal{S}(\kappa))^{-1/3}$. The time that an eddy within the inertial range takes to break down into eddies with half their linear size is called the *eddy turnover time*.

The cascade mechanism asserts that the turnover time is about the same as the turnaround time for an eddy within the inertial range. Hence, the enstrophy flux should be such that

$$\eta_\kappa = \begin{pmatrix} \text{enstrophy transferred} \\ \text{through wavenumber } 2\kappa \\ \text{per unit time} \end{pmatrix} \sim \begin{pmatrix} \text{total} \\ \text{enstrophy of } \mathbf{u}_\kappa \\ \text{per turnaround time} \end{pmatrix} = \frac{\kappa^3\mathcal{S}(\kappa)}{\tau_\kappa}.$$

Since we are assuming a state of statistical equilibrium and that all enstrophy is produced in the largest scales and dissipated in the smallest scales, the enstrophy flux η_κ within the inertial range should be the same as the rate of enstrophy dissipation per unit time and unit mass due to the viscous effects, i.e. $\eta_\kappa \approx \eta$. Therefore, we deduce that

$$\mathcal{S}(\kappa) \sim \eta^{2/3}\kappa^{-3}.$$

2. Mathematical setting for the Navier-Stokes equations

We recall briefly the L^2-setting for the mathematical theory of the incompressible Navier-Stokes equations as initiated by J. Leray (see [32], [37], [49], [9]). We consider the three- and two-dimensional cases with two types of boundary conditions: i) *no-slip* condition $\mathbf{u} = 0$ on the boundary $\partial\Omega$ of a bounded domain Ω (not necessarily smooth) in $\mathbb{R}^d$, $d = 2, 3$; and ii) *periodic* conditions in $\mathbb{R}^d$, with a period L_i, $i = 1, \ldots, d$, in each direction x_i, in which case we set the flow domain to be $\Omega = \Pi_{i=1}^d(-L_i/2, L_i/2)$, and we assume that the average flow on Ω vanishes: $\int_\Omega \mathbf{u}(\mathbf{x}, t)\,\mathrm{d}\mathbf{x} = 0$.

We consider the space $\mathcal{V}$ of divergence free test functions:

$$\begin{aligned} \mathcal{V} &= \mathcal{V}_{\text{nsp}} = \{\mathbf{u} \in \mathcal{C}_c^\infty(\Omega)^d;\ \ \boldsymbol{\nabla}\cdot\mathbf{u} = 0\}, \\ \mathcal{V} &= \dot{\mathcal{V}}_{\text{per}} = \left\{\mathbf{u} \in \mathcal{C}_{\text{per}}^\infty(\mathbb{R}^d)^d;\ \int_\Omega \mathbf{u}(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0,\ \ \boldsymbol{\nabla}\cdot\mathbf{u} = 0\right\}, \end{aligned}$$

in the no-slip case and in the space-periodic case with zero space average, respectively. Here $\mathcal{C}_c^\infty(\Omega)$ is the space of $\mathcal{C}^\infty$ functions with compact support in Ω, and $\mathcal{C}_{\text{per}}^\infty(\mathbb{R}^d)$

is the space of $\mathcal{C}^\infty$ functions on $\mathbb{R}^d$ which are periodic with the period L_i in each direction x_i.

In either case we let H be the closure of $\mathcal{V}$ in the Lebesgue space $L^2(\Omega)^d$, and V the closure of $\mathcal{V}$ in the Sobolev space $H^1(\Omega)^d$. In H and V we consider respectively the inner products

$$(\mathbf{u},\mathbf{v}) = \int_\Omega \mathbf{u}(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})\,\mathrm{d}\mathbf{x}, \quad (\!(\mathbf{u},\mathbf{v})\!) = \int_\Omega \sum_{i=1}^d \frac{\partial \mathbf{u}(\mathbf{x})}{\partial x_i}\cdot\frac{\partial \mathbf{v}(\mathbf{x})}{\partial x_i}\,\mathrm{d}\mathbf{x},$$

and the associated norms $|\mathbf{u}| = (\mathbf{u},\mathbf{u})^{1/2}$ and $\|\mathbf{v}\| = (\!(\mathbf{v},\mathbf{v})\!)^{1/2}$. We denote by $H_{\mathbf{w}}$ the space H endowed with its weak topology.

The boundedness of the domain guarantees that the following Poincaré inequality holds: there exists $\lambda_1 > 0$ such that $|\mathbf{u}|^2 \leqslant \lambda_1^{-1}\|\mathbf{u}\|^2$ for all $\mathbf{u}$ in V. The importance of this condition, regardless of the boundedness (one could consider an infinite channel-like domain) or the smoothness of the domain, is that the system is dissipative, and many of the results in the dynamical system approach to the NSE, usually done for bounded, smooth domains, can be extended to such domains [1], [2], [33], [23], [44], [38].

We define the (abstract) Stokes operator $A\colon V \to V'$ by duality,

$$(A\mathbf{u},\mathbf{v}) = (\!(\mathbf{u},\mathbf{v})\!), \quad \forall\, \mathbf{u},\mathbf{v} \in V,$$

where the same symbol $(\cdot,\cdot)$ is used for the duality product between V' and V.

In the weak formulation of the NSE, given $\mathbf{u}_0$ in H and $\mathbf{f}$ in V', we look for a function $\mathbf{u}$ in $L^\infty(0,T;H)\cap L^2(0,T;V)$, $T>0$, such that

$$\frac{\mathrm{d}}{\mathrm{dt}}(\mathbf{u},\mathbf{v}) + \nu(\!(\mathbf{u},\mathbf{v})\!) + b(\mathbf{u},\mathbf{u},\mathbf{v}) = (\mathbf{f},\mathbf{v}), \quad \forall\, \mathbf{v}\in V,$$

in the distribution sense on $(0,T)$, and with $\mathbf{u}(0)=\mathbf{u}_0$ in some suitable sense. Here

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = \sum_{i,j=1}^d \int_\Omega u_i(\mathbf{x})\frac{\partial v_j(\mathbf{x})}{\partial x_i}\,w_j(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

This trilinear form $b = b(\mathbf{u},\mathbf{v},\mathbf{w})$ can be extended to a continuous trilinear form defined on V. We use the Ladyzhenskaya inequalities in dimensions two and three:

$$\|\mathbf{u}\|_{L^4(\Omega^3)} \leqslant c_1|\mathbf{u}|^{1/4}\|\mathbf{u}\|^{3/4}, \quad \|\mathbf{u}\|_{L^4(\Omega)^2} \leqslant c_1|\mathbf{u}|^{1/2}, \ \forall\,\mathbf{u}\in V,$$

and the following Agmon inequality in the two-dimensional periodic case (valid also for 2D smooth domains):

$$\|\mathbf{u}\|_{L^\infty(\Omega)^2} \leqslant c_1|\mathbf{u}|^{1/2}|A\mathbf{u}|^{1/2}, \ \forall\,\mathbf{u}\in D(A).$$

These two inequalities, together with various applications of Hölder's inequality, yield a number of important estimates for the inertial term.

The following orthogonality property of the nonlinear term is fundamental and expresses the conservation of energy by the inertial forces:

$$b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0, \ \ \forall\, \mathbf{u}, \mathbf{v} \in V.$$

In the two-dimensional periodic case, we also have the following fundamental orthogonality property expressing the conservation of enstrophy of the inertial term:

$$b(\mathbf{u}, \mathbf{u}, A\mathbf{u}) = 0, \quad \forall\, \mathbf{u} \in D(A).$$

An equivalent functional-equation formulation for the NSE can be written. The form $b(\mathbf{u}, \mathbf{v}, \mathbf{w})$ defines a bilinear operator $B\colon\ V \times V \to V'$ by

$$(B(\mathbf{u}, \mathbf{v}), \mathbf{w}) = b(\mathbf{u}, \mathbf{v}, \mathbf{w}), \quad \forall\, \mathbf{u}, \mathbf{v}, \mathbf{w} \in V.$$

Then we have the following functional-equation formulation: Given $\mathbf{u}_0$ in H and $\mathbf{f}$ in V', find $\mathbf{u}$ in $L^\infty(0,T;H) \cap L^2(0,T;V)$, $T > 0$, with $\mathbf{u}' = \mathrm{d}\mathbf{u}/\mathrm{d}t \in L^1(0,T;V')$, such that

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \nu A\mathbf{u} + B(\mathbf{u}, \mathbf{u}) = \mathbf{f}$$

and $\mathbf{u}(0) = \mathbf{u}_0$ in some suitable sense. For simplicity of notation, we define $\mathbf{F}(\mathbf{u}) = \mathbf{f} - \nu A\mathbf{u} - B(\mathbf{u}, \mathbf{u})$, and we often write the NSE in the form

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{F}(\mathbf{u}).$$

Under the sole assumption that the Poincaré inequality holds on Ω, the operator A can be regarded as a positive, self-adjoint closed operator $A\colon\ D(A) \subseteq H \to H$ with the domain $D(A) = \{\mathbf{v} \in V;\ A\mathbf{v} \in H\}$. Then the fractional powers A^α, $\alpha \in \mathbb{R}$, ($\alpha \geqslant 0$ or $\alpha < 0$) can be defined, with $D(A^{1/2}) = V$, $D(A^0) = H$ and $D(A^{-1/2}) = V'$.

Under the condition that the domain Ω is bounded it follows by the Rellich Lemma that V is compactly embedded into H, so that A^{-1} is compact as a closed operator in H. Hence, there exists an orthonormal basis $\{\mathbf{w}_m\}_{m=1}^\infty$ in H such that

$$A\mathbf{w}_m = \lambda_m \mathbf{w}_m, \quad 0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_m \to +\infty \quad \text{as } m \to +\infty.$$

We let P_m be the orthogonal projector of H onto the space spanned by $\mathbf{w}_1, \ldots, \mathbf{w}_m$.

The notion of the ensemble average is reconciled with the mathematical theory of the Navier-Stokes equations through the concept of statistical solution. Two such concepts exist, one is made of probability measures in the phase space H, [17], [11], [12], [22] and the other is a probability measure in the time-dependent space of weak solutions $\mathcal{C}([0,T), H_{\mathrm{w}})$ of the Navier-Stokes equations [52], [53]. Due to the lack of space and since we are mostly considering the stationary case we only deal with the first concept here.

Ensemble averages.

We have seen earlier that ensemble averages are interpreted as averages over a number of experiments. This can be made more formal with use of the probability theory. In the formalism we consider, the probability space is taken to be the space H defined above, the corresponding σ-algebra of measurable sets is that of the Borel sets of H, and the measure is a Borel probability measure on H. In the three-dimensional case, we will be often working with the weak topology. Fortunately the Borel σ-algebra generated by the weakly open sets coincides with that for the open sets in the strong topology, which saves some work concerning the measurability of sets and functions.

Since H is a separable Hilbert space every Borel measure is automatically (inner and outer) regular. An important consequence of the regularity of a Borel probability measure is the density of continuous functions (or just weakly continuous functions) in the space of integrable functions.

We say that a measure μ is *carried* by a measurable set E when E has *full measure*, i.e. $\mu(H \setminus E) = 0$. The *support* of a Borel probability measure μ is the smallest closed set which carries μ. Note that a carrier is usually not unique, while the support is always unique.

The ensemble averages are regarded as averages with respect to a Borel probability measure μ. If $\varphi\colon H \to \mathbb{R}$ is a Borel function representing some physical information $\varphi(\mathbf{u})$ extracted from a velocity field $\mathbf{u}$, such as kinetic energy, velocity, correlation, enstrophy, etc., then its mean value is

$$\langle \varphi(\mathbf{u}) \rangle = \int_H \varphi(\mathbf{u}) \, \mathrm{d}\mathbf{u}.$$

On the right-hand side, $\mathbf{u}$ is just a dummy variable for the integral on H, and $\langle \varphi(\mathbf{u}) \rangle$ is just a more explicit notation for $\langle \varphi \rangle$ to indicate that it is some information concerning a random velocity field.

A particularly useful class of Borel functions are the *cylindrical test functions* $\Phi\colon H \to \mathbb{R}$ of the form $\Phi(\mathbf{u}) = \varphi((\mathbf{u}, \mathbf{g}_1), \ldots, (\mathbf{u}, \mathbf{g}_m))$, where φ is a $\mathcal{C}^1$ scalar function on $\mathbb{R}^m$, $m \in \mathbb{N}$, with compact support, and $\mathbf{g}_1, \ldots, \mathbf{g}_m$ belong to V. For such a Φ we denote by Φ' its differential in H, which has the form

$$\Phi'(\mathbf{u}) = \sum_{j=1}^{m} \partial_j \varphi((\mathbf{u}, \mathbf{g}_1), \ldots, (\mathbf{u}, \mathbf{g}_m))\mathbf{g}_j,$$

where $\partial_j \varphi$ denotes the derivative of φ with respect to the j-th variable. It follows that $\Phi'(\mathbf{u}) \in V$ since it is a linear combination of the $\mathbf{g}_j$'s.

Time-dependent statistical solutions.

We look for the evolution of a probability distribution for the velocity field of a flow. We are given a Borel probability measure μ_0 on H representing the initial probability distribution of the velocity field, and we want to obtain a time-dependent family $\{\mu_t\}_{t\geqslant 0}$ of Borel probability measures, with each measure μ_t representing the probability distribution of the velocity field at time t.

We may derive an equation for the evolution of those measures as follows. If we think of the ensemble average as an average over a number of experiments $\mathbf{u}^{(n)}(\mathbf{x}, t)$, $n = 1, \ldots, N$, with possibly different weights $\theta_n \in (0, 1)$, $\sum\limits_n \theta_n = 1$, then the average of a quantity $\Phi\colon H \to \mathbb{R}$ is $\sum\limits_{n=1}^{N} \theta_n \Phi(\mathbf{u}^{(n)}(t))$. By formally differentiating this expression in time and assuming that the velocity field satisfies the Navier-Stokes equations $\mathrm{d}\mathbf{u}/\mathrm{d}t = \mathbf{F}(\mathbf{u})$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{n=1}^{N} \theta_n \Phi(\mathbf{u}^{(n)}(t)) = \sum_{n=1}^{N} \theta_n \big(\mathbf{F}(\mathbf{u}^{(n)}(t)), \Phi'(\mathbf{u}^{(n)}(t))\big).$$

This can be written in terms of Borel probability measures on H. Indeed, let $\delta_{\mathbf{v}}$ denote the Dirac measure at an arbitrary $\mathbf{v}$ in H. The family $\{\mu_t\}_{t\geqslant 0}$ can be written as $\mu_t = \sum\limits_{n=1}^{N} \theta_n \delta_{\mathbf{u}^{(n)}(t)}$. The ensemble average becomes

$$\langle \Phi(\mathbf{u}(t)) \rangle = \sum_{n=1}^{N} \theta_n \Phi(\mathbf{u}^{(n)}(t)) = \int_H \Phi(\mathbf{u})\, \mathrm{d}\mu_t(\mathbf{u}).$$

Thus, we can write the previous differential equation as the *statistical Navier-Stokes equation*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_H \Phi(\mathbf{u})\, \mathrm{d}\mu_t(\mathbf{u}) = \int_H (\mathbf{F}(\mathbf{u}), \Phi'(\mathbf{u}))\, \mathrm{d}\mu_t(\mathbf{u})$$

with the initial condition $\mu_0 = \sum\limits_{n=1}^{N} \theta_n \delta_{\mathbf{u}^{(n)}(0)}$.

More generally, given an arbitrary initial Borel probability measure μ_0 on H (with finite-kinetic energy, i.e. $\int_H |\mathbf{u}|^2 \, \mathrm{d}\mu_0(\mathbf{u}) < \infty$), we may look for a family $\{\mu_t\}_{t \geqslant 0}$ of Borel probability measures on H satisfying the statistical Navier-Stokes equation defined above and the initial condition $\mu_t|_{t=0} = \mu_0$ in some suitable sense [11]. Note the remarkable fact that this equation is linear in μ_t.

Thus, we are led to the following definition: A family $\{\mu_t\}_{t \geqslant 0}$ of Borel probability measures on H is called a *statistical solution* of the Navier-Stokes equations on H with the initial condition μ_0 if

i) $t \mapsto \int_H \varphi(\mathbf{u}) \, \mathrm{d}\mu_t(\mathbf{u})$ is continuous on $[0, \infty)$ for all $\varphi \in \mathcal{C}(H_{\mathrm{w}})$ bounded;

ii) $t \mapsto \int_H |\mathbf{u}|^2 \, \mathrm{d}\mu_t(\mathbf{u})$ belongs to $L^\infty(0, \infty)$ and is continuous at $t = 0$;

iii) $t \mapsto \int_H \|\mathbf{u}\|^2 \, \mathrm{d}\mu_t(\mathbf{u})$ belongs to $L^1_{\mathrm{loc}}(0, \infty)$.

iv) The following energy inequality holds in the distribution sense on $(0, \infty)$:

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_H |\mathbf{u}|^2 \, \mathrm{d}\mu_t(\mathbf{u}) + \nu \int_H \|\mathbf{u}\|^2 \, \mathrm{d}\mu_t(\mathbf{u}) \leqslant \int_H (\mathbf{f}, \mathbf{u}) \, \mathrm{d}\mu_t(\mathbf{u}).$$

v) The family $\{\mu_t\}_{t \geqslant 0}$ satisfies the statistical Navier-Stokes equation in the distribution sense on $(0, \infty)$, for all cylindrical test functions Φ.

Stationary statistical solutions.

In the case of a turbulent flow in statistical equilibrium in time, the statistical information of the flow does not vary with time. In the above framework, this is reflected by the family $\{\mu_t\}_{t \geqslant 0}$ of statistical solutions being independent of t. Hence, μ_t is identically equal to a Borel probability measure μ on H, and this measure must satisfy the *stationary statistical Navier-Stokes equation*

$$\int_H (\mathbf{F}(\mathbf{u}), \Phi'(\mathbf{u})) \, \mathrm{d}\mu(\mathbf{u}) = 0$$

for all cylindrical test functions Φ on H. Moreover, the mean kinetic energy and the mean enstrophy are finite,

$$\frac{1}{2} \int_H |\mathbf{u}|^2 \, \mathrm{d}\mu(\mathbf{u}) < \infty, \qquad \frac{1}{2} \int_H \|\mathbf{u}\|^2 \, \mathrm{d}\mu(\mathbf{u}) < \infty,$$

and the following energy inequality holds:

$$\int_H \{\nu \|\mathbf{u}\|^2 - (\mathbf{f}, \mathbf{u})\} \, \mathrm{d}\mu(\mathbf{u}) \leqslant 0.$$

A stronger energy inequality is actually assumed, namely

$$\int_{\{e_1 \leqslant \frac{1}{2}|\mathbf{u}|^2 < e_2\}} \{\nu \|\mathbf{u}\|^2 - (\mathbf{f}, \mathbf{u})\} \, \mathrm{d}\mu(\mathbf{u}) \leqslant 0,$$

for all energy levels $0 \leqslant e_1 < e_2 \leqslant \infty$. A Borel probability measure μ satisfying those properties is called a *stationary statistical solution*. One can deduce from the stronger energy inequality that the support of a stationary statistical solution is bounded in H by $\nu^{-1}\lambda_1^{-1/2}\|\mathbf{f}\|_{V'}$.

The concept of a stationary statistical solution is regarded as a generalization of the notion of an invariant measure. It is relevant in the three-dimensional case, in which the semigroup associated with the NSE is not available. In the two-dimensional case, in which the semigroup can be defined, the two notions are in fact equivalent [12], [14].

Generalized time-average limits.

A particular type of a stationary statistical solution is obtained via time averages, in a way akin to the ergodic hypothesis. The ergodicity, in our context, would imply that for "most" weak solutions $\mathbf{u} = \mathbf{u}(t)$, $t \geqslant 0$, of the NSE we have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \varphi(\mathbf{u}(t))\,\mathrm{d}t = \int_H \varphi(\mathbf{u})\,\mathrm{d}\mu(\mathbf{u}),$$

for some stationary statistical solution μ. The term "most" here could be made rigorous by using stationary statistical solutions in the trajectory space $\mathcal{C}([0,\infty), H_{\mathrm{w}})$. In order to avoid the ergodic hypothesis the notion of a generalized limit is considered in [6]. A *generalized limit* on $(0,\infty)$ is a positive linear functional which extends the classical limit to the space of bounded real-valued functions defined on $(0,\infty)$. We denote this generalized limit by $\mathrm{LIM}_{T\to\infty}$. Then it is possible to show that for each global weak solution $\tilde{\mathbf{u}} = \tilde{\mathbf{u}}(t)$, $t \geqslant 0$, of the NSE, the following generalized limit is well-defined for every bounded, weakly continuous function φ in H, and defines a regular Borel probability measure $\mu_{\tilde{\mathbf{u}}}$ satisfying

$$\mathop{\mathrm{LIM}}_{T\to\infty} \frac{1}{T} \int_0^T \varphi(\mathbf{u}(t))\,\mathrm{d}t = \int_H \varphi(\mathbf{u})\,\mathrm{d}\mu_{\tilde{\mathbf{u}}}(\mathbf{u}).$$

Such a measure $\mu_{\tilde{\mathbf{u}}}$ is called a *time-average measure* on H associated with the weak solution $\tilde{\mathbf{u}}$. As the notation indicates, it may vary with $\tilde{\mathbf{u}}$. This is true, for instance, when ergodicity does not hold. This measure also depends on the choice of the generalized limit, but this dependence is usually omitted from the notation. One can show that this time-average measure is a stationary statistical solution. In the applications that follow, we consider arbitrary stationary statistical solutions, which includes the particular time-average measures usually exploited in actual (or numerical) fluid flow experiments.

4. Stationary statistical solutions applied to turbulence in statistical equilibrium

We now apply the concept of stationary statistical solutions to deduce results for the statistical mean value of some flow properties relevant to turbulence theory [14], [15], [16], [13]. First, let us introduce a notation related to the decomposition of the flow with respect to wavenumbers. Recall the spectral representation of a velocity field $\mathbf{u}$ in H: $\mathbf{u} = \sum_{j=0}^{\infty} \hat{u}_j \mathbf{w}_j$, where the $\mathbf{w}_j$ are the eigenfunctions of the Stokes operator A, each associated with an eigenvalue λ_j counted according to its multiplicity. With each eigenvalue $\lambda = \lambda_j$ we associate a wavenumber $\kappa = \lambda^{1/2}$. The wavenumbers form a discrete set, and the smallest possible one is $\lambda_1^{1/2}$. In what follows, κ is implicitly assumed to be equal to the square root of some eigenvalue of the Stokes operator.

For a wavenumber κ we define the component $\mathbf{u}_\kappa$ of a vector field $\mathbf{u}$ by $\mathbf{u}_\kappa = \sum_{\lambda_j=\kappa^2} \hat{u}_j \mathbf{w}_j$, and the component $\mathbf{u}_{\kappa',\kappa''}$ with a range of wavenumbers $(\kappa', \kappa'']$ by $\mathbf{u}_{\kappa',\kappa''} = \sum_{\kappa'<\kappa\leqslant\kappa''} \mathbf{u}_\kappa$.

Ensemble averages.

The ensemble averages in our context are taken to be averages with respect to statistical solutions of the NSE. In the situation of statistical equilibrium, we consider stationary statistical solutions. In other words, the flow is assumed to have reached a state of statistical equilibrium represented by some stationary statistical solution μ on H.

Due to the regularity properties obtained for stationary statistical solutions (finite enstrophy and a support bounded in the metric of H), the mean value $\langle\varphi(\mathbf{u})\rangle$ can be defined not only for weakly continuous functions bounded in H but for any real-valued function φ which is continuous on V and satisfies the estimate

$$|\varphi(\mathbf{u})| \leqslant C(|\mathbf{u}|)(1 + \nu^{-2}\kappa_0^{-1}\|\mathbf{u}\|^2), \quad \forall\, \mathbf{u} \in V,$$

where $C(|\mathbf{u}|)$ is bounded on bounded subsets of H. Important examples of such φ which will be used in the sequel are $|\mathbf{u}|^2$, $\|\mathbf{u}\|^2$, $b(\mathbf{u}_{0,\kappa}, \mathbf{u}_{0,\kappa}, \mathbf{u}_{\kappa,\infty})$ and $b(\mathbf{u}_{\kappa,\infty}, \mathbf{u}_{\kappa,\infty}, \mathbf{u}_{0,\kappa})$. The corresponding mean quantities are well-defined and finite.

The ensemble averages need not be restricted to scalar quantities, represented by real-valued functions φ. Indeed, by a duality argument, we can define the mean velocity field $\langle \mathbf{u} \rangle$ and the mean value $\langle B(\mathbf{u},\mathbf{u}) \rangle$ of the inertial term by

$$(\langle \mathbf{u} \rangle, \mathbf{v}) = \int_H (\mathbf{u},\mathbf{v})\, \mathrm{d}\mu(\mathbf{u}), \ \ \forall\, \mathbf{v} \in V',$$
$$(\langle B(\mathbf{u},\mathbf{u}) \rangle, \mathbf{v}) = \int_H (B(\mathbf{u},\mathbf{u}),\mathbf{v})\, \mathrm{d}\mu(\mathbf{u}), \ \ \forall\, \mathbf{v} \in D(A^{3/8}).$$

The mean flow $\langle \mathbf{u} \rangle$ is a vector field on Ω with $\langle \mathbf{u} \rangle \in V$, while $\langle B(\mathbf{u},\mathbf{u}) \rangle \in D(A^{-3/8})$.

Stationary Reynolds equations.

The Reynolds equations can be recovered within this framework. Since we assume statistical equilibrium, we obtain the stationary form of the Reynolds equations.

Let ψ be a $\mathcal{C}^1$ real-valued function with compact support on $\mathbb{R}$. For any $\mathbf{v} \in V$ and any $m \in \mathbb{N}$, the function $\Phi(\mathbf{u}) = \psi((\mathbf{u}, P_m\mathbf{v}))$ is a cylindrical test function. Thus,

$$\int_H \psi'((\mathbf{u}, P_m\mathbf{v}))\{(\mathbf{f}, P_m\mathbf{v}) - \nu(A\mathbf{u}, P_m\mathbf{v}) - b(\mathbf{u},\mathbf{u},P_m\mathbf{v})\}\, \mathrm{d}\mu(\mathbf{u}) = 0.$$

Let ψ' converge pointwise to 1 while being uniformly bounded, so that at the limit we find

$$\int_H \{(\mathbf{f}, P_m\mathbf{v}) - \nu(A\mathbf{u}, P_m\mathbf{v}) - b(\mathbf{u},\mathbf{u},P_m\mathbf{v})\}\, \mathrm{d}\mu(\mathbf{u}) = 0.$$

For each fixed $\mathbf{v} \in V$ we may let m go to infinity to find (since μ has finite enstrophy and a support bounded in H)

$$\int_H \{(\mathbf{f}, \mathbf{v}) - \nu(A\mathbf{u}, \mathbf{v}) - b(\mathbf{u},\mathbf{u},\mathbf{v})\}\, \mathrm{d}\mu(\mathbf{u}) = 0.$$

Thus, we obtain the stationary Reynolds equations

$$\nu A\langle \mathbf{u} \rangle + \langle B(\mathbf{u},\mathbf{u}) \rangle = \mathbf{f},$$

which hold in V'. By decomposing the flow into the mean flow $\langle \mathbf{u} \rangle$ and the fluctuating part $\tilde{\mathbf{u}} = \mathbf{u} - \langle \mathbf{u} \rangle$, we obtain the equation

$$\nu A\langle \mathbf{u} \rangle + B(\langle \mathbf{u} \rangle, \langle \mathbf{u} \rangle) = \mathbf{f} - \langle B(\tilde{\mathbf{u}},\tilde{\mathbf{u}}) \rangle.$$

The term $\langle B(\tilde{\mathbf{u}},\tilde{\mathbf{u}}) \rangle$ is associated with the Reynolds stress tensor. A mean pressure term P can be recovered and we find the common form of the Reynolds equations,

$$-\nu\Delta\langle \mathbf{u} \rangle + (\langle \mathbf{u} \rangle \cdot \boldsymbol{\nabla})\langle \mathbf{u} \rangle + \boldsymbol{\nabla} P = \mathbf{f} - \boldsymbol{\nabla} \cdot \langle \tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}} \rangle, \quad \boldsymbol{\nabla} \cdot \langle \mathbf{u} \rangle = 0.$$

Here we have used the fact that a stationary statistical solution is carried by V and by a bounded set in H, so that it makes sense to write $\langle(\tilde{\mathbf{u}}\cdot\boldsymbol{\nabla})\tilde{\mathbf{u}}\rangle = \boldsymbol{\nabla}\cdot\langle\tilde{\mathbf{u}}\otimes\tilde{\mathbf{u}}\rangle$ for the last term in the momentum equation, where $\langle\tilde{\mathbf{u}}\otimes\tilde{\mathbf{u}}\rangle = (\langle\tilde{u}_i\tilde{u}_j\rangle)_{ij}$ is the Reynolds stress tensor.

Characteristic dimensions and nondimensional numbers.

The macroscopic characteristic length is set to be a quantity ℓ_0 which would typically be of the order of $\lambda_1^{-1/2}$, where λ_1 is the first eigenvalue of the Stokes operator (it has the physical dimension $(\text{length})^{-2}$). The corresponding large-scale wavenumber is $\kappa_0 = \ell_0^{-1}$. For simplicity, we take the unit mass to be that contained in a characteristic cube, namely ϱ_0/κ_0^3, where ϱ_0 denotes the uniform mass density of the fluid. Then the mean kinetic energy per unit mass and the mean energy dissipation rate per unit time and unit mass are given respectively by

$$e = \frac{\kappa_0^3}{2}\langle|\mathbf{u}|^2\rangle, \qquad \epsilon = \nu\kappa_0^3\langle\|\mathbf{u}\|^2\rangle.$$

The characteristic average velocity U is given by $e = U^2/2$, and we define the Reynolds number

$$\mathrm{Re} = \frac{\ell_0 U}{\nu} = \frac{\kappa_0^{1/2}\langle|\mathbf{u}|^2\rangle^{1/2}}{\nu}.$$

With our definition of ϵ, Kolmogorov's dissipation wavenumber reads $\kappa_\epsilon = (\epsilon/\nu^3)^{1/4}$. The Taylor wavenumber is defined in our context as

$$\kappa_\tau = \left(\frac{\langle\|\mathbf{u}\|^2\rangle}{\langle|\mathbf{u}|^2\rangle}\right)^{1/2} = \left(\frac{\epsilon}{2\nu e}\right)^{1/2}.$$

Note that this is not exactly the characteristic wavenumber $\kappa_T = \ell_T^{-1}$ defined originally by Taylor, but if homogeneity and isotropy were to hold they would differ only by a universal constant, namely $\kappa_\tau = \sqrt{15}\kappa_T$.

Energy flux and energy-budget equation.

Let ψ be a $\mathcal{C}^1$ real-valued function with a compact support on $\mathbb{R}$. The function $\Phi(\mathbf{u}) = \psi(|\mathbf{u}_{\kappa',\kappa''}|^2/2)$ is a cylindrical test function for any $0 \leqslant \kappa' < \kappa'' < \infty$. Thus,

$$\int_H \psi'\Big(\frac{1}{2}|\mathbf{u}_{\kappa',\kappa''}|^2\Big)\{(\mathbf{f},\mathbf{u}_{\kappa',\kappa''}) - \nu\|\mathbf{u}_{\kappa',\kappa''}\|^2 - b(\mathbf{u},\mathbf{u},\mathbf{u}_{\kappa',\kappa''})\}\,\mathrm{d}\mu(\mathbf{u}) = 0.$$

Let ψ' converge pointwise to 1 while being uniformly bounded, so that at the limit we find

$$\int_H \{(\mathbf{f},\mathbf{u}_{\kappa',\kappa''}) - \nu\|\mathbf{u}_{\kappa',\kappa''}\|^2 - b(\mathbf{u},\mathbf{u},\mathbf{u}_{\kappa',\kappa''})\}\,\mathrm{d}\mu(\mathbf{u}) = 0.$$

Thus, using the orthogonality property of the nonlinear term, we find the energy-budget relation

$$\nu\kappa_0^3\langle\|\mathbf{u}_{\kappa',\kappa''}\|^2\rangle = \kappa_0^3\langle(\mathbf{f}_{\kappa',\kappa''},\mathbf{u}_{\kappa',\kappa''})\rangle + \langle\mathfrak{e}_{\kappa'}(\mathbf{u})\rangle - \langle\mathfrak{e}_{\kappa''}(\mathbf{u})\rangle$$

where

$$\mathfrak{e}_\kappa(\mathbf{u}) = -\kappa_0^3 b(\mathbf{u}_{0,\kappa},\mathbf{u}_{0,\kappa},\mathbf{u}_{\kappa,\infty}) + \kappa_0^3 b(\mathbf{u}_{\kappa,\infty},\mathbf{u}_{\kappa,\infty},\mathbf{u}_{0,\kappa}).$$

The term $\mathfrak{e}_\kappa(\mathbf{u})$ represents the net flux per unit time of kinetic energy per unit mass transferred into the higher modes $\mathbf{u}_{\kappa,\infty}$ by the inertial effects.

In the particular case $\kappa' = 0$ and $\kappa'' = \kappa$, the energy-budget relation reduces to

$$\nu\kappa_0^3\langle\|\mathbf{u}_{0,\kappa}\|^2\rangle = \kappa_0^3\langle(\mathbf{f}_{0,\kappa},\mathbf{u}_{0,\kappa})\rangle - \langle\mathfrak{e}_\kappa(\mathbf{u})\rangle.$$

The energy inequality for the stationary statistical solutions can be written as

$$\nu\kappa_0^3\langle\|\mathbf{u}\|^2\rangle \leqslant \kappa_0^3\langle(\mathbf{f},\mathbf{u})\rangle.$$

Thus, by subtracting from it the previous relation, we find

$$\nu\kappa_0^3\langle\|\mathbf{u}_{\kappa,\infty}\|^2\rangle \leqslant \kappa_0^3\langle(\mathbf{f}_{\kappa,\infty},\mathbf{u}_{\kappa,\infty})\rangle + \langle\mathfrak{e}_\kappa(\mathbf{u})\rangle.$$

The inequality can be interpreted as coming from a possible loss of regularity, "leaking" kinetic energy to infinity. An equality can be recovered if we take proper account of this loss. Note that the following limits hold:

$$\lim_{\kappa\to\infty}\langle\|\mathbf{u}_{0,\kappa}\|^2\rangle = \langle\|\mathbf{u}\|^2\rangle, \qquad \lim_{\kappa\to\infty}\langle(\mathbf{f}_{0,\kappa},\mathbf{u}_{0,\kappa})\rangle = \langle(\mathbf{f},\mathbf{u})\rangle.$$

Therefore, we may define the limit

$$\begin{aligned}\langle\mathfrak{e}_\infty\rangle &\stackrel{\text{def}}{=} \lim_{\kappa\to\infty}\langle\mathfrak{e}_\kappa(\mathbf{u})\rangle = \lim_{\kappa\to\infty}\{\kappa_0^3\langle(\mathbf{f}_{0,\kappa},\mathbf{u}_{0,\kappa})\rangle - \nu\kappa_0^3\langle\|\mathbf{u}_{0,\kappa}\|^2\rangle\}\\ &= \kappa_0^3\langle(\mathbf{f},\mathbf{u})\rangle - \nu\kappa_0^3\langle\|\mathbf{u}\|^2\rangle \geqslant 0.\end{aligned}$$

For regular stationary statistical solutions (regularity in the sense of being supported by a bounded set in V, such as time-average measures obtained from globally regular solutions) one has $\langle\mathfrak{e}(\mathbf{u})\rangle_\infty = 0$. For general stationary statistical solutions which are not known to be regular, it is appropriate to consider the quantity

$$\mathfrak{e}_\kappa^*(\mathbf{u}) = \mathfrak{e}_\kappa(\mathbf{u}) - \langle\mathfrak{e}(\mathbf{u})\rangle_\infty,$$

which accounts for the possible loss of kinetic energy, in average, due to the lack of regularity of the solution. We term $\mathfrak{e}_\kappa^*(\mathbf{u})$ the *restricted energy flux* through the wave number κ. Then, considering $\kappa' = \kappa \geqslant 0$ and taking the limit $\kappa'' \to \infty$, we obtain the following identity for all $\kappa \geqslant 0$:

$$\nu\kappa_0^3\langle\|\mathbf{u}_{\kappa,\infty}\|^2\rangle = \kappa_0^3\langle(\mathbf{f}, \mathbf{u}_{\kappa,\infty})\rangle + \langle\mathfrak{e}_\kappa^*(\mathbf{u})\rangle.$$

Energy cascade.

Since

$$\nu\kappa_0^3\langle\|\mathbf{u}_{\kappa,\infty}\|^2\rangle \searrow \nu\kappa_0^3\langle\|\mathbf{u}\|^2\rangle = \epsilon \quad \text{as } \kappa \searrow 0, \quad \lim_{\kappa\to\infty} \kappa_0^3\langle(\mathbf{f}, \mathbf{u}_{\kappa,\infty})\rangle = 0,$$

we may define wavenumbers $\underline{\kappa}_\epsilon$ and $\overline{\kappa}_\epsilon$ respectively as the smallest and largest ones such that

$$|\kappa_0^3\langle(\mathbf{f}, \mathbf{u}_{\kappa,\infty})\rangle| \ll \epsilon \quad \text{for all } \kappa \geqslant \underline{\kappa}_\epsilon \quad \text{and} \quad \nu\kappa_0^3\langle\|\mathbf{u}_{\overline{\kappa}_\epsilon,\infty}\|^2\rangle \approx \epsilon.$$

We quantify these conditions more precisely with help of a small nondimensional parameter δ representing the order of accuracy in the relations. Then $\overline{\kappa}_\epsilon$ is the largest wavenumber for which

$$\nu\kappa_0^3\langle\|\mathbf{u}_{\overline{\kappa}_\epsilon,\infty}\|^2\rangle \geqslant (1-\delta)\epsilon,$$

while $\underline{\kappa}_\epsilon$ is the smallest wavenumber such that

$$|\kappa_0^3\langle(\mathbf{f}, \mathbf{u}_{\kappa,\infty})\rangle| \leqslant \delta\epsilon \quad \text{for all } \kappa \geqslant \underline{\kappa}_\epsilon.$$

The wavenumber $\overline{\kappa}_\epsilon$ is interpreted as a lower bound for the energy-dissipative scales, and $\underline{\kappa}_\epsilon$ as an upper bound on the energy-productive scales.

Kolmogorov's theory is based on the assumption that there is a significant separation between the (large) energy-productive scales and the (small) energy-dissipative scales. This is expressed by the assumption that $\overline{\kappa}_\epsilon \gg \underline{\kappa}_\epsilon$. Under this assumption, we may show that there exists an energy cascade within an extensive interval given by $[\underline{\kappa}_\epsilon, \overline{\kappa}_\epsilon]$, with $\underline{\kappa}_\epsilon \ll \overline{\kappa}_\epsilon$. Indeed, consider a wavenumber κ in this interval. From the relation

$$\nu\kappa_0^3\langle\|\mathbf{u}_{\kappa,\infty}\|^2\rangle = \kappa_0^3\langle(\mathbf{f}, \mathbf{u}_{\kappa,\infty})\rangle + \langle\mathfrak{e}_\kappa^*(\mathbf{u})\rangle$$

we obtain

$$(1-2\delta)\epsilon \leqslant \langle\mathfrak{e}_\kappa^*(\mathbf{u})\rangle = \nu\kappa_0^3\langle\|\mathbf{u}_{\kappa,\infty}\|^2\rangle - \kappa_0^3\langle(\mathbf{f}, \mathbf{u}_{\kappa,\infty})\rangle \leqslant (1+\delta)\epsilon.$$

Thus,

$$-\delta \leqslant 1 - \frac{\langle\mathfrak{e}_\kappa^*(\mathbf{u})\rangle}{\epsilon} \leqslant 2\delta.$$

In other words, the following energy-cascade relation holds within the range $[\underline{\kappa}_\epsilon, \overline{\kappa}_\epsilon]$:

$$\langle \mathfrak{e}_\kappa^*(\mathbf{u}) \rangle \approx \epsilon.$$

Note that this relation is still valid with the mere assumption that $\underline{\kappa}_\epsilon \leqslant \overline{\kappa}_\epsilon$, but this does not guarantee an extensive energy-cascade range.

Conditions related to the energy cascade.

Sufficient conditions pertaining to the energy cascade can be derived in terms of the Taylor wavenumber $\kappa_\mathcal{T}$. Indeed, note that for any wavenumber $\kappa > 0$,

$$\nu\kappa_0^3\langle \|\mathbf{u}_{0,\kappa}\|^2\rangle \leqslant \nu\kappa_0^3\kappa^2\langle |\mathbf{u}_{0,\kappa}|^2\rangle \leqslant \nu\kappa_0^3\kappa^2\langle |\mathbf{u}|^2\rangle \leqslant \left(\frac{\kappa}{\kappa_\mathcal{T}}\right)^2 \epsilon.$$

Thus, for $\kappa^2 \ll \kappa_\mathcal{T}^2$ we have $\nu\kappa_0^3\langle \|\mathbf{u}_{0,\kappa}\|^2\rangle \ll \epsilon$. Hence, $\overline{\kappa}_\epsilon \geqslant \delta^{1/2}\kappa_\mathcal{T}$. For the scale separation, if $\kappa_\mathcal{T}^2 \gg \underline{\kappa}_\epsilon^2$, then $\overline{\kappa}_\epsilon \geqslant \underline{\kappa}_\epsilon$, with a possibly short energy-cascade range. If $\kappa_\mathcal{T} \gg \underline{\kappa}_\epsilon$, then $\delta \geqslant \underline{\kappa}_\epsilon/\kappa_\mathcal{T}$, so that $\overline{\kappa}_\epsilon \geqslant \underline{\kappa}_\epsilon^{1/2}\kappa_\mathcal{T}^{1/2}$, and a wider energy-cascade range is ensured, with $\overline{\kappa}_\epsilon^2 \gg \underline{\kappa}_\epsilon^2$. If $\kappa_\mathcal{T}^{2/3} \gg \underline{\kappa}_\epsilon^{2/3}$, then $\delta \geqslant \underline{\kappa}_\epsilon^{2/3}/\kappa_\mathcal{T}^{2/3}$, so that $\overline{\kappa}_\epsilon \geqslant \underline{\kappa}_\epsilon^{1/3}\kappa_\mathcal{T}^{2/3}$, and an extensive energy-cascade range exists, with $\overline{\kappa}_\epsilon \gg \underline{\kappa}_\epsilon$.

Notice that $\kappa_\mathcal{T}$ is related to a separation between the energy-containing and the energy-dissipative scales. Indeed, if a considerable amount of energy is concentrated within wavenumbers $\underline{\kappa}_e$ and $\overline{\kappa}_e$ below those associated with energy dissipation, in the sense that $\kappa_0^3\langle |\mathbf{u}_{\underline{\kappa}_e,\overline{\kappa}_e}|^2\rangle/2 \sim e$, with $\underline{\kappa}_e \leqslant \overline{\kappa}_e \leqslant \underline{\kappa}_\epsilon$, then

$$\kappa_\mathcal{T}^2 = \frac{\epsilon}{2\nu e} \sim \frac{\epsilon}{\nu\kappa_0^3\langle |\mathbf{u}_{\underline{\kappa}_e,\overline{\kappa}_e}|^2\rangle} \gg \frac{\nu\kappa_0^3\langle \|\mathbf{u}_{\underline{\kappa}_e,\overline{\kappa}_e}\|^2\rangle}{\nu\kappa_0^3\langle |\mathbf{u}_{\underline{\kappa}_e,\overline{\kappa}_e}|^2\rangle} \geqslant \underline{\kappa}_e^2.$$

However, necessary conditions on $\kappa_\mathcal{T}$ for the energy cascade do not seem to be readily available since the energy cascade was obtained under the condition of a separation between the energy-dissipative scales and the energy-productive scales instead of the energy-containing scales. It is natural though to expect that the sufficient conditions given above for $\kappa_\mathcal{T}$ are reflected somehow in the necessary conditions.

The statement that the energy-cascade range begins around $\underline{\kappa}_\epsilon$, which may be too close to wavenumbers associated with the anisotropic large-scale eddies, is not in contradiction with the basis for the Kolmogorov universality theory. In our case, $\underline{\kappa}_\epsilon$ represents the beginning of the spectral region in which the transfer of energy occurs at a nearly constant flux close to ϵ. It does not mean that this transfer occurs necessarily in a universal, homogeneous and isotropic fashion. The small-scale eddies may lose the anisotropic information carried by the energy flux from the larger scales only at higher wavenumbers, above $\overline{\kappa}_\epsilon$, in which we may then see a more distinct Kolmogorov power spectrum.

In Kolmogorov's theory the energy cascade is associated with a power spectrum $\mathcal{S}(\kappa) \sim \epsilon^{2/3}\kappa^{-5/3}$ through an *inertial range* $[\underline{\kappa}_i, \overline{\kappa}_i]$. In this case, one can derive the relation

$$\begin{aligned}\kappa_\epsilon^{4/3} &= \left(\frac{\epsilon}{\nu^3}\right)^{1/3} = \frac{\epsilon}{\nu\epsilon^{2/3}} = \frac{\kappa_0^3}{\epsilon^{2/3}}\langle\|\mathbf{u}\|^2\rangle \sim \frac{1}{\epsilon^{2/3}}\int \kappa^2\mathcal{S}(\kappa)\,\mathrm{d}\kappa \\ &\geqslant \frac{1}{\epsilon^{2/3}}\int_{\underline{\kappa}_i}^{\overline{\kappa}_i} \mathcal{S}(\kappa)\,\mathrm{d}\kappa \sim \int_{\underline{\kappa}_i}^{\overline{\kappa}_i} \kappa^{1/3}\,\mathrm{d}\kappa.\end{aligned}$$

Thus,

$$\kappa_\epsilon^{4/3} \gtrsim \overline{\kappa}_i^{4/3} - \underline{\kappa}_i^{4/3},$$

which strictly speaking means that there exists a constant c of order unity such that $\kappa_\epsilon^{4/3} \geqslant c(\overline{\kappa}_i^{4/3} - \underline{\kappa}_i^{4/3})$. How close (or how far) κ_ϵ is from $\overline{\kappa}_i$ depends on how much energy dissipation occurs inside the inertial range. It is often argued that the inertial range extends to a wider interval in such a way that indeed a large amount of energy dissipation occurs within it, thus $\epsilon \sim \nu\int_{\underline{\kappa}_i}^{\overline{\kappa}_i} \kappa^2\mathcal{S}(\kappa)\,\mathrm{d}\kappa \sim \nu\epsilon^{2/3}(\overline{\kappa}_i^{4/3} - \underline{\kappa}_i^{4/3})$. Hence, $\kappa_\epsilon^{4/3} \sim \overline{\kappa}_i^{4/3} - \underline{\kappa}_i^{4/3}$. Similarly, it is argued that the inertial range also contains a significant amount of energy, so that $e \sim \int_{\underline{\kappa}_i}^{\overline{\kappa}_i} \mathcal{S}(\kappa)\,\mathrm{d}\kappa \sim \epsilon^{2/3}(\underline{\kappa}_i^{-2/3} - \overline{\kappa}_i^{-2/3})$. In this case, it is found that $\kappa_\tau^2 = \epsilon(2\nu e)^{-1} \sim \underline{\kappa}_i^{2/3}\overline{\kappa}_i^{2/3}(\underline{\kappa}_i^{2/3} + \overline{\kappa}_i^{2/3})$, which requires a relatively large Taylor wavenumber for an extensive inertial range. If, however, the inertial range is such that little energy dissipation occurs within it, then $\kappa_\epsilon^{4/3} \gg \overline{\kappa}_i^{4/3} - \underline{\kappa}_i^{4/3}$, and no lower bound for the Taylor wavenumber can be deduced from the above.

The energy cascade was proved in [16] for stationary statistical solutions obtained as generalized limits of time averages of weak solutions and under the assumptions that $\mathbf{f}$ be made of a finite number of modes and that $\kappa_\tau \gg \overline{\kappa}_f$, where $\overline{\kappa}_f$ denotes the highest mode in $\mathbf{f}$. One can verify that under these assumptions, we have $\underline{\kappa}_\epsilon \leqslant \overline{\kappa}_f$ and $\overline{\kappa}_f^2 \ll \overline{\kappa}_\epsilon^2$. Indeed, the first relation is trivial. As for the latter one, we note that for $\kappa > \overline{\kappa}_f$,

$$\begin{aligned}\epsilon &\geqslant \langle\mathfrak{e}_\kappa^*(\mathbf{u})\rangle = \nu\kappa_0^3\langle\|\mathbf{u}_{\kappa,\infty}\|^2\rangle = \epsilon - \nu\kappa_0^3\langle\|\mathbf{u}_{0,\kappa}\|^2\rangle \\ &\geqslant \epsilon - \nu\kappa_0^3\kappa^2\langle|\mathbf{u}_{0,\kappa}|^2\rangle \geqslant \epsilon - \nu\kappa_0^3\kappa^2\langle|\mathbf{u}|^2\rangle = \left(1 - \left(\frac{\kappa}{\kappa_\tau}\right)^2\right)\epsilon.\end{aligned}$$

This is relation (11) in [16]. It yields $\langle\mathfrak{e}_\kappa^*(\mathbf{u})\rangle \approx \epsilon$ under the condition that $\kappa_\tau^2 \gg \kappa_f^2$. But this condition does not guarantee a very extensive energy-cascade range. It is similar to the assumption $\overline{\kappa}_\epsilon \geqslant \underline{\kappa}_\epsilon$ in our case. In fact, $\kappa_\tau^2 \gg \kappa_f^2$ implies only $\overline{\kappa}_\epsilon \geqslant \underline{\kappa}_\epsilon$ for some $\overline{\kappa}_\epsilon \geqslant \overline{\kappa}_f \geqslant \underline{\kappa}_\epsilon$.

For a wider energy-cascade range, it is assumed in [16] that $\kappa_\tau \gg \overline{\kappa}_f$. And this implies by the above that $\overline{\kappa}_\epsilon \geqslant \overline{\kappa}_f^{1/2}\kappa_\tau^{1/2}$. This does not imply that the energy-cascade range is so large (with respect to the order of magnitude implicit in δ) as to yield $\overline{\kappa}_\epsilon \gg \underline{\kappa}_\epsilon$, but only $\overline{\kappa}_\epsilon^2 \gg \underline{\kappa}_\epsilon^2$. For a more extensive energy-cascade range we may require the assumption $\kappa_\tau^{2/3} \gg \overline{\kappa}_f^{2/3}$. In this case, $\overline{\kappa}_\epsilon \geqslant \overline{\kappa}_f^{1/3}\kappa_\tau^{2/3} \gg \overline{\kappa}_f \geqslant \underline{\kappa}_\epsilon$.

Some estimates for the charateristic dimensions and the nondimensional numbers.

We mention here some estimates derived in [16] for smooth domains and time-average measures. Those estimates are based on the Reynolds equation, and they were obtained under a further assumption that $\mathbf{f}$ belongs to V. In this case, we define the following characteristic wavenumber associated with the forcing term: $\kappa_f = (|A^{1/2}\mathbf{f}|/|A^{-1/2}\mathbf{f}|)^{1/2}$. Since we have seen that the Reynolds equation also holds in our context for a nonsmooth domain and for an arbitrary stationary statistical solution, the same proofs as in [16] apply here provided $\mathbf{f} \in V$. Then, for κ_f of the order of the large-scale wavenumber κ_0 and for sufficiently large Reynolds numbers, we find that

$$\epsilon \leqslant c\kappa_0 U^3, \quad \kappa_\epsilon \leqslant c\kappa_0 \mathrm{Re}^{3/4}, \quad \kappa_\tau \leqslant c\kappa_0^{1/3}\kappa_\epsilon^{2/3}, \quad \kappa_\tau \leqslant c\kappa_0 \mathrm{Re}^{1/2}.$$

We recall that Kolmogorov's heuristic approach yields $\epsilon \sim \kappa_0 U^3$, $\kappa_\epsilon \sim \kappa_0\,\mathrm{Re}^{3/4}$, $\kappa_\tau \sim \kappa_0^{1/3}\kappa_\epsilon^{2/3}$, and $\kappa_\tau \sim \kappa_0\,\mathrm{Re}^{1/2}$. Hence, the above estimates can be viewed as a rigorous partial confirmation of those results. An inequality at this level is expected since simple laminar flows and weakly turbulent flows are not ruled out in our context. Kolmogorov's estimates were derived assuming fully-developed turbulent flows.

Enstrophy cascade in the two-dimensional case.

We can proceed in a similar way in the two-dimensional case. However, the nature of turbulence in two dimensions is different from that in the three-dimensional case. In Kraichnan's theory this is seen to force an inverse energy cascade (to lower wavenumbers). On the other hand, the conservation of enstrophy is argued to force a direct cascade of enstrophy towards smaller scales.

Kraichnan's theory was obtained for periodic flows, for an idealized model of turbulence. Similarly, we may proceed assuming periodic conditions for the 2D NSE, in which case the conservation of enstrophy of the inertial term holds. The rate of enstrophy dissipation by viscous effects per unit time and unit mass is defined in this context as

$$\eta = \nu\kappa_0^2\langle|A\mathbf{u}|^2\rangle.$$

In dimension two an important role is played by the Taylor-like wavenumber

$$\kappa_\sigma = \left(\frac{\langle|A\mathbf{u}|^2\rangle}{\langle\|\mathbf{u}\|^2\rangle}\right)^{1/2} = \left(\frac{\eta}{\epsilon}\right)^{1/2}.$$

In the two-dimensional case there is enough regularity for handling the nonlinear term, and we need not consider restricted energy or enstrophy fluxes. We obtain directly the enstrophy-budget relation

$$\nu\kappa_0^2\langle|A\mathbf{u}_{\kappa',\kappa''}|^2\rangle = \kappa_0^2\langle(\mathbf{f}_{\kappa',\kappa''}, A\mathbf{u}_{\kappa',\kappa''})\rangle + \langle\mathfrak{E}_{\kappa'}(\mathbf{u})\rangle - \langle\mathfrak{E}_{\kappa''}(\mathbf{u})\rangle$$

for all $0 \leqslant \kappa' < \kappa'' \leqslant \infty$ and for any stationary statistical solution. We recall that in the two-dimensional case, any stationary statistical solution is in fact an invariant measure for the semigroup generated by the 2D NSE, and vice-versa. In the above,

$$\mathfrak{E}_\kappa(\mathbf{u}) = -\kappa_0^2 b(\mathbf{u}_{0,\kappa}, \mathbf{u}_{0,\kappa}, A\mathbf{u}_{\kappa,\infty}) + \kappa_0^2 b(\mathbf{u}_{\kappa,\infty}, \mathbf{u}_{\kappa,\infty}, A\mathbf{u}_{0,\kappa}).$$

The term $\mathfrak{E}_\kappa(\mathbf{u})$ represents the net flux per unit time of enstrophy per unit mass transferred into the higher modes $\mathbf{u}_{\kappa,\infty}$ by the inertial effects.

Kraichnan's theory relies on the assumption that there exists a wide separation between the enstrophy-productive scales and the enstrophy-dissipative scales. In this regard let us define the wavenumbers $\overline{\kappa}_\eta$ and $\underline{\kappa}_\eta$, respectively, as the largest and the smallest wavenumbers such that

$$\nu\kappa_0^2\langle|A\mathbf{u}_{\overline{\kappa}_\eta,\infty}|^2\rangle \approx \eta, \quad |\kappa_0^2\langle(\mathbf{f}, A\mathbf{u}_{\kappa,\infty})\rangle| \ll \eta \quad \text{for all } \kappa \geqslant \underline{\kappa}_\eta.$$

We quantify these conditions again with a small nondimensional parameter δ with

$$\nu\kappa_0^2\langle|A\mathbf{u}_{\overline{\kappa}_\eta,\infty}|^2\rangle \geqslant (1-\delta)\eta, \quad |\kappa_0^2\langle(\mathbf{f}, A\mathbf{u}_{\kappa,\infty})\rangle| \leqslant \delta\eta \quad \text{for all } \kappa \geqslant \underline{\kappa}_\eta.$$

The wavenumber $\overline{\kappa}_\eta$ is interpreted as a lower bound for the enstrophy-dissipative scales, and $\underline{\kappa}_\eta$ as an upper bound on the enstrophy-productive scales.

Then, for $\underline{\kappa}_\eta \leqslant \kappa \leqslant \overline{\kappa}_\eta$, we find from the relation

$$\nu\kappa_0^2\langle|A\mathbf{u}_{\kappa,\infty}|^2\rangle = \kappa_0^2\langle(\mathbf{f}, A\mathbf{u}_{\kappa,\infty})\rangle + \langle\mathfrak{E}_\kappa(\mathbf{u})\rangle$$

that

$$(1-\delta)\eta \leqslant \delta\eta + \kappa_0^2\langle\mathfrak{E}_\kappa(\mathbf{u})\rangle \leqslant (1+\delta)\eta,$$

and we obtain the enstrophy-cascade relation

$$\kappa_0^2\langle\mathfrak{E}_\kappa(\mathbf{u})\rangle \approx \eta.$$

Conditions related to the enstrophy cascade.

Conditions for the enstrophy cascade can be given in terms of κ_σ, as is similarly done in dimension three for κ_τ. For any $\kappa > 0$ we have

$$\nu\kappa_0^2\langle|A\mathbf{u}_{0,\kappa}|^2\rangle \leqslant \nu\kappa_0^2\kappa^2\langle\|\mathbf{u}_{0,\kappa}\|^2\rangle \leqslant \nu\kappa_0^2\kappa^2\langle\|\mathbf{u}\|^2\rangle \leqslant \left(\frac{\kappa}{\kappa_\sigma}\right)^2\eta.$$

Thus, for $\kappa^2 \ll \kappa_\sigma^2$, we have $\nu\kappa_0^2\langle|A\mathbf{u}_{0,\kappa}|^2\rangle \ll \eta$. Hence, $\overline{\kappa}_\eta \geqslant \delta^{1/2}\kappa_\sigma$. For the scale separation, if $\kappa_\sigma^2 \gg \underline{\kappa}_\eta^2$, then $\underline{\kappa}_\eta \leqslant \overline{\kappa}_\eta$, with a possibly short enstrophy-cascade range. If $\kappa_\sigma \gg \underline{\kappa}_\eta$, then a wider enstrophy-cascade range is assured, with $\overline{\kappa}_\eta^2 \gg \underline{\kappa}_\eta^2$. If $\kappa_\sigma^{2/3} \gg \underline{\kappa}_\eta^{2/3}$, then an extensive enstrophy-cascade range exists, with $\overline{\kappa}_\eta \gg \underline{\kappa}_\eta$.

The wavenumber κ_σ is related to a separation between the enstrophy-containing and the enstrophy-dissipative scales. As is similarly done for dimension three, if a considerable amount of enstrophy is concentrated within wavenumbers $\underline{\kappa}_E$ and $\overline{\kappa}_E$ below those associated with energy dissipation, in the sense that $\kappa_0^2\langle\|\mathbf{u}_{\underline{\kappa}_E,\overline{\kappa}_E}\|^2\rangle/2 \sim E = \nu\kappa_0^2\langle\|\mathbf{u}\|^2\rangle$ with $\underline{\kappa}_E \leqslant \overline{\kappa}_E \leqslant \underline{\kappa}_\eta$, then

$$\kappa_\sigma^2 = \frac{\eta}{\epsilon} \sim \frac{\eta}{\nu\kappa_0^2\langle\|\mathbf{u}_{\underline{\kappa}_E,\overline{\kappa}_E}\|^2\rangle} \gg \frac{\nu\kappa_0^2\langle|A\mathbf{u}_{\underline{\kappa}_E,\overline{\kappa}_E}\|^2\rangle}{\nu\kappa_0^2\langle\|\mathbf{u}_{\underline{\kappa}_E,\overline{\kappa}_E}\|^2\rangle} \geqslant \underline{\kappa}_E^2.$$

In Kraichnan's theory the enstrophy cascade is associated with a power spectrum $\mathcal{S}(\kappa) \sim \eta^{2/3}\kappa^{-3}$ within an *inertial range* $[\underline{\kappa}_i, \overline{\kappa}_i]$. Similarly to the three-dimensional case, one can derive the relation

$$\begin{aligned}\kappa_\eta^2 = \left(\frac{\eta}{\nu^3}\right)^{1/3} = \frac{\eta}{\nu\eta^{2/3}} = \frac{\kappa_0^2}{\eta^{2/3}}\langle|A\mathbf{u}|^2\rangle \sim \frac{1}{\eta^{2/3}}\int \kappa^4\mathcal{S}(\kappa)\,\mathrm{d}\kappa \\ \geqslant \frac{1}{\eta^{2/3}}\int_{\underline{\kappa}_i}^{\overline{\kappa}_i}\mathcal{S}(\kappa)\,\mathrm{d}\kappa \sim \int_{\underline{\kappa}_i}^{\overline{\kappa}_i}\kappa\,\mathrm{d}\kappa.\end{aligned}$$

Thus,

$$\kappa_\eta^2 \gtrsim \overline{\kappa}_i^2 - \underline{\kappa}_i^2,$$

which strictly speaking means that there exists a constant c of order unity such that $\kappa_\eta^2 \geqslant c(\overline{\kappa}_i^2 - \underline{\kappa}_i^2)$. Again, how close (or how far) κ_η is from $\overline{\kappa}_i$ depends on how much enstrophy dissipation occurs inside the inertial range. If a significant amount of enstrophy dissipation occurs within the inertial range, then $\eta \sim \nu\int_{\underline{\kappa}_i}^{\overline{\kappa}_i}\kappa^4\mathcal{S}(\kappa)\,\mathrm{d}\kappa \sim \nu\eta^{2/3}(\overline{\kappa}_i^2 - \underline{\kappa}_i^2)$, hence, $\kappa_\eta^2 \sim \overline{\kappa}_i^2 - \underline{\kappa}_i^2$. Similarly, if the inertial range also contains a significant amount of enstrophy, then $\kappa_0^2\langle\|\mathbf{u}\|^2\rangle \sim \int_{\underline{\kappa}_i}^{\overline{\kappa}_i}\kappa^2\mathcal{S}(\kappa)\,\mathrm{d}\kappa \sim \eta^{2/3}\ln(\overline{\kappa}_i/\underline{\kappa}_i)$, and in this case, it is found that $\kappa_\sigma^2 = \eta/\epsilon \sim \overline{\kappa}_i^2\underline{\kappa}_i^2/\ln(\overline{\kappa}_i/\underline{\kappa}_i)$, which requires a relatively large wavenumber κ_σ for an extensive inertial range. If, however, the inertial range is such that little enstrophy dissipation occurs within it, then $\kappa_\eta^2 \gg \overline{\kappa}_i^2 - \underline{\kappa}_i^2$, and no lower bound for the wavenumber κ_σ can be deduced from the above.

Consider now the positive and negative energy injection rates

$$r_+ = \kappa_0^2\sum_{\kappa>0}\langle(\mathbf{f}_\kappa, \mathbf{u}_\kappa)\rangle^+, \qquad r_- = \kappa_0^2\sum_{\kappa>0}\langle(\mathbf{f}_\kappa, \mathbf{u}_\kappa)\rangle^-,$$

where the superscripts '+' and '−' indicate the operations $a^+ = \max\{0, a\}$ and $a^- = \max\{0, -a\}$ for a real number a. With this notation we have the relation

$$\kappa_0^2\langle(\mathbf{f}, \mathbf{u})\rangle + r_- = r_+.$$

Clearly, $r_+ - r_- = \kappa_0^2\langle(\mathbf{f}, \mathbf{u})\rangle = \epsilon > 0$, otherwise $\mathbf{u} = 0$ almost everywhere with respect to μ and, hence, by the Reynolds equations, $\mathbf{f} = 0$. This situation is discarded since it is not interesting for turbulence in statistical equilibrium. Thus, $0 \leqslant r_- < r_+$. Define the wavenumbers

$$\kappa_+^2 = \frac{\kappa_0^2 \sum_{\kappa>0} \kappa^2\langle(\mathbf{f}_\kappa, \mathbf{u}_\kappa)\rangle^+}{r_+}, \qquad \kappa_-^2 = \frac{\kappa_0^2 \sum_{\kappa>0} \kappa^2\langle(\mathbf{f}_\kappa, \mathbf{u}_\kappa)\rangle^-}{r_-},$$

with $\kappa_- = 0$ if $r_- = 0$. Then

$$\kappa_\sigma^2(r_+ - r_-) = \kappa_\sigma^2\nu\kappa_0^2\langle\|\mathbf{u}\|^2\rangle = \nu\kappa_0^2\langle|A\mathbf{u}|^2\rangle = \kappa_0^2\langle(\mathbf{f}, A\mathbf{u})\rangle = r_+\kappa_+^2 - r_-\kappa_-^2.$$

Solving for κ_σ^2 yields

$$\kappa_\sigma^2 = \frac{r_+\kappa_+^2 - r_-\kappa_-^2}{r_+ - r_-}.$$

Note that if $r_- = 0$, then $\langle(\mathbf{f}_\kappa, \mathbf{u}_\kappa)\rangle \geqslant 0$ for all κ, $r_+ = \epsilon$, and

$$\begin{aligned}
\kappa_\sigma^2 = \kappa_+^2 &= \frac{1}{\epsilon}\left\{\kappa_0^2 \sum_{0<\kappa\leqslant\underline{\kappa}_\eta} \kappa^2\langle(\mathbf{f}_\kappa, \mathbf{u}_\kappa)\rangle + \kappa_0^2\langle(\mathbf{f}, A\mathbf{u}_{\underline{\kappa}_\eta,\infty})\rangle\right\} \\
&\leqslant \frac{1}{\epsilon}\left\{\kappa_0^2\underline{\kappa}_\eta^2 \sum_{0<\kappa\leqslant\underline{\kappa}_\eta} \langle(\mathbf{f}_\kappa, \mathbf{u}_\kappa)\rangle + \delta\eta\right\} \\
&\leqslant \frac{1}{\epsilon}\{\kappa_0^2\underline{\kappa}_\eta^2\langle(\mathbf{f}, \mathbf{u})\rangle + \delta\eta\} \leqslant \underline{\kappa}_\eta^2 + \delta\kappa_\sigma^2.
\end{aligned}$$

Thus, if $r_- = 0$, there is no separation between the enstrophy-containing and the enstrophy-dissipative scales since

$$\kappa_\sigma^2 \leqslant \frac{1}{1-\delta}\,\underline{\kappa}_\eta^2 \lesssim \underline{\kappa}_\eta^2.$$

If $\mathbf{f}$ has only two wavenumber components, say κ_{hi} and κ_{low}, with $\kappa_{\text{hi}} > \kappa_{\text{low}}$, then we must have $\kappa_+ = \kappa_{\text{hi}}$ and $\kappa_- = \kappa_{\text{low}}$ for $\kappa_\sigma^2 \gg \underline{\kappa}_\eta^2 \geqslant \kappa_{\text{hi}}$ to hold. Hence, the higher wavenumber mode must act as a source of energy, while the lower wavenumber mode must act as a sink and, of course, the source must be stronger than the sink in the sense that $r_+ > r_-$.

Another consequence is that we must have at least two modes in $\mathbf{f}$, otherwise we would have necessarily $r_- = 0$, which contradicts $\kappa_\sigma \gg \underline{\kappa}_\eta$ (see also [10], [13]).

It is not clear at this point what kinds of forcing terms would yield a Taylor-like wavenumber such that $\kappa_\sigma \gg \underline{\kappa}_\eta$. This is a particularly important issue for the numerical simulation of turbulence (similar considerations apply to κ_τ in three dimensions). It is related to the fact that most simulations do not yield the correct spectral power laws under forcing on a single mode without resorting to inverse viscosity and other artificial mechanisms (see further discussions in [51]).

The enstrophy cascade was proved in [13] under the assumptions that $\mathbf{f}$ is made of a finite number of modes and that $\kappa_\sigma \gg \overline{\kappa}_f$, where $\overline{\kappa}_f$ is the highest mode in $\mathbf{f}$. As is similarly done in dimension three, this implies that $\kappa_\eta \geqslant \overline{\kappa}_f^{1/2}\kappa_\sigma^{1/2}$ with $\overline{\kappa}_\eta^2 \gg \overline{\kappa}_f^2 \geqslant \underline{\kappa}_\eta^2$. For a more extensive enstrophy-cascade range with $\overline{\kappa}_\eta \gg \underline{\kappa}_\eta$, one may require that $\kappa_\sigma^{2/3} \gg \overline{\kappa}_f^{2/3}$.

Another result proved in [13] under those assumptions pertains to the relation between the energy and enstrophy fluxes for $\kappa > \overline{\kappa}_f$. For such κ, it turns out that the energy flux is also upwards in wavenumber, i.e. $\langle \mathfrak{e}_\kappa(\mathbf{u})\rangle = \nu\kappa_0^2\langle\|\mathbf{u}_{\kappa,\infty}\|^2\rangle > 0$. This does not mean necessarily that the flux is uniform in the sense of an energy cascade. If κ_τ is relatively large, then a direct energy cascade does occur as in the three-dimensional case, but the nature of two-dimensional turbulence may be such that κ_τ is typically relatively small. We show next that indeed a necessary condition for the commonly accepted inverse energy cascade below the forced wavenumbers is that κ_τ be relatively small. But if $\kappa_\sigma^2 \gg \overline{\kappa}_f^2$, despite the value of κ_τ and even if an energy cascade also occurs, the enstrophy cascade is much stronger in the sense that

$$\frac{1 - \frac{\langle \mathfrak{E}_\kappa(\mathbf{u})\rangle}{\eta}}{1 - \frac{\langle \mathfrak{e}_\kappa(\mathbf{u})\rangle}{\epsilon}} = \frac{\epsilon}{\eta}\frac{\langle |A\mathbf{u}_{0,\kappa}|^2\rangle}{\langle \|\mathbf{u}_{0,\kappa}\|^2\rangle} \leqslant \left(\frac{\kappa}{\kappa_\sigma}\right)^2 \ll 1$$

for $\overline{\kappa}_f^2 \leqslant \kappa^2 \ll \kappa_\sigma^2$. If one accepts that the enstrophy-cascade mechanism is the fundamental aspect leading to the Kraichnan spectrum $\eta^{2/3}\kappa^{-1/3}$ while the energy-cascade mechanism is that leading to the Kolmogorov spectrum $\epsilon^{2/3}\kappa^{-5/3}$, then the estimate above strongly supports the Kraichnan spectrum beyond the forced modes in two-dimensional turbulence.

Inverse energy cascade in two-dimensional turbulence.

As we have mentioned at the beginning of the discussion, the nature of turbulence in two dimensions is such that energy seems to cascade towards lower wavenumbers. This *inverse energy cascade* yields an accumulation of energy in the large scales in such a way that a significant amount of energy dissipation may occur in those scales. The forcing term acts on some mesoscale range, with the enstrophy cascading up (in wavenumber space) to be dissipated at higher modes and with the energy cascading down to be dissipated at lower modes.

Assume then that a significant amount of energy dissipation occurs below a range $[\underline{\kappa}_{-\epsilon}, \overline{\kappa}_{-\epsilon}]$, $\underline{\kappa}_{-\epsilon} \leqslant \overline{\kappa}_{-\epsilon}$, and with very little dissipation within this interval, in such a way that

$$\nu\kappa_0^2\langle\|\mathbf{u}_{0,\underline{\kappa}_{-\epsilon}}\|^2\rangle \sim \epsilon, \qquad \nu\kappa_0^2\langle\|\mathbf{u}_{\underline{\kappa}_{-\epsilon},\overline{\kappa}_{-\epsilon}}\|^2\rangle \ll \epsilon.$$

Assume also that most of the energy-injection occurs at some mesoscale range beyond $\overline{\kappa}_{-\epsilon}$ so that

$$|\kappa_0^2\langle(\mathbf{f}, \mathbf{u}_{0,\kappa})\rangle| \ll \epsilon$$

for all $\kappa \geqslant \overline{\kappa}_{-\epsilon}$. Then from the relation

$$\nu\kappa_0^2\langle\|\mathbf{u}_{0,\kappa}\|^2\rangle = \kappa_0^2\langle(\mathbf{f}_{0,\kappa}, \mathbf{u})\rangle - \langle\mathfrak{e}_\kappa(\mathbf{u})\rangle$$

we see that, for κ within $[\underline{\kappa}_{-\epsilon}, \overline{\kappa}_{-\epsilon}]$, we have the inverse-energy-cascade relation

$$\langle\mathfrak{e}_\kappa(\mathbf{u})\rangle \approx -\epsilon', \qquad \text{where } \epsilon' \stackrel{\text{def}}{=} \nu\kappa_0^2\langle\|\mathbf{u}_{0,\underline{\kappa}_{-\epsilon}}\|^2\rangle.$$

(Notice that in the particular case that $\nu\kappa_0^2\langle\|\mathbf{u}_{0,\underline{\kappa}_{-\epsilon}}\|^2\rangle \approx \epsilon$ we obtain $\langle\mathfrak{e}_\kappa(\mathbf{u})\rangle \approx -\epsilon$.)

A necessary condition on κ_τ for this inverse energy cascade comes from the relation $\kappa_\tau^2 = \epsilon(2\nu e)^{-1} \sim \kappa_0^2\langle\|\mathbf{u}_{0,\underline{\kappa}_{-\epsilon}}\|\rangle^2(2e)^{-1} \leqslant \underline{\kappa}_{-\epsilon}^2\kappa_0^2\langle|\mathbf{u}_{0,\underline{\kappa}_{-\epsilon}}|^2\rangle(2e)^{-1} \leqslant \underline{\kappa}_{-\epsilon}^2$, i.e.

$$\kappa_\tau \lesssim \underline{\kappa}_{-\epsilon}.$$

References

[1] *F. Abergel*: Attractor for a Navier-Stokes flow in an unbounded domain. Attractors, Inertial Manifolds and Their Approximation (Marseille-Luminy, 1987). RAIRO Modél. Math. Anal. Numér. *23* (1989), 359–370.

[2] *A. V. Babin*: The attractor of a Navier-Stokes system in an unbounded channel-like domain. J. Dynam. Differential Equations *4* (1992), 555–584.

[3] *G. I. Barenblatt, A. J. Chorin*: New perspectives in turbulence: scaling laws, asymptotics, and intermittency. SIAM Rev. *40* (1998), 265–291.

[4] *G. K. Batchelor*: The Theory of Homogeneous Turbulence. Cambridge Monographs on Mechanics and Applied Mathematics. Cambridge University Press, New York, 1953.

[5] *G. K. Batchelor*: Computation of the energy spectrum in homogeneous two-dimensional turbulence. Phys. Fluids Suppl. II *12* (1969), 233–239.

[6] *H. Bercovici, P. Constantin, C. Foias and O. P. Manley*: Exponential decay of the power spectrum of turbulence. J. Statist. Phys. *80* (1995), 579–602.

[7] *A. J. Chorin*: Vorticity and Turbulence. Applied Mathematical Sciences 103. Springer-Verlag, New York, 1994.

[8] *P. Constantin*: Geometric statistics in turbulence. SIAM Rev. *36* (1994), 73–98.

[9] *P. Constantin, C. Foias*: Navier-Stokes Equation. University of Chicago Press, Chicago, 1989.

[10] *P. Constantin, C. Foias and O. Manley*: Effects of the forcing function spectrum on the energy spectrum in 2-D turbulence. Phys. Fluids *6* (1994), 427–429.

[11] *C. Foias*: Statistical study of Navier-Stokes equations I. Rend. Sem. Mat. Univ. Padova *48* (1972), 219–348..
[12] *C. Foias*: Statistical study of Navier-Stokes equations II. Rend. Sem. Mat. Univ. Padova *49* (1973), 9–123.
[13] *C. Foias, M. S. Jolly, O. P. Manley and R. Rosa*: Statistical estimates for the Navier-Stokes equations and the Kraichnan theory of 2-D fully developed turbulence. J. Statist. Phys. *108* (2002), 591–646.
[14] *C. Foias, O. P. Manley, R. Rosa and R. Temam*: Navier-Stokes Equations and Turbulence. Encyclopedia of Mathematics and Its Applications, Vol. 83. Cambridge University Press, Cambridge, 2001.
[15] *C. Foias, O. P. Manley, R. Rosa and R. Temam*: Cascade of energy in turbulent flows. C. R. Acad. Sci. Paris, Série I Math. *332* (2001), 509–514.
[16] *C. Foias, O. P. Manley, R. Rosa and R. Temam*: Estimates for the energy cascade in three-dimensional turbulent flows. C. R. Acad. Sci. Paris, Série I Math. *333* (2001), 499–504.
[17] *C. Foias, G. Prodi*: Sur le comportement global des solutions non-stationnaires des équations de Navier-Stokes en dimension 2. Rend. Sem. Mat. Univ. Padova *39* (1967), 1–34.
[18] *S. Friedlander, L. Topper*: Turbulence. Classic Papers on Statistical Theory. Interscience Publisher, New York, 1961.
[19] *U. Frisch*: Turbulence. The Legacy of A. N. Kolmogorov. Cambridge University Press, Cambridge, 1995.
[20] *J. O. Hinze*: Turbulence. McGraw-Hill Series in Mechanical Engineering. McGraw-Hill, New York, 1975.
[21] *P. Holmes, J. L. Lumley and G. Berkooz*: Turbulence, Coherent Structures, Dynamical Systems, and Symmetry. Cambridge University Press, Cambridge, 1996.
[22] *E. Hopf*: Statistical hydromechanics and functional calculus. J. Rat. Mech. Analysis *1* (1952), 87–123.
[23] *A. A. Ilyin*: Attractors for Navier-Stokes equations in domains with finite measure. Nonlinear Anal. *27* (1996), 605–616.
[24] *J. Jiménez, A. A. Wray, P. G. Saffman and R. S. Rogallo*: The structure of intense vorticity in isotropic turbulence. J. Fluid Mech. *255* (1993), 65–90.
[25] *T. von Karman, L. Howarth*: On the statistical theory of isotropic turbulence. Proc. Roy. Soc. London A164. 1938, pp. 192–215.
[26] *A. N. Kolmogorov*: The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers. C. R. (Dokl.) Acad. Sci. URSS *30* (1941), 301–305.
[27] *A. N. Kolmogorov*: On degeneration of isotropic turbulence in an incompressible viscous liquid. C. R. (Dokl.) Acad. Sci. URSS *31* (1941), 538–540.
[28] *A. N. Kolmogorov*: Dissipation of energy in locally isotropic turbulence. C. R. (Doklady) Acad. Sci. URSS *32* (1941), 16–18.
[29] *A. N. Kolmogorov*: A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number. J. Fluid Mech. *13* (1962), 82–85.
[30] *R. H. Kraichnan*: Inertial ranges in two-dimensional turbulence. Phys. Fluids *10* (1967), 1417–1423.
[31] *R. H. Kraichnan*: Some modern developments in the statistical theory of turbulence. Statistical Mechanics: New Concepts, New Problems, New Applications (S. A. Rice, K. F. Freed and J. C. Light, eds.). 1972, pp. 201–227.
[32] *O. Ladyzhenskaya*: The Mathematical Theory of Viscous Incompressible Flow. Revised English edition. Gordon and Breach Science Publishers, New York-London-Paris, 1963.

[33] *O. Ladyzhenskaya*: First boundary value problem for the Navier-Stokes equations in domains with non smooth boundaries. C. R. Acad. Sci. Paris, Sér. I Math. *314* (1992), 253–258.
[34] *L. Landau, E. Lifchitz*: Mécanique des Fluides, Physique Théorique, Tome 6. Editions Mir, Moscow, 1971.
[35] *C. E. Leith*: Diffusion approximation for two-dimensional turbulence. Phys. Fluids *11* (1968), 671–673.
[36] *M. Lesieur*: Turbulence in Fluids. Third edition. Fluid Mechanics and its Applications, 40. Kluwer Academic Publishers Group, Dordrecht, 1997.
[37] *J.-L. Lions*: Quelques méthodes de résolution des problèmes aux limites non linéaires. Dunod, Gauthier-Villars, Paris, 1969.
[38] *I. Moise, R. Rosa and X. Wang*: Attractor for noncompact semigroups via energy equations. Nonlinearity *11* (1998), 1369–1393.
[39] *A. S. Monin, A. M. Yaglom*: Statistical Fluid Mechanics: Mechanics of Turbulence 2. MIT Press, Cambridge, 1975.
[40] *E. A. Novikov, R. V. Stewart*: The intermittency of turbulence and the spectrum of energy dissipation. Izv. Akad. Nauk SSSR, Ser. Geoffiz *3* (1964), 408–413.
[41] *A. M. Obukhoff*: On the energy distribution in the spectrm of turbulent flow. C. R. (Dokl.) Acad. Sci. USSR *32* (1941), 19–21.
[42] *A. M. Obukhoff*: Some specific features of atmospheric turbulence. J. Fluid Mech. *13* (1962), 77–81.
[43] *L. F. Richardson*: Weather Prediction by Numerical Process. Cambridge University Press, Cambridge, 1922.
[44] *R. Rosa*: The global attractor for the 2D Navier-Stokes flow on some unbounded domains. Nonlinear Anal. *32* (1998), 71–85.
[45] *Z. S. She, E. Jackson and S. A. Orszag*: Structure and dynamics of homogeneous turbulence: models and simulations. Proc. Roy. Soc. London Ser. A *434*, 101–124.
[46] *G. I. Taylor*: Diffusion by continuous movements. Proc. London Math. Soc. Ser. 2 *20* (1921), 196–211.
[47] *G. I. Taylor*: Statistical theory of turbulence. Proc. Roy. Soc. London Ser. A *151* (1935), 421–478.
[48] *G. I. Taylor*: The spectrum of turbulence. Proc. Roy. Soc. London Ser. A *164* (1938), 476–490.
[49] *R. Temam*: Navier-Stokes Equations. Theory and numerical analysis. Studies in Mathematics and its Applications, 2. 3rd edition. North-Holland Publishing Co., Amsterdam-New York, 1984. Reedition in 2001 in the AMS Chelsea series, AMS, Providence.
[50] *H. Tennekes, J. L. Lumley*: A First Course in Turbulence. MIT Press, Cambridge, Mass., 1972.
[51] *C. V. Tran, T. G. Shepherd*: Constraints on the spectral distribution of energy and enstrophy dissipation in forced two-dimensional turbulence. Physica D. To appear.
[52] *M. I. Vishik, A. V. Fursikov*: Translationally homogeneous statistical solutions and individual solutions with infinite energy of a system of Navier-Stokes equations. Sibirsk. Mat. Zh. *19* (1978), 1005–1031.
[53] *M. I. Vishik, A. V. Fursikov*: Mathematical Problems of Statistical Hydrodynamics. Kluwer, Dordrecht, 1988.

Author's address: *R. M. S. Rosa*, Departamento de Matemática Aplicada, Instituto de Matemática, Universidade Federal do Rio de Janeiro, Caixa Postal 68530 Ilha do Fundão, RJ 21945-970 Rio de Janeiro, Brazil, email: `rrosa@ufrj.br`.