Eduard Feireisl; Hana Petzeltová

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ON THE DOMAIN DEPENDENCE OF SOLUTIONS TO THE TWO-PHASE STEFAN PROBLEM*

Eduard Feireisl and Hana Petzeltová, Praha

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Abstract. We prove that solutions to the two-phase Stefan problem defined on a sequence of spatial domains $\Omega_n \subset \mathbb{R}^N$ converge to a solution of the same problem on a domain Ω where Ω is the limit of Ω_n in the sense of Mosco. The corresponding free boundaries converge in the sense of Lebesgue measure on $\mathbb{R}^N$.

Keywords: Stefan problem, domain dependence, Mosco-type covergence of domains

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1. Introduction

In many situations arising in mathematical physics, one is to study the behaviour of solutions to partial differential equations defined on domains approximated in some sense by simpler one. In particular, it is of interest whether or not the different domains yield solutions which are close to each other. In principle, two types of problems may be found in literature:

(i) *Domain perturbation*

A typical example is the so-called Crushed Ice Problem (see Rauch and Taylor [14]). For a given spatial domain, the perturbation is obtained by removing closed balls of radius r_n. A physical problem this would model is the heat flow in the domain where the balls are little coolers maintained at the temperature zero. The question is how fast must r_n decrease in order to render the balls negligible and what happens if this condition fails.

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(ii) *Optimal shape design*

A classical problem in optimal design is to prove the existence of minimizers for shape functionals. This leads to solving certain partial differential equations defined on a sequence of domains, extracting convergent subsequences etc. Consequently, it is useful to introduce a topology for which some interesting classes of open sets are compact and the solutions of the corresponding equations converge to the solution of the limit problem (see Dal Maso and Mosco [5], Pironneau [13], Šverák [15] etc.).

The above problems gave rise to the concept of Γ-*convergence* for a sequence of open sets related directly to the equation and the boundary conditions. Equivalently, the Mosco conditions describe the same type of convergence in terms of the underlying function spaces. This is the weakest hypothesis which still guarantees convergence of solutions of the corresponding equations but, on the other hand, it is often very difficult to verify in a direct fashion. Moreover, the family of open sets need not be compact with respect to this type of convergence. For this reason, several attempts have been made to obtain some sufficient conditions for the Γ-convergence expressed directly in terms of the geometrical properties of the domains (see Šverák [15], Bucur and Zolesio [3], Henrot [11] and the references and examples cited therein).

Most of the existing theory deals with elliptic problems while much less seems to be known for evolutionary equations. There are results for parabolic equations in Rauch and Taylor [14], Dancer [6], Hale and Vegas [10] and, most recently, Daners [7]. The methods are based on the linear elliptic theory and the variation of constants formula for the corresponding semigroup solutions, which restricts their use to basically semilinear nondegenerate parabolic equations. The conditions on convergence of the domains are more restrictive than those mentioned above. The limiting domain should be stable, which excludes open sets with cracks etc.

The aim of the present paper is to introduce a different approach yielding positive convergence results for a wide class of nonlinear problems including quasilinear and possibly degenerate parabolic equations under the weakest conditions on convergence of the underlying spatial domains. The method is based on a rather simple fact that the problem may be reformulated as an equation on the whole space $\mathbb{R}^N$ with a measure on the right-hand side.

The domain dependence of solutions to the two-phase Stefan problem reflects all the features mentioned above as the equation is quasilinear and degenerate parabolic where, in addition, the phases are divided by the free boundary, which makes the analysis even more delicate.

The *enthalpy* U and the *temperature* ϑ in the two-phase Stefan problem satisfy the equation

$$U_t - \Delta\vartheta = f \quad \text{in } \mathscr{D}'(Q) \tag{1.1}$$

on a bounded cylindrical domain $Q=(0,T)\times\Omega\subset\mathbb{R}^{N+1}$.

We assume that U and ϑ are tied together by means of the constitutive relation

$$\vartheta=\varphi(U) \tag{1.2}$$

with

$$\varphi(U)=\begin{cases}\beta_-(U+\nu) & \text{for } U\in(-\infty,-\nu],\\ 0 & \text{for } U\in(-\nu,0),\\ \beta_+U & \text{for } U\in[0,\infty)\end{cases} \tag{1.3}$$

where the *diffusion coefficients* β_+, β_- are positive constants.

The *heat source* $f=f(t,x,\vartheta,\nabla\vartheta)$ will be a Carathéodory function, i.e. $f(t,x,\cdot,\cdot)$ is continuous for a.e. t, x and $f(\cdot,\cdot,v,w)$ is measurable for any v, w.

Finally, we suppose that ϑ attains the melting temperature on the boundary

$$\vartheta|_{(0,T)\times\partial\Omega}\equiv 0. \tag{1.4}$$

Let Ω_n be a sequence of open sets contained in a fixed ball, $\overline{\Omega}_n\subset B\subset\mathbb{R}^N$.

We will say that the sequence $\{\Omega_n\}$ converges to an open set Ω, $\overline{\Omega}\subset B$ in the sense of Mosco, $\Omega_n\xrightarrow{M}\Omega$ iff

(M1) *for any $\psi\in\mathscr{D}(\Omega)$ there exists a sequence $\psi_n\in W_0^{1,2}(\Omega_n)$ such that $\psi_n\to\psi$ strongly in $W^{1,2}(B)$;*

(M2) *if $w_n\to w$ weakly in $W^{1,2}(B)$, $w_n\in W_0^{1,2}(\Omega_n)$, then $w\in W_0^{1,2}(\Omega)$.*

Remark. Here and in what follows we will not distinguish between a function from $W_0^{1,2}(\Omega_n)$, $W_0^{1,2}(\Omega)$ and its prolongation by zero onto B. Similarly, functions from L^2 will be prolonged by zero outside their natural domain of definition.

Our main result reads as follows:

Theorem 1.1. *Let $\Omega_n\subset B$ be a sequence of domains such that*

$$\Omega_n\xrightarrow{M}\Omega.$$

Assume that U_n, ϑ_n are weak solutions of the problem (1.1), (1.2) *on $(0,T)\times\Omega_n$ satisfying the boundary conditions* (1.4).

Finally, let

$$\|U_n\|_{L^\infty(0,T;L^2(\Omega_n))}+\|\vartheta_n\|_{L^2(0,T;W_0^{1,2}(\Omega_n))}+\|f(t,x,\vartheta_n,\nabla\vartheta_n)\|_{L^q(Q)}\leqslant c \tag{1.5}$$

for a certain $q>1$ and $n=1,2,\ldots$

Then, passing to subsequences as the case may be we have

$$
\begin{aligned}
&U_n \to U \text{ weakly star in } L^\infty\big(0,T;L^2(B)\big),\\
(1.6)\qquad &\vartheta_n \to \vartheta \text{ weakly in } L^2\big(0,T;W^{1,2}(B)\big),\ \text{strongly in } L^2\big((0,T)\times B\big),\\
&\nabla\vartheta_n \to \nabla\vartheta \text{ strongly in } L^1\big((0,T)\times B\big)
\end{aligned}
$$

and

$$
(1.7)\qquad U_n(t) \to U(t) \text{ weakly in } L^2(\Omega) \text{ for any } t\in[0,T]
$$

where $U\in L^\infty\big(0,T;L^2(\Omega)\big)$, $\vartheta\in L^2(0,T;W_0^{1,2}(\Omega))$ *is a weak solution of* (1.1), (1.2), (1.4) *on* $(0,T)\times\Omega$.

Remark. Observe that (1.7) makes sense as both U_n and U belong to $C\big([0,T],W^{-1,2}(\Omega_n)\oplus L^q(\Omega_n)\big)$ and $C\big([0,T],W^{-1,2}(\Omega)\oplus L^q(\Omega)\big)$, respectively. On the other hand, the limit function U is not necessarily zero outside Ω.

The condition (1.5) is easily verified in most cases of interest as, typically, the temperature ϑ is bounded (even continuous) independently of the underlying domain and, consequently, the bound on $\nabla\vartheta$ and f follows from the standard energy estimates provided some suitable growth conditions are imposed on f (see e.g. the monograph Meirmanov [12]). In particular, we do not require any bound for the time derivative of ϑ as, in some cases, this is not natural for this kind of problem (cf. Di Benedetto [8]).

Finally, an additional piece of information can be obtained concerning the behaviour of the free boundaries. To this end, it is necessary to introduce some notation:

(i) *the ice region:*

$$
\mathscr{I}_n \equiv \{(t,x)\in\Omega_n\times(0,T);\quad U_n(t,x)\leqslant -\nu\};
$$

(ii) *the water region:*

$$
\mathscr{W}_n \equiv \{(t,x)\in\Omega_n\times(0,T);\quad U_n(t,x)\geqslant 0\};
$$

(iii) *the mushy region:*

$$
\mathscr{M}_n \equiv \{(t,x)\in\Omega_n\times(0,T);\quad U_n(t,x)\in(-\nu,0)\}.
$$

Similarly, the sets $\mathscr{I}$, $\mathscr{W}$, $\mathscr{M}$ are defined for the limit problem on Ω.

We claim the following statement:

Theorem 1.2. *Under the hypotheses of Theorem* 1.1, *we have*

$$\operatorname{meas}(\mathscr{I}_n \cap \mathscr{W}) \to 0, \tag{1.8}$$

$$\operatorname{meas}(\mathscr{W}_n \cap \mathscr{I}) \to 0 \tag{1.9}$$

as $n \to \infty$.

Note that nothing is said about the mushy region $\mathscr{M}$ formed by a mixture of the two phases. We recall that the mushy region is void for any positive t provided it does not exist for $t = 0$ and the heat source is of the form

$$f = f(t, x, \vartheta),\ f(t, x, 0) \equiv 0, \quad f \text{ Lipschitz in } \vartheta$$

(see Meirmanov [12], Chapter 1, Theorem 11). Naturally, the mushy region may develop for the limit problem as a result of oscillations in the sequence $U_n(0)$. Probably the most elegant way to describe such a phenomenon is to introduce the Young measure $\mu_{t,x}$, $(t, x) \in (0, T) \times B$ associated to the sequence U_n in $L^2((0, T) \times B)$. It will be clear from the analysis presented in Section 5 that

$$\mu_{t,x} = -\frac{U(t,x)}{\nu}\delta_{-\nu} + \Big(1 + \frac{U(t,x)}{\nu}\Big)\delta_0 \quad \text{for a.e. } (t,x) \in \mathscr{M}$$

provided $\mathscr{M}_n = \emptyset$ for all $n = 1, 2, \dots$ where the symbol δ_y stands for the Dirac mass centered at y.

The paper is organized as follows:

In Section 2, we decompose the temperature ϑ_n into its positive and negative part and show that both satisfy a semilinear nondegenerate parabolic equation on $(0, T) \times B$ with a measure on the right-hand side which is locally bounded independently of Ω_n. To this end, the concept of viscosity and entropy solutions is introduced.

This fact together with some ideas of Boccardo and Murat [2] will be used in Section 3 to show compactness of the sequences ϑ_n, $\nabla\vartheta_n$ in appropriate function spaces.

Finally, the proof of Theorem 1.1 is given in Section 4 while Section 5 contains the proof of Theorem 1.2.

2. Preliminaries, the decomposition

In what follows, we will always assume that any weak solution U, ϑ of the problem (1.1), (1.2), (1.4) is a *viscosity* solution, i.e. it can be constructed as the limit

$$
\begin{aligned}
&U^\varepsilon \to U \text{ weakly star in } L^\infty\big(0,T;L^2(\Omega)\big),\\
&\vartheta^\varepsilon \to \vartheta \text{ weakly in } L^2\big(0,T;W_0^{1,2}(\Omega)\big),\\
&f^\varepsilon \to f \text{ weakly in } L^q(Q)
\end{aligned}
\tag{2.1}
$$

where

$$
\begin{aligned}
&U^\varepsilon \text{ bounded in } L^\infty\big(0,T;L^2(\Omega)\big),\\
&\vartheta^\varepsilon \text{ bounded in } L^2\big(0,T;W_0^{1,2}(\Omega)\big),\\
&f^\varepsilon \text{ bounded in } L^q(Q)
\end{aligned}
\tag{2.2}
$$

are classical solutions of the regularized problems:

$$U_t^\varepsilon - \Delta\vartheta^\varepsilon = f^\varepsilon \text{ on } (0,T)\times\Omega^\varepsilon, \tag{2.3}$$

$$\vartheta^\varepsilon = \varphi^\varepsilon(U^\varepsilon), \tag{2.4}$$

$$\vartheta^\varepsilon|_{(0,T)\times\partial\Omega^\varepsilon} \equiv 0 \tag{2.5}$$

with

$$
\begin{cases}
\varphi^\varepsilon \in C^\infty(\mathbb{R}),\ \ \varphi^\varepsilon(0)=0,\ \ 0<\varepsilon\leqslant(\varphi^\varepsilon)'\leqslant 2\max\{\beta_-,\beta_+\}\\
\varphi^\varepsilon\equiv\varphi \quad \text{on} \quad (-\infty,-\nu-\varepsilon)\cup(\varepsilon,\infty)
\end{cases}
\tag{2.6}
$$

and $\Omega^\varepsilon \subset \Omega$ regular domains.

This assumption is not restrictive in view of the available existence theory (cf. e.g. Friedman [9], Meirmanov [12]). On the other hand, our definition of viscosity solutions should not be confused with that of Crandall, Lions and Ishii [4].

Lemma 2.1. *Let U, ϑ be a (viscosity) solution of the problem* (1.1), (1.2) *on the set $(0,T)\times\Omega$ belonging to the class*

$$U \in L^\infty\big(0,T;L^2(\Omega)\big), \quad \vartheta \in L^2\big(0,T;W_0^{1,2}(\Omega)\big).$$

Then we have

$$S(\vartheta)_t - \beta_+\Delta S(\vartheta) \leqslant \beta_+ 1_\Omega|f| \quad \text{in } \mathscr{D}'\big((0,T)\times B\big) \tag{2.7}$$

for any convex S such that

$$S \equiv 0 \text{ on the interval } (-\infty, 0], \quad S(z) \leqslant |z|.$$

Similarly,

$$S(\vartheta)_t - \beta_- \Delta S(\vartheta) \leqslant \beta_- 1_\Omega |f| \quad \text{in } \mathscr{D}'\big((0,T)\times B\big) \tag{2.8}$$

for any convex S satisfying

$$S \equiv 0 \text{ on } [0, \infty), \quad S(z) \leqslant |z|.$$

Remark. The conclusion of Lemma 2.1 is related to the fact that any viscosity solution of the problem is also an *entropy* solution in the sense of the theory of nonlinear conservation laws. Note that there is a stronger concept of the so-called *renormalized* solutions for this type of equations introduced by Blanchard and Redwane [1].

Proof. The temperature ϑ can be obtained as a limit of the approximate solutions:

$$\vartheta^\varepsilon \equiv \varphi^\varepsilon(U^\varepsilon) \to \vartheta \text{ weakly in } L^2\big(0,T;W^{1,2}(B)\big). \tag{2.9}$$

Consequently, multiplying (2.3) by the expression $S'(\vartheta^\varepsilon)$ and using the boundary conditions we obtain

$$S'(\vartheta^\varepsilon)U_t^\varepsilon - \Delta S(\vartheta^\varepsilon) \leqslant S'(\vartheta^\varepsilon) f^\varepsilon \text{ on } (0,T)\times B$$

for any "regular" convex S, i.e. $S \in C^\infty(\mathbb{R})$, $S \equiv 0$ on $(-\infty, \delta]$ for a certain $\delta > 0$, $S' \leqslant 1$.

Now, since S vanishes on the interval $(-\infty, \delta]$, the first term in the above expression may be treated as follows:

$$S'(\vartheta^\varepsilon)U_t^\varepsilon = \frac{S'(\vartheta^\varepsilon)\varphi^{\varepsilon\prime}(U^\varepsilon)U_t^\varepsilon}{\varphi^{\varepsilon\prime}(U^\varepsilon)} = \frac{1}{\beta_+} S(\vartheta^\varepsilon)_t$$

for all $\varepsilon > 0$ sufficiently small.

Thus, we can write

$$S(\vartheta^\varepsilon)_t - \beta_+ \Delta S(\vartheta^\varepsilon) = \beta_+ S'(\vartheta^\varepsilon) f^\varepsilon - \mu^\varepsilon \text{ on } (0,T)\times B$$

where μ^ε are nonnegative Radon measures supported in $[0,T]\times\overline{\Omega}$. Since

$$\langle \mu^\varepsilon, \psi\rangle = \int_0^T \int_B S(\vartheta^\varepsilon)\psi_t + \beta_+ S(\vartheta^\varepsilon)\Delta\psi + \beta_+ S'(\vartheta^\varepsilon) f^\varepsilon \psi \,\mathrm{d}x\,\mathrm{d}t$$

for any

$$\psi \geqslant 0, \quad \psi \in \mathscr{D}\big((0,T)\times B\big), \quad \psi \equiv 1 \text{ on } [t_1,t_2]\times\overline{\Omega},$$

we can use (2.2) to obtain

$$\|\mu^\varepsilon\|_{\mathscr{M}([t_1,t_2]\times\bar{\Omega})} \leqslant c(t_1,t_2), \quad 0<t_1<t_2<T$$

independently of ε.

Since $S(\vartheta^\varepsilon)$ are bounded in $L^2\big(0,T;W^{1,2}(B)\big)\cap L^\infty\big(0,T;L^2(B)\big)$, we can use Lemma 4.2 of Boccardo and Murat [2] to conclude

$$\begin{aligned} &S(\vartheta^\varepsilon) \to \mathscr{S} \text{ strongly in } L^2\big((0,T)\times B\big),\\ &S'(\vartheta^\varepsilon) f^\varepsilon \to F \text{ weakly in } L^q\big((0,T)\times B\big)\end{aligned} \tag{2.10}$$

where

$$\mathscr{S}_t - \beta_+\Delta\mathscr{S} \leqslant \beta_+ F \text{ in } \mathscr{D}'\big((0,T)\times B\big).$$

As S is monotone, (2.9) and (2.10) together with Minty's trick can be used to obtain

$$\mathscr{S} = S(\vartheta)$$

and, since $S'(\vartheta^\varepsilon) = S'\big(S^{-1}S(\vartheta^\varepsilon)\big)$,

$$F = S'(\vartheta) f.$$

Thus we have verified (2.7) for all regular S. Approximating an arbitrary S by regular ones we complete the proof of (2.7). Note that $0 \leqslant S' \leqslant 1$.

The proof of (2.8) follows similar arguments. □

At this stage, we decompose ϑ into its positive and negative part

$$\vartheta = \vartheta^+ - \vartheta^-$$

and use Lemma 2.1 for $S(z)=z^+$, $S(z)=z^-$ respectively to deduce

$$\begin{cases} \vartheta^+_t - \beta_+\Delta\vartheta^+ = & \beta_+ 1_\Omega |f| - \mu^+ \\ \vartheta^-_t - \beta_-\Delta\vartheta^- = & \beta_- 1_\Omega |f| - \mu^- \end{cases} \quad \text{in } \mathscr{D}'\big((0,T)\times B\big) \tag{2.11}$$

where μ_+, μ_- are nonnegative Radon measures with support in $[0,T]\times\overline{\Omega}$.

3. Compactness of the temperature gradient

We start with some auxiliary results. Let us introduce a mollifier ϱ,

$$\varrho \in \mathscr{D}\big((-1,1)\big), \quad \varrho \geqslant 0, \quad \varrho(-z) = \varrho(z),$$
$$\int_{-1}^{1} \varrho(z)\,\mathrm{d}z = 1, \quad \varrho \text{ nonincreasing on } [0,1)$$

and set $\varrho_h(z) = \frac{1}{h}\varrho(\frac{z}{h}),\ h > 0$.

Moreover, we define $F_h(t) \in \mathscr{D}'(\mathbb{R}^N)$ as

$$\langle F_h(t), \psi\rangle \equiv \langle F, \varrho_h(t-\cdot)\otimes\psi\rangle \ , \quad t \in (h, T-h), \quad \psi \in \mathscr{D}(\mathbb{R}^N)$$

for any $F \in \mathscr{D}'\big((0,T)\times\mathbb{R}^N\big)$.

Lemma 3.1. *Let*

$$\mu \in \mathscr{D}'\big((0,T)\times\mathbb{R}^N\big)$$

be a nonnegative distribution.

Then $[t \mapsto \mu_h(t)] \in L^\infty(t_1,t_2;\mathscr{M}(K))$ *for all* $0 < h < h_0,\ h_0 < t_1 < t_2 < T - h_0$ *and for all compacts* $K \subset \mathbb{R}^N$ *and*

$$\|\mu_h\|_{L^1(t_1,t_2;\mathscr{M}(K))} \leqslant \langle \mu, \eta\otimes\psi\rangle$$

where

$$\eta(s) \equiv \int_{t_1}^{t_2} \varrho_h(t-s)\,\mathrm{d}t$$

and

$$\psi \in \mathscr{D}(\mathbb{R}^N), \quad \psi \geqslant 0, \quad \psi = 1 \text{ on } K.$$

P r o o f. We compute

$$\int_{t_1}^{t_2} \|\mu_h(t)\|_{\mathscr{M}(K)}\,\mathrm{d}t = \int_{t_1}^{t_2} \sup_{\|\omega\|_{C(K)}\leqslant 1} \langle \mu_h(t), \omega\rangle\ \mathrm{d}t \leqslant \int_{t_1}^{t_2} \langle \mu_h(t), \psi\rangle\ \mathrm{d}t$$
$$= \int_{t_1}^{t_2} \langle \mu, \varrho_h(t-\cdot)\otimes\psi\rangle\ \mathrm{d}t = \langle \mu, \eta\otimes\psi\rangle\,.$$

$\square$

The following assertion is the heart of the paper:

Proposition 3.1. *Let U_n, ϑ_n be (viscosity) solutions of the problem* (1.1), (1.2), (1.4) *on the set* $(0,T)\times\Omega_n$ *satisfying the hypotheses of Theorem* 1.1.

Then there is $\vartheta\in L^2\big(0,T;W_0^{1,2}(B)\big)$ *such that*

$$\begin{aligned}&\vartheta_n\to\vartheta \text{ strongly in } L^2\big((0,T)\times B\big),\\&\nabla\vartheta_n\to\nabla\vartheta \text{ strongly in } L^1\big((0,T)\times B\big)\end{aligned}\tag{3.1}$$

passing to a subsequence if necessary.

Proof. We restrict ourselves to showing compactness of the sequence $\{\vartheta_n^+\}$. To begin, fix $0<t_1<t_2<T$. By virtue of (2.11) the functions ϑ_n^+ satisfy

$$(\vartheta_n^+)_t=h_n+k_n\quad\text{in } \mathscr{D}'\big((t_1,t_2)\times B\big)$$

where $h_n\equiv\beta_+\Delta\vartheta_n^+$ are bounded in $L^2\big(t_1,t_2;W^{-1,2}(B)\big)$ and $k_n=1_\Omega\beta_+|f_n|-\mu_n^+$ are bounded in $\mathscr{M}([t_1,t_2]\times B)$.

By virtue of Boccardo and Murat ([2], Lemma 4.2) and (1.5),

$$\vartheta_n^+\to\vartheta^+ \text{ strongly in } L^2\big((0,T)\times B\big).\tag{3.2}$$

At this stage, we regularize (2.11) to obtain

$$\big((\vartheta_n^+)_h\big)_t-\beta_+\Delta(\vartheta_n^+)_h=1_{\Omega_n}\beta_+|f_n|_h-(\mu_n^+)_h\tag{3.3}$$

in $\mathscr{D}'(B)$ for any $t\in[t_1,t_2]$.

Taking the difference of (3.3) for $(\vartheta_n^+)_h$, $(\vartheta_m^+)_h$, multiplying the resulting expression by $\gamma(t)g\big((\vartheta_n^+)_h-(\vartheta_m^+)_h\big)$ where

$$\gamma\in\mathscr{D}\big((t_1,t_2)\big),\quad g\in C^1(\mathbb{R}^1),\quad g,\ g' \text{ bounded },$$

and integrating by parts we obtain

$$\begin{aligned}&\int_{t_1}^{t_2}\int_B\gamma g'\big((\vartheta_n^+)_h-(\vartheta_m^+)_h\big)\beta_+|\nabla\big((\vartheta_n^+)_h-(\vartheta_m^+)_h\big)|^2\,\mathrm{d}x\,\mathrm{d}t\\&=\int_{t_1}^{t_2}\int_B\gamma'G\big((\vartheta_n^+)_h-(\vartheta_m^+)_h\big)+\gamma[(k_n)_h-(k_m)_h]g\big((\vartheta_n^+)_h-(\vartheta_m^+)_h\big)\,\mathrm{d}x\,\mathrm{d}t\end{aligned}\tag{3.4}$$

where $G(v)\equiv\int_0^v g(z)\,\mathrm{d}z$.

Thus we may use Lemma 3.1 together with (1.5), boundedness of the supports of μ_n^+, μ_m^+ and (3.2) to pass to the limit, first for $h \to 0+$ then for $n, m \to \infty$ and to conclude

$$\limsup_{n,m\to\infty} \int_{t_1}^{t_2} \int_B g'(\vartheta_n^+ - \vartheta_m^+)\beta_+ |\nabla(\vartheta_n^+ - \vartheta_m^+)|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant c\|g\|_{L^\infty} \tag{3.5}$$

for any g as above, which is the same as the formula (4.27) in Boccardo and Murat [2]. Consequently, the rest of the proof is identical to that of [2], Theorem 4.3.

4. The proof of Theorem 1.1

We start with the following auxiliary result:

Lemma 4.1. *Let* $\Omega_n \xrightarrow{M} \Omega$, $w_n \in L^2\big(0,T; W_0^{1,2}(\Omega_n)\big)$ *and*

$$w_n \to w \text{ weakly in } L^2\big(0,T; W^{1,2}(B)\big).$$

Then $w \in L^2\big(0,T; W_0^{1,2}(\Omega)\big)$.

P r o o f. First we regularize w_n in the t variable to obtain

$$\varrho_h * w_n \to \varrho_h * w \text{ weakly in } L^2\big(0,T; W^{1,2}(B)\big) \quad \text{for } n \to \infty$$

where $*$ denotes the convolution with respect to t.

Since $\varrho_h w_n$ are bounded in $C\big([0,T]; W^{1,2}(B)\big)$ independently of n for any fixed $h > 0$, we get, by virtue of (M2),

$$\varrho_h * w(t) \in W_0^{1,2}(\Omega) \quad \text{for all } t, h > 0.$$

Finally, letting $h \to 0+$ we obtain $w \in L^2\big(0,T; W_0^{1,2}(\Omega)\big)$. □

Now, we are ready to prove Theorem 1.1.

By virtue of Proposition 3.1, there are functions U, ϑ such that (1.6) and (1.2) hold, where the latter is verified using monotonicity of φ and Minty's trick.

Next, by virtue of (M1), any test function $\Psi \in \mathscr{D}\big((0,T)\times\Omega\big)$ may be approximated as

$$\Psi_n \to \Psi \text{ strongly in } L^2\big(0,T; W^{1,2}(B)\big)$$

where $\Psi_n \in L^2\big(0,T; W_0^{1,2}(\Omega_n)\big)$. Moreover, it is easy to see that Ψ_n may be chosen bounded by a constant independent of n. Consequently, U, ϑ satisfy (1.1).

Using (M1) again we approximate any $\psi \in \mathscr{D}(\Omega)$ by a sequence $\psi_n \in \mathscr{D}(\Omega_n)$ and, consequently,

$$l^n(t) \equiv \int_{\Omega_n} U_n(t)\psi_n \,\mathrm{d}x \to l(t) \equiv \int_{\Omega} U(t)\psi \,\mathrm{d}x$$

strongly in $C([0,T])$, which implies (1.7).

Finally, $\vartheta \in L^2\big(0,T;W_0^{1,2}(\Omega)\big)$ by virtue of (M2) and Lemma 4.1.

Theorem 1.1 has been proved.

5. Convergence of free boundaries

The aim of this section is to prove Theorem 1.2. To begin, observe that (1.6) yields

$$\begin{cases} U_n \to U \text{ weakly in } L^2\big((0,T)\times B\big), \\ \varphi(U_n) \to \varphi(U) \text{ strongly in } L^2\big((0,T)\times B\big). \end{cases} \tag{5.1}$$

The *Young measure* associated to the sequence $\{U_n\}$ is a family $\mu_{t,x}$ of probability measures defined on $\mathbb{R}$ for a.e. $(t,x) \in (0,T)\times B$ such that

$$\langle \mu_{t,x}, h\rangle = \tilde{h}(t,x) \text{ a.e. on } (0,T)\times B$$

for any $h \in C(\mathbb{R})$ satisfying

$$\limsup_{|z|\to\infty} \frac{|h(z)|}{z^2} = 0 \tag{5.2}$$

where $\tilde{h}$ stands for the weak limit of the sequence $h(U_n)$.

Let $H \equiv \{h_1,\dots,h_m\}$ be a finite system of continuous functions satisfying (5.2) and such that

$$h_i(U_n) \to h_i(U) \text{ weakly in } L^1\big((0,T)\times B\big) \text{ as } n\to\infty \tag{5.3}$$

for all $i = 1,2,\dots,m$.

We will say that a point $z \in \mathbb{R}$ is H-regular if there exists a function $\chi_z \in \operatorname{span}\{H\}$ such that

$$\chi_z \geqslant 0 \text{ on } \mathbb{R}, \quad \chi_z(z) = 0 \tag{5.4}$$

and

$$\text{either } \chi_z > 0 \text{ on } (z,\infty) \quad \text{or } \chi_z > 0 \text{ on } (-\infty,z). \tag{5.5}$$

Lemma 5.1. *Let $\{\mu_{t,x}\}$ be a Young measure on $(0,T)\times B$ generated by a bounded sequence*

$$U_n \to U \text{ weakly in } L^2\big((0,T)\times B\big).$$

Let H be a system of functions satisfying (5.2), (5.3) and denote by $\mathscr{R}$ the set of all H-regular points in $\mathbb{R}$.

Then

$$\mu_{t,x} = \delta_{U(t,x)}$$

for a.e. $(t,x) \in U^{-1}(\mathscr{R})$.

P r o o f. There exists a full measure set $G \subset (0,T)\times B$ such that

$$\langle \mu_{t,x}, \chi\rangle = \chi\big(U(t,x)\big) \quad \text{for all } (t,x) \in G$$

and for any $\chi \in \operatorname{span}\big\{H \cup \{1, Id\}\big\}$.

Now, for *any* $(t,x) \in G \cap U^{-1}(\mathscr{R})$ we have

$$\langle \mu_{t,x}, Id\rangle = U(t,x)$$

and, denoting $z = U(t,x)$, we get

$$\langle \mu_{t,x}, \chi_z\rangle = \chi_z\big(U(t,x)\big) = 0$$

where χ_z is as in (5.4), (5.5). Necessarily, $\mu_{t,x} = \delta_{U(t,x)}$. □

Now, for

$$H \equiv \{|\varphi(z)|^\gamma, z, 1\}, \quad \gamma \in (1,2)$$

the conditions (5.2), (5.3) follow from (5.1). Moreover, since the function $|\varphi(z)|^\gamma$ is strictly convex on the set $(-\infty,-\nu)\cup(0,\infty)$, it is easy to see that

$$\mathscr{R} = (-\infty,-\nu]\cup[0,\infty).$$

Consequently, using Lemma 5.1, we conclude

$$U_n \to U \text{ strongly in } L^1(\mathscr{W}\cup\mathscr{I}),$$

which implies (1.8), (1.9).

Theorem 1.2 has been proved.

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Authors' addresses: *Eduard Feireisl, Hana Petzeltová*, Institute of Mathematics of the Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Praha 1, Czech Republic, e-mail: `feireisl@math.cas.cz`.