Jan Eisner; Milan Kučera

Spatial patterns for reaction-diffusion systems with conditions described by inclusions

Applications of Mathematics, Vol. 42 (1997), No. 6, 421–449

Persistent URL: `http://dml.cz/dmlcz/134368`

SPATIAL PATTERNS FOR REACTION-DIFFUSION SYSTEMS WITH CONDITIONS DESCRIBED BY INCLUSIONS

Jan Eisner, Milan Kučera, Praha

(Received December 21, 1996)

Abstract. We consider a reaction-diffusion system of the activator-inhibitor type with boundary conditions given by inclusions. We show that there exists a bifurcation point at which stationary but spatially nonconstant solutions (spatial patterns) bifurcate from the branch of trivial solutions. This bifurcation point lies in the domain of stability of the trivial solution to the same system with Dirichlet and Neumann boundary conditions, where a bifurcation of this classical problem is excluded.

Keywords: reaction-diffusion systems, variational inequalities, inclusions, bifurcation, stationary solutions, spatial patterns

MSC 2000: 35B32, 35K57, 35K58, 47H04

1. Introduction

We will study a reaction-diffusion system

$$u_t = d_1\Delta u + f(u,v),\ v_t = d_2\Delta v + g(u,v) \text{ on } (0,\infty)\times\Omega \tag{1.1}$$

with boundary conditions of the type

$$u=\bar u, v=\bar v \text{ on } \Gamma_D,\ \frac{\partial u(x)}{\partial n}\in -\frac{m_1(x,u(x))}{d_1}, \frac{\partial v(x)}{\partial n}\in -\frac{m_2(x,v(x))}{d_2} \quad \text{on } \Gamma_N. \tag{1.2}$$

It will be always supposed that f, g are real differentiable functions on $\mathbb{R}^2$, $\bar u,\bar v$ are constants such that $f(\bar u,\bar v)=g(\bar u,\bar v)=0$, Ω is a bounded domain in $\mathbb{R}^N$ with a lipschitzian boundary $\partial\Omega$, Γ_D, Γ_N are open (in $\partial\Omega$) disjoint subsets of $\partial\Omega$ such that

$$\operatorname{meas}\Gamma_D>0,\ \operatorname{meas}(\partial\Omega\setminus\Gamma_D\cup\Gamma_N)=0, \tag{1.3}$$

The authors are supported by the grant No. 201/95/0630 of the Grant Agency of the Czech Republic

m_1, m_2 are multivalued functions of a certain type. (For instance, m_1 can be described by a real continuous function on $(-\infty, \bar{u}) \cup (\bar{u}, +\infty)$, $m_1(s) = 0$ for $s > \bar{u}$, $m_1(s) \leqslant 0$ for $s < \bar{u}$, $\lim\limits_{s \to \bar{u}_-} m_1(s) = m_1 < 0$, $m_1(\bar{u}) = [m_1, 0]$, similarly for m_2.)

Simultaneously we will study (1.1) with classical boundary conditions

$$u = \bar{u}, v = \bar{v} \text{ on } \Gamma_D, \ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_N. \tag{1.4}$$

We will denote $b_{11} = \frac{\partial f}{\partial u}(\bar{u}, \bar{v}), b_{12} = \frac{\partial f}{\partial v}(\bar{u}, \bar{v}), b_{21} = \frac{\partial g}{\partial u}(\bar{u}, \bar{v}), b_{22} = \frac{\partial g}{\partial v}(\bar{u}, \bar{v})$ and suppose

$$b_{11} > 0, b_{12} < 0, b_{21} > 0, b_{22} < 0, b_{11} + b_{22} < 0, \ \det \ b_{ij} > 0. \tag{1.5}$$

This assumption corresponds to systems of activator-inhibitor (prey-predator) type for which diffusion-driven instability occurs: $\bar{u}, \bar{v}$ is a stable solution of the ordinary differential equations $u_t = f(u, v), v_t = g(u, v)$ but it is stable as a solution of the problem (1.1), (1.4) only for some parameters d_1, d_2 (the domain of stability) and unstable for the other $d_1, d_2 \in \mathbb{R}^2_+$ (domain of instability). See Fig. 1, Proposition 2.1. Moreover, stationary spatially nonhomogeneous solutions of (1.1), (1.4) (spatial patterns) bifurcate at the border between the domain of stability and instability. This effect was discovered in the simplest form by A. M. Turing in [23] and studied by many authors from the point of view of applications in biology (see e.g. [7], [16]) as well as from the purely mathematical point of view (see e.g. [15], [17]).

It was studied in the papers [3], [4], [20], [10], [12], [13], [14], [5] how the situation changes if the classical boundary conditions (1.4) are replaced (or supplemented) by some unilateral conditions, e.g. by boundary conditions (1.2) (some additional conditions in the interior of Ω can be also considered). Unilateral conditions were prescribed only for one of the functions u or v in all papers mentioned, that means particularly $m_1 \equiv 0$ or $m_2 \equiv 0$ in the case of boundary conditions (1.2). A certain destabilizing effect of such conditions prescribed only for the inhibitor v and a stabilizing effect of such conditions for the activator u was proved. See [10], [11] for a brief survey. In the paper [11], the first result concerning the destabilizing effect (in terms of bifurcations) of unilateral conditions prescribed for both u and v is given. However, the unilateral conditions considered in [11] are described by variational inequalities (which can be understood as a special case of inclusions) and a certain simplicity assumption concerning critical points of the classical problem is considered. The aim of the present paper is to give a generalization to the inclusions and to remove the simplicity assumption.

We will always suppose without loss of generality that $\bar{u} = \bar{v} = 0$.

N o t a t i o n 2.1. R_+—the set of all positive reals, $\mathbb{R}^2_+ = \mathbb{R}_+ \times \mathbb{R}_+$

$$D(d) = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \; D^{-1}(d) = \begin{pmatrix} 1/d_1 & 0 \\ 0 & 1/d_2 \end{pmatrix},$$
$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \; B^* = \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{pmatrix}$$

$\mathbb{V}$, A, N_j—a real Hilbert space and operators satisfying (2.1) (see also Weak Formulation 2.1)

$\widetilde{\mathbb{V}} = \mathbb{V} \times \mathbb{V}$, $AU = [Au, Av]$, $U^* = [u^*, v^*] = [\frac{b_{21}}{b_{12}} u, v]$ for $U = [u, v] \in \widetilde{\mathbb{V}}$

$\langle .\,, . \rangle$, $\|.\|$—the inner product and the norm in $\mathbb{V}$ or in $\widetilde{\mathbb{V}}$, i.e. $\langle U, W \rangle = \langle u, w \rangle + \langle v, z \rangle$, $\|U\|^2 = \|u\|^2 + \|v\|^2$ for $U = [u, v]$, $W = [w, z] \in \widetilde{\mathbb{V}}$

L_δ—linear completely continuous operator defined in Notation 4.2

M, M_0—multivalued mappings of $\widetilde{\mathbb{V}}$ into $2^{\widetilde{\mathbb{V}}}$ satisfying (2.14)–(2.18)

P^τ—mappings of $\widetilde{\mathbb{V}}$ into $\widetilde{\mathbb{V}}$ satisfying (2.19)–(2.23)

$K = \{U \in \widetilde{\mathbb{V}},\; 0 \in M_0(U)\}$—closed convex cone (see the assumption (2.15))

$K^- = \{U \in K;\; \langle P^\tau V, U \rangle < 0$ for all $V \notin K$, $\tau > 0$ and for any $Z \in \widetilde{\mathbb{V}}$, $Z \neq 0$ there exists $F \in \widetilde{\mathbb{V}}$ such that $\langle Z, F \rangle > 0$, $U \pm F \in K\}$

κ_j, e_j $(j = 1, 2, \ldots)$—the characteristic values and characteristic vectors of the operator A, i.e. eigenvalues and eigenvectors of $-\Delta u = \kappa u$ with the boundary conditions (1.4) in the special case of the operator from Weak Formulation 2.1

$C_j = \{d = [d_1, d_2] \in \mathbb{R}^2_+;\, d_2 = \frac{b_{12} b_{21}/\kappa_j^2}{d_1 - b_{11}/\kappa_j} + \frac{b_{22}}{\kappa_j}\}$, $j = 1, 2, 3, \ldots$ (see Fig. 1)

C—the envelope of the hyperbolas C_j, $j = 1, 2, 3, \ldots$ (see Fig. 1)

$\mathcal{T}$—joint tangent to all hyperbolas C_j, $j = 1, 2, 3, \ldots$ (see Fig. 1)

D_S—domain of stability—the set of all $d \in \mathbb{R}^2_+$ lying to the right from C (see Fig. 1)

D_U—domain of instability—the set of all $d \in \mathbb{R}^2_+$ lying to the left from C (see Fig. 1)

$\to$, $\rightharpoonup$ —strong convergence, weak convergence

$E_B(d) = \{U \in \widetilde{\mathbb{V}};$ (2.3) holds$\}$, $E_I(d) = \{U \in \widetilde{\mathbb{V}};$ (2.12) holds$\}$

critical point of (3.2) *or* (3.3)—a parameter $s \in \mathbb{R}$ for which $E_B(\sigma(s)) \neq \{0\}$ or $E_I(\sigma(s)) \neq \{0\}$, respectively

bifurcation point of (3.2) *or* (3.3)—a parameter $s_0 \in \mathbb{R}$ such that in any neighbourhood of $[s_0, 0]$ in $\mathbb{R} \times \widetilde{\mathbb{V}}$ there is $[s, U] = [s, u, v]$, $\|U\| \neq 0$ satisfying (3.2) or (3.3), respectively.

W e a k F o r m u l a t i o n 2.1. Set $\mathbb{V} = \{\varphi \in W^1_2(\Omega);\; \varphi = 0 \text{ on } \Gamma_D\}$, $\widetilde{\mathbb{V}} = \mathbb{V} \times \mathbb{V}$. Then $\mathbb{V}$ is a Hilbert space with the inner product $\langle u, \varphi \rangle = \int_\Omega \nabla u \cdot \nabla \varphi \,\mathrm{d}x$ and the

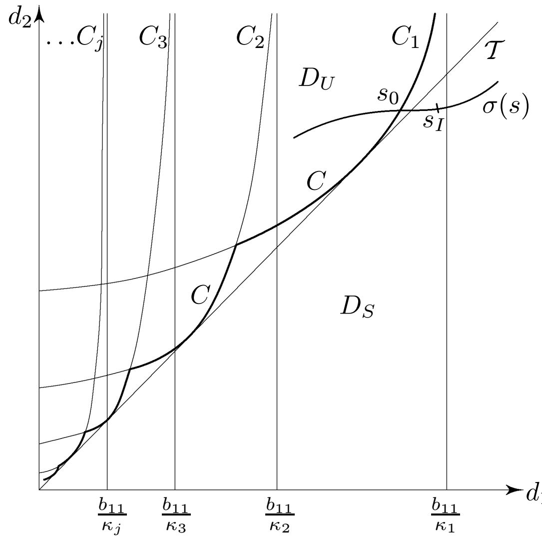

Fig. 1

corresponding norm $\|\cdot\|$ is equivalent to the usual Sobolev norm under the assumption (1.3). Set $n_1(u,v) = b_{11}u + b_{12}v - f(u,v)$, $n_2(u,v) = b_{21}u + b_{22}v - g(u,v)$,

$$\langle Au, \varphi\rangle = \int_\Omega u\varphi \, \mathrm{d}x, \quad \langle N_j(u,v), \varphi\rangle = \int_\Omega n_j(u,v)\varphi \, \mathrm{d}x \quad \text{for all } u, v, \varphi \in \mathbb{V},$$
$$N(U) = [N_1(U), N_2(U)] \text{ for all } U = [u,v] \in \widetilde{\mathbb{V}}.$$

Then

$$\text{(2.1)} \qquad \begin{aligned} &A\colon\ \mathbb{V} \to \mathbb{V},\ N_j\colon\ \widetilde{\mathbb{V}} \to \mathbb{V} \text{ are completely continuous operators,} \\ &A \text{ is linear, symmetric, positive, } \lim_{\|U\|\to 0} \frac{\|N(U)\|}{\|U\|} = 0 \end{aligned}$$

under standard growth conditions on n_j (see e.g. [6], for the last condition see [10], Appendix). A weak solution of the stationary problem corresponding to (1.1), (1.4) is a solution of the system of operator equations

$$d_1u - b_{11}Au - b_{12}Av + N_1(u,v) = 0, \ d_2v - b_{21}Au - b_{22}Av + N_2(u,v) = 0$$

which can be written in the vector form as

$$D(d)U - BAU + N(U) = 0. \tag{2.2}$$

Remark 2.1. We shall use the linearized equation

$$D(d)U - BAU = 0 \tag{2.3}$$

and the corresponding adjoint equation

$$D(d)U - B^*AU = 0. \tag{2.4}$$

Clearly $U = [u,v]$ satisfies (2.3) if and only if $U^* = [u^*,v^*]$, $u^* = \frac{b_{21}}{b_{12}}u, v^* = v$ satisfies (2.4).

Recall that if $\operatorname{Re}\lambda \leqslant -\varepsilon < 0$ for all eigenvalues of the problem

$$d_1\Delta u + b_{11}u + b_{12}v = \lambda u,\ d_2\Delta v + b_{21}u + b_{22}v = \lambda v \tag{2.5}$$

with the boundary conditions (1.4) (with $\bar{u} = \bar{v} = 0$) then the trivial solution of (1.1), (1.4) is stable and if there exists an eigenvalue of (2.5), (1.4) satisfying $\operatorname{Re}\lambda > 0$ then the trivial solution of (1.1), (1.4) is unstable (see e.g. [8], [22]). If A is from Weak Formulation 2.1 then the weak formulation of (2.5), (1.4) is

$$D(d)U - BAU + \lambda AU = 0. \tag{2.6}$$

The eigenvalues and the eigenvectors of (2.5), (1.4) coincide with those of (2.6). Hence, our definition of the domains of stability and instability (see Notation 2.1) is justified by the following statement.

Proposition 2.1. *Let the assumptions* (1.5), (2.1) *be fulfilled. Then* $\bigcup_{j=1}^{\infty} C_j$ *is the set of all* $d \in \mathbb{R}^2_+$ *such that* $E_B(d) \neq \{0\}$. *If* p *is such that the characteristic value* κ_p *of* A *(i.e. the eigenvalue* κ_p *of* $-\Delta$, (1.4) *in the situation of Weak Formulation* 2.1*) has the multiplicity* k, $\kappa_p = \ldots = \kappa_{p+k-1}$ *then* $C_p = \ldots = C_{p+k-1}$, $C_p \neq C_j$ *for all* $j \notin \{p,\ldots,p+k-1\}$. *If* $d \in C_p$, $d \notin C_j$ *for all* $j \notin \{p,\ldots,p+k-1\}$ *then*

$$E_B(d) = \operatorname{Lin}\{U_i(d)\}_{i=p}^{p+k-1} \text{ where } U_i(d) = [\alpha_p(d)e_i, e_i]$$

with $\alpha_p(d) = \frac{d_2\kappa_p - b_{22}}{b_{21}} > 0$. *If* $d \in C_p \cap C_q$ *for some* q *satisfying* $C_p \neq C_q$, $\kappa_p \neq \kappa_q = \ldots = \kappa_{q+l-1}$, *where* κ_q *has the multiplicity* l *then*

$$E_B(d) = \operatorname{Lin}\{U_i(d)\}_{i=p,\ldots,p+k-1,q,\ldots,q+l-1},$$
$$U_i(d) = [\alpha_p(d)e_i, e_i] \text{ for } i = p,\ldots,p+k-1,$$
$$U_i(d) = [\alpha_q(d)e_i, e_i] \text{ for } i = q,\ldots,q+l-1.$$

If $d \in D_S$ *then all eigenvalues of* (2.6) *(i.e. particularly of* (2.5), (1.4)*) satisfy* $\operatorname{Re}\lambda < -\varepsilon < 0$. *If* $d \in D_U$ *then there exists at least one positive (real) eigenvalue of* (2.6) *(i.e. also of* (2.5), (1.4)*).*

P r o o f can be done by the same considerations as in Observations 4.1, 4.2 where we will study a slightly different and more complicated eigenvalue problem (see also [17] for the special case $N = 1$ and $\Gamma_D = \emptyset$ or [4] for the general case). □

E x a m p l e 2.1. Let $\mathbb{V}$, A, N be from Weak Formulation 2.1. Consider multivalued mappings $m_j\colon \Gamma_N \times \mathbb{R} \to 2^{\overline{\mathbb{R}}}$, $j = 1, 2$ defined by

$$\left.\begin{aligned} &m_j(x,\xi) = 0 \text{ for } \xi > 0,\\ &m_j(x,\xi) \leqslant 0 \text{ for } \xi < 0,\ \lim_{\xi\to 0_-} m_j(x,\xi) = m_j^0(x),\\ &m_j(x,0) = [m_j^0(x), 0] \quad \text{with some } m_j^0(x) \in [-\infty, 0] \end{aligned}\right\} \quad \text{for all } x \in \Gamma_N.$$

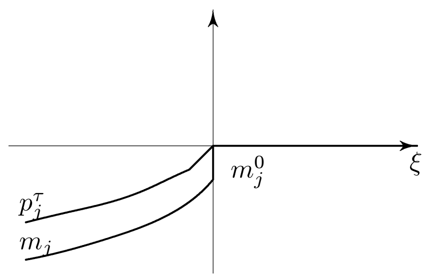

Fig. 2

Let us suppose for the simplicity that

$$\text{if } m_j^0(x) = 0 \text{ for some } x \in \Gamma_N \text{ then}$$
$$\text{either } m_j(x,\xi) = 0 \text{ for all } \xi \in \mathbb{R} \text{ or } m_j \text{ has the left derivative } \frac{\partial_- m_j(x,0)}{\partial \xi} = +\infty.$$

(Of course, this assumption is not necessary. We could consider more general m_j under suitable growth conditions.) Set

$$\left.\begin{aligned} &\underline{m}_j(x,\xi) = \overline{m}_j(x,\xi) = m_j(x,\xi) \quad \text{for } \xi \neq 0,\\ &\underline{m}_j(x,0) = m_j^0(x), \quad \overline{m}_j(x,0) = 0 \end{aligned}\right\} \quad \text{for all } x \in \Gamma_N.$$

Let us denote

$$\Gamma_j = \Big\{ x \in \Gamma_N\,;\ \text{either } \underline{m}_j(x,0) < 0 \text{ or } \underline{m}_j(x,0) = 0 \text{ and } \frac{\partial_- m_j(x,0)}{\partial\xi} = +\infty \Big\}$$

and suppose

(2.7) $$\operatorname{meas}\Gamma_1 > 0 \text{ and } \operatorname{meas}\Gamma_2 > 0.$$

For $x \in \Gamma_N \setminus \Gamma_j$ we have $m_j(x,\xi) = 0$ for all $\xi \in \mathbb{R}$. The multivalued condition (1.2) with such m_j describes for instance a semipermeable membrane on Γ_j allowing the flux only in the direction from the outside source into the domain Ω. If the concentration of u or v at the point $x \in \Gamma_j$ is great enough ($u(x) > \overline{u}$ or $v(x) > \overline{v}$) then the membrane is closed, i.e. there is no flux through the boundary at this point. If the concentration is low ($u(x) < \overline{u}$ or $v(x) < \overline{v}$) then there is a certain amount of flux prescribed by $m_1(x, u(x))$ or $m_2(x, v(x))$. The interval $[m_j^0(x), 0]$ corresponds to the opening/closing of the membrane at the point x when the concentration reaches the prescribed value $\overline{u}$ or $\overline{v}$, respectively. There is no flux through the part of the boundary $\Gamma_N \setminus \Gamma_j$.

Further, define a multivalued mapping $M\colon \widetilde{\mathbb{V}} \to 2^{\widetilde{\mathbb{V}}}$, $M(U) = [M_1(u), M_2(v)]$ for $U = [u, v]$ by

$$M_j(\psi) = \Big\{ z \in \mathbb{V};\ \int_{\Gamma_N} \underline{m}_j(x, \psi(x))\varphi(x)\,\mathrm{d}\Gamma \leqslant \langle z, \varphi\rangle \leqslant \int_{\Gamma_N} \overline{m}_j(x, \psi(x))\varphi(x)\,\mathrm{d}\Gamma \text{ for all } \varphi \in \mathbb{V}, \varphi \geqslant 0 \text{ on } \Gamma_N \Big\}. \tag{2.8}$$

Then a weak solution of the stationary problem corresponding to (1.1), (1.2) can be introduced as a solution of the inclusion written in the vector form

$$D(d)U - BAU + N(U) \in -M(U). \tag{2.9}$$

Let us define multivalued mappings $m_{0j}\colon \Gamma_N \times \mathbb{R} \to 2^{\overline{\mathbb{R}}}$, $j = 1, 2$ and corresponding $\underline{m}_{0j}, \overline{m}_{0j}\colon \Gamma_N \times \mathbb{R} \to \overline{\mathbb{R}}$ by

$$\begin{aligned} &m_{0j}(x,\xi) = \underline{m}_{0j}(x,\xi) = \overline{m}_{0j}(x,\xi) = 0 \ \text{ for } \ \xi > 0,\ x \in \Gamma_N,\\ &m_{0j}(x,0) = [-\infty, 0],\ \underline{m}_{0j}(x,0) = -\infty,\ \overline{m}_{0j}(x,0) = 0 \ \text{ for } x \in \Gamma_j,\\ &m_{0j}(x,\xi) = \underline{m}_{0j}(x,\xi) = \overline{m}_{0j}(x,\xi) = -\infty \ \text{ for } \xi < 0,\ x \in \Gamma_j,\\ &m_{0j}(x,\xi) = \underline{m}_{0j}(x,\xi) = \overline{m}_{0j}(x,\xi) = 0 \ \text{ for all } \xi \in \mathbb{R},\ x \in \Gamma_N \setminus \Gamma_j. \end{aligned} \tag{2.10}$$

Further, consider positively homogeneous multivalued mapping $M_0\colon \widetilde{\mathbb{V}} \to 2^{\widetilde{\mathbb{V}}}$, $M_0(U) = [M_{01}(u), M_{02}(v)]$ corresponding to M and defined by

$$M_{0j}(\psi) = \Big\{ z \in \mathbb{V};\ \int_{\Gamma_N} \underline{m}_{0j}(x, \psi(x))\varphi(x)\,\mathrm{d}\Gamma \leqslant \langle z, \varphi\rangle \leqslant \int_{\Gamma_N} \overline{m}_{0j}(x, \psi(x))\varphi(x)\,\mathrm{d}\Gamma \text{ for all } \varphi \in \mathbb{V}, \varphi \geqslant 0 \text{ on } \Gamma_N \Big\}. \tag{2.11}$$

(Note that $M_{0j}(\psi) = \emptyset$ if $\psi < 0$ on a subset of Γ_j of a positive measure.) The "homogenized" inclusion

$$D(d)U - BAU \in -M_0(U) \tag{2.12}$$

is a weak formulation of the problem (2.5) with $\lambda = 0$ and with the boundary conditions

$$u = v = 0 \text{ on } \Gamma_D,\ \frac{\partial u}{\partial n}(x) \in -m_{01}(x, u(x)),\ \frac{\partial v}{\partial n}(x) \in -m_{02}(x, v(x)) \text{ on } \Gamma_N \tag{2.13}$$

which are equivalent to

$$\begin{aligned}
&u = v = 0 \text{ on } \Gamma_D,\\
&\frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_N \setminus \Gamma_1,\ \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_N \setminus \Gamma_2,\\
&u \geqslant 0,\ \frac{\partial u}{\partial n} \geqslant 0,\ u \cdot \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_1,\ v \geqslant 0,\ \frac{\partial v}{\partial n} \geqslant 0,\ v \cdot \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_2.
\end{aligned}$$

Of course, the problem (2.12) is positively homogeneous but nonlinear again. One of the basic difficulties in the study of our bifurcation problem consists in the fact that it cannot be approximated by a linear problem.

It is not hard to show that the following conditions are fulfilled for such M and M_0 (cf. [14] and [5]):

$$0 \in M(0); \tag{2.14}$$

$$\begin{cases} K = \{U \in \widetilde{\mathbb{V}},\ 0 \in M_0(U)\} \text{ is a closed convex cone in } \widetilde{\mathbb{V}} \\ \text{with the vertex at the origin, } \{0\} \neq K \neq \widetilde{\mathbb{V}}; \end{cases} \tag{2.15}$$

$$M_0(tV) = tM_0(V) \text{ for all } t > 0,\ V \in \widetilde{\mathbb{V}}; \tag{2.16}$$

$$\begin{cases} \text{if } U_n \to 0,\ W_n = \frac{U_n}{\|U_n\|} \rightharpoonup W,\ Z_n \to Z,\ d_n \to d \in \mathbb{R}^2_+, \\ D(d_n)W_n + Z_n \in -\frac{M(U_n)}{\|U_n\|} \text{ then } W_n \to W,\ D(d)W + Z \in -M_0(W); \end{cases} \tag{2.17}$$

$$\text{if } U \in \widetilde{\mathbb{V}} \text{ then } \langle Z, V\rangle \geqslant 0 \text{ for all } V \in K,\ Z \in -M_0(U). \tag{2.18}$$

We have $K = K_1 \times K_2$, $K_j = \{\varphi \in \mathbb{V};\ \varphi \geqslant 0 \text{ on } \Gamma_j\}$ in our situation.

Moreover, there exists a system of real functions $p_j^\tau \colon \Gamma_N \times \mathbb{R} \to \mathbb{R}$ with a parameter $\tau \in [0, +\infty)$ (see Fig. 2) such that the operators $P^\tau U = [P_1^\tau u, P_2^\tau v]$ with

$$\langle P_j^\tau u, \varphi\rangle = \int_{\Gamma_N} p_j^\tau(x, u(x))\varphi(x)\,\mathrm{d}\Gamma \text{ for all } u, \varphi \in \mathbb{V}$$

satisfy the following conditions (2.19)–(2.23) (see [5] for details):

$$\begin{cases} \langle P^\tau(U), U\rangle \geqslant 0 \text{ for all } U \in \widetilde{\mathbb{V}},\ P^\tau(U) = 0 \text{ for all } U \in K,\ \tau \in [0, +\infty), \\ \langle P^\tau(U), V\rangle \leqslant 0 \text{ for all } U \in \widetilde{\mathbb{V}},\ V \in K,\ \tau \in [0, +\infty); \end{cases} \tag{2.19}$$

$$
(2.20)\qquad \begin{cases} \text{if } U_n \rightharpoonup U,\ \tau_n \geqslant 0,\ d_n \to d \in \mathbb{R}^2_+ \\ \qquad\qquad\qquad\qquad \text{then } \liminf \langle D^{-1}(d_n) P^{\tau_n}(U_n), U_n - U\rangle \geqslant 0; \\ \text{if moreover } U = 0,\ \frac{P^{\tau_n}(U_n)}{\|U_n\|} \text{ are bounded and } W_n = \frac{U_n}{\|U_n\|} \rightharpoonup W \\ \qquad\qquad\qquad\qquad \text{then } \liminf \left\langle \frac{D^{-1}(d_n)P^{\tau_n}(U_n)}{\|U_n\|}, W_n - W\right\rangle \geqslant 0; \end{cases}
$$

$$
(2.21)\qquad \begin{cases} \text{if } U_n \rightharpoonup U,\ \tau_n \to \tau \in [0,+\infty) \text{ then } P^{\tau_n}(U_n) \to P^{\tau}(U); \\ \text{if } U_n \to U,\ \tau_n \to +\infty,\ P^{\tau_n}(U_n) \to Z \ \text{ then } \ Z \in M(U); \\ \text{if, moreover, } U = 0,\ W_n = \frac{U_n}{\|U_n\|} \rightharpoonup W,\ \tau_n \to +\infty \\ \ \text{and } \frac{P^{\tau_n}(U_n)}{\|U_n\|} \to Z \text{ then } \ Z \in M_0(W); \end{cases}
$$

$$
(2.22)\qquad \text{if } U_n \to 0,\ \tau_n \to 0 \text{ then } \frac{P^{\tau_n}(U_n)}{\|U_n\|} \to 0.
$$

The interior of K is empty and therefore it is useful to introduce the "pseudointerior" K^- (see Notation 2.1). Particularly, $\varphi \in K_j^-$ for any $\varphi \in \mathbb{V}$ satisfying $\varphi \geqslant \varepsilon$ on Γ_j with some $\varepsilon > 0$. Further,

$$
(2.23)\qquad \begin{cases} \text{if } U_n \to 0,\ W_n = \frac{U_n}{\|U_n\|} \rightharpoonup W \notin K, \tau_n \to \tau \in [0,+\infty),\ V \in K^- \\ \ \text{then } \limsup \langle \frac{P^{\tau_n}(U_n)}{\tau_n \|U_n\|}, V\rangle < 0. \end{cases}
$$

Let us consider a fixed $d^0 \in C_p$ such that there is an eigenfunction e_p of $-\Delta$, (1.4) corresponding to κ_p satisfying

$$
(2.24)\qquad \begin{aligned} & e_p \leqslant -\varepsilon \text{ on } \Gamma_1, e_p \geqslant \varepsilon \text{ on } \Gamma_2 \text{ with some } \varepsilon > 0, \\ & \text{any eigenfunction of } -\Delta,\ (1.4) \text{ corresponding to } \kappa_p \\ & \text{changes its sign on } \Gamma_1 \cup \Gamma_2. \end{aligned}
$$

(Of course, this is possible only if $\Gamma_1 \cap \Gamma_2 = \emptyset$.) Particularly, it follows from Proposition 2.1 that the following condition is fulfilled with $U_0 = [\alpha(d^0)e_p, e_p]$:

$$
(2.25)\qquad E_B(d^0) \cap K = \{0\} \text{ and there exists } U_0 \in E_B(d^0),\ U_0^* \in K^-.
$$

This condition will be essential for our abstract considerations.

R e m a r k 2.2. It is easy to see that $-U \notin K$ for any $U \in K^-$ under the assumption $K \neq \widetilde{\mathbb{V}}$. If $M(U) = [M_1(u), M_2(v)]$, $M_0(U) = [M_{01}(u), M_{02}(v)]$ then M, M_0 satisfy the assumptions (2.14)–(2.18) if and only if M_j, M_{0j} satisfy the same conditions in $\mathbb{V}$. If there are operators $P_1^\tau, P_2^\tau \colon \mathbb{V} \to \mathbb{V}$ satisfying conditions analogous to (2.19)–(2.23) in $\mathbb{V}$ then $P^\tau(U) = [P_1^\tau(u), P_2^\tau(v)]$ satisfy (2.19)–(2.23). Of course, if $K = K_1 \times K_2$, $K_1, K_2 \subset \mathbb{V}$ then $K^- = K_1^- \times K_2^-$.

R e m a r k 2.3. If the operators A and N are from Weak Formulation 2.1, then it follows from Proposition 2.1 that $\dim E_B(d^0) = 1$ for $d^0 \in C_1 \setminus \bigcup_{k=2}^{\infty} C_k$ because the first eigenvalue of $-\Delta$ with (1.4) is simple. Further, if $\dim E_B(d^0) = 1$ and if $K = K_1 \times K_2$, $K_1 \neq \mathbb{V}, K_2 \neq \mathbb{V}$ then $U_0 \notin K, -U_0 \notin K$ is fulfilled automatically under the assumption $U_0^* \in K^-$. Indeed, $U_0^* \in K^-$ means $u_0^* = \frac{b_{21}}{b_{12}} u_0 \in K_1^-, v_0^* = v_0 \in K_2^-$. But $\frac{b_{21}}{b_{12}} < 0$ by (1.5) and therefore $u_0 \notin K_1$, i.e. $U_0 \notin K$. Simultaneously $-v_0 \notin K_2$, i.e. $-U_0 \notin K$.

E x a m p l e 2.2. Let $\mathbb{V}$, A, N be from Weak Formulation 2.1. Consider multivalued mappings $m_j \colon \Gamma_N \times \mathbb{R} \to 2^{\overline{\mathbb{R}}}$, $j = 1, 2$ defined by

$$\left.\begin{array}{l} m_1(x,\xi) = 0 \text{ for } \xi < 0, \\ m_1(x,\xi) \geqslant 0 \text{ for } \xi > 0, \ \lim\limits_{\xi \to 0_+} m_1(x,\xi) = m_1^0(x), \\ m_1(x,0) = [0, m_1^0(x)] \quad \text{with some } m_1^0(x) \in [0,+\infty] \end{array}\right\} \quad \text{for all } x \in \Gamma_N,$$

$$\left.\begin{array}{l} m_2(x,\xi) = 0 \text{ for } \xi > 0, \\ m_2(x,\xi) \leqslant 0 \text{ for } \xi < 0, \ \lim\limits_{\xi \to 0_-} m_2(x,\xi) = m_2^0(x), \\ m_2(x,0) = [m_2^0(x), 0] \quad \text{with some } m_2^0(x) \in [-\infty,0] \end{array}\right\} \quad \text{for all } x \in \Gamma_N.$$

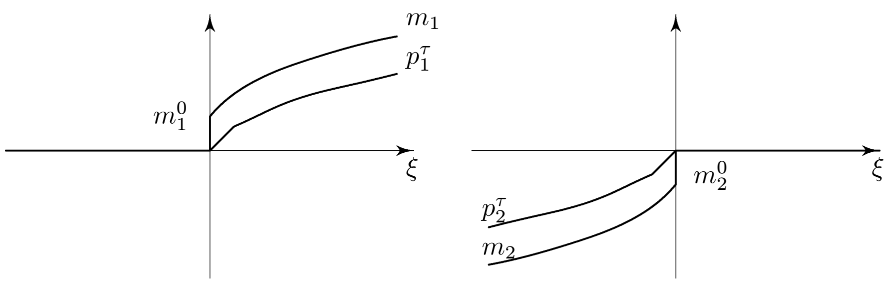

Fig. 3

Set

$$\left.\begin{array}{l} \underline{m}_j(x,\xi) = \overline{m}_j(x,\xi) = m_j(x,\xi) \quad \text{for } \xi \neq 0 \\ \underline{m}_1(x,0) = 0, \quad \overline{m}_1(x,0) = m_1^0(x) \\ \underline{m}_2(x,0) = m_2^0(x), \quad \overline{m}_2(x,0) = 0 \end{array}\right\} \quad \text{for all } x \in \Gamma_N.$$

Let us suppose for the simplicity that

$$\text{if } m_1^0(x) = 0 \text{ for some } x \in \Gamma_N \text{ then}$$
$$\text{either } m_1(x,\xi) = 0 \text{ for all } \xi \in \mathbb{R} \text{ or } m_1 \text{ has the right derivative } \frac{\partial_+ m_1(x,0)}{\partial \xi} = +\infty,$$
$$\text{if } m_2^0(x) = 0 \text{ for some } x \in \Gamma_N \text{ then}$$
$$\text{either } m_2(x,\xi) = 0 \text{ for all } \xi \in \mathbb{R} \text{ or } m_2 \text{ has the left derivative } \frac{\partial_- m_2(x,0)}{\partial \xi} = +\infty.$$

Let us denote

$$\Gamma_1 = \{x \in \Gamma_N;\ \text{either } \overline{m}_1(x,0) > 0 \text{ or } \overline{m}_1(x,0) = 0 \text{ and } \frac{\partial_+ m_1(x,0)}{\partial \xi} = +\infty\},$$
$$\Gamma_2 = \{x \in \Gamma_N;\ \text{either } \underline{m}_2(x,0) < 0 \text{ or } \underline{m}_2(x,0) = 0 \text{ and } \frac{\partial_- m_2(x,0)}{\partial \xi} = +\infty\}$$

and suppose (2.7) again. Let us define multivalued mappings $m_{0j}\colon \Gamma_N \times \mathbb{R} \to 2^{\overline{\mathbb{R}}}$, $j = 1,2$ and corresponding $\underline{m}_{0j}, \overline{m}_{0j}\colon \Gamma_N \times \mathbb{R} \to \overline{\mathbb{R}}$ by

$$m_{01}(x,\xi) = \underline{m}_{01}(x,\xi) = \overline{m}_{01}(x,\xi) = 0 \quad \text{for } \xi < 0,\ x \in \Gamma_N,$$
$$m_{01}(x,0) = [0,+\infty],\ \underline{m}_{01}(x,0) = 0,\ \overline{m}_{01}(x,0) = +\infty \quad \text{for } x \in \Gamma_1,$$
$$m_{01}(x,\xi) = \underline{m}_{01}(x,\xi) = \overline{m}_{01}(x,\xi) = +\infty \quad \text{for } \xi > 0,\ x \in \Gamma_1,$$
$$m_{01}(x,\xi) = \underline{m}_{01}(x,\xi) = \overline{m}_{01}(x,\xi) = 0 \quad \text{for all } \xi \in \mathbb{R},\ x \in \Gamma_N \setminus \Gamma_1,$$

$$m_{02}(x,\xi) = \underline{m}_{02}(x,\xi) = \overline{m}_{02}(x,\xi) = 0 \quad \text{for } \xi > 0,\ x \in \Gamma_N,$$
$$m_{02}(x,0) = [-\infty,0],\ \underline{m}_{02}(x,0) = -\infty,\ \overline{m}_{02}(x,0) = 0 \quad \text{for } x \in \Gamma_2,$$
$$m_{02}(x,\xi) = \underline{m}_{02}(x,\xi) = \overline{m}_{02}(x,\xi) = -\infty \quad \text{for } \xi < 0,\ x \in \Gamma_2,$$
$$m_{02}(x,\xi) = \underline{m}_{02}(x,\xi) = \overline{m}_{02}(x,\xi) = 0 \quad \text{for all } \xi \in \mathbb{R},\ x \in \Gamma_N \setminus \Gamma_2.$$

Further, define multivalued mappings $M, M_0\colon \widetilde{\mathbb{V}} \to 2^{\widetilde{\mathbb{V}}}$, $M(U) = [M_1(u), M_2(v)]$ and $M_0(U) = [M_{01}(u), M_{02}(v)]$ by (2.8) and (2.11), respectively, as in Example 2.1. The corresponding convex cones are $K_1 = \{\varphi \in \mathbb{V};\ \varphi \leqslant 0 \text{ on } \Gamma_1\}$, $K_2 = \{\varphi \in \mathbb{V};\ \varphi \geqslant 0 \text{ on } \Gamma_2\}$, $K = K_1 \times K_2$.
Such M and M_0 satisfy the conditions (2.14)–(2.18) and there exist operators P^τ such that the conditions (2.19)–(2.23) are fulfilled. If $d^0 \in C_p$ and there exists an eigenfunction e_p of $-\Delta$ with (1.4) corresponding to κ_p such that

$$e_p \geqslant \varepsilon > 0 \text{ on } \Gamma_1 \cup \Gamma_2,$$
$$\text{there is no eigenfunction } e \text{ of } -\Delta,\ (1.4) \text{ corresponding to } \kappa_p \qquad (2.26)$$
$$\text{satisfying } e \leqslant 0 \text{ on } \Gamma_1, e \geqslant 0 \text{ on } \Gamma_2$$

then it follows from Proposition 2.1 that the assumption (2.25) is fulfilled. This is automatically true in the case $d^0 \in C_1 \setminus \bigcup_{k=2}^{\infty} C_k$, because the first eigenfunction of the Laplacian is positive.

E x a m p l e 2.3. Analogously as in Example 2.1 we can consider multivalued mappings $m_j\colon \Omega \times \mathbb{R} \to 2^{\overline{\mathbb{R}}}$, $j = 1, 2$ defined by

$$\left.\begin{aligned} &m_j(x,\xi) = 0 \text{ for } \xi > 0,\\ &m_j(x,\xi) \leqslant 0 \text{ for } \xi < 0,\ \lim_{\xi \to 0_-} m_j(x,\xi) = m_j^0(x),\\ &m_j(x,0) = [m_j^0(x), 0] \quad \text{with some } m_j^0(x) \in [-\infty, 0] \end{aligned}\right\} \text{ for all } x \in \Omega.$$

Let us suppose for the simplicity that

$$\text{if } m_j^0(x) = 0 \text{ for some } x \in \Omega \text{ then}$$
$$\text{either } m_j(x,\xi) = 0 \text{ for all } \xi \in \mathbb{R} \text{ or } m_j \text{ has the left derivative } \frac{\partial_- m_j(x,0)}{\partial \xi} = +\infty.$$

Set

$$\left.\begin{aligned} &\underline{m}_j(x,\xi) = \overline{m}_j(x,\xi) = m_j(x,\xi) \quad \text{for } \xi \neq 0\\ &\underline{m}_j(x,0) = m_j^0(x), \quad \overline{m}_j(x,0) = 0 \end{aligned}\right\} \text{ for all } x \in \Omega.$$

Further, define a multivalued mapping $M\colon \widetilde{\mathbb{V}} \to 2^{\widetilde{\mathbb{V}}}$, $M(U) = [M_1(u), M_2(v)]$ by

$$\begin{aligned} M_j(\psi) = \{z \in \mathbb{V};\ \int_\Omega \underline{m}_j(x,\psi(x))\varphi(x)\,\mathrm{d}x \leqslant \langle z,\varphi\rangle \leqslant \int_\Omega \overline{m}_j(x,\psi(x))\varphi(x)\,\mathrm{d}x\\ \text{for all } \varphi \in \mathbb{V}, \varphi \geqslant 0 \text{ on } \Omega\}. \end{aligned} \tag{2.27}$$

If $G_j = \{x \in \Omega;$ either $\underline{m}_j(x,0) < 0$ or $\underline{m}_j(x,0) = 0$ and $\frac{\partial_- m_j(x,0)}{\partial \xi} = +\infty\}$, $j = 1, 2$ are subsets of Ω such that $\overline{G_j} \subset \Omega$,

$$\operatorname{meas} G_1 > 0 \text{ and } \operatorname{meas} G_2 > 0 \tag{2.28}$$

then a solution of (2.9) is a weak solution of the problem

$$d_1 \Delta u + f(u,v) \in m_1(x,u)\ d_2 \Delta v + g(u,v) \in m_2(x,v) \text{ on } \Omega \tag{2.29}$$

with the boundary conditions (1.4). Particularly

$$d_1 \Delta u + f(u,v) = 0 \text{ on } \Omega \setminus \overline{G}_1,\ d_2 \Delta v + g(u,v) = 0 \text{ on } \Omega \setminus \overline{G}_2.$$

Such a model describes a similar situation as in Example 2.1 with the source in the interior of the domain Ω.

Analogously as in Example 2.1, we have $K = K_1 \times K_2$ with $K_j = \{\varphi \in \mathbb{V};\ \varphi \geqslant 0 \text{ on } G_j\}$. Further, if $d^0 \in C_p$ and there exists an eigenfunction e_p of $-\Delta$ with (1.4) corresponding to κ_p such that

$$\begin{aligned} &e_p \leqslant -\varepsilon \text{ on } G_1 \text{ and } e_p \geqslant \varepsilon \text{ on } G_2 \text{ with some } \varepsilon > 0,\\ &\text{any eigenfunction of } -\Delta,\ (1.4) \text{ corresponding to } \kappa_p \\ &\text{changes its sign on } G_1 \cup G_2 \end{aligned} \tag{2.30}$$

then the assumption (2.25) is fulfilled by Proposition 2.1. The penalty operator corresponding to M is $P^\tau U = [P_1^\tau u, P_2^\tau v]$ with $\langle P_j^\tau u, \varphi\rangle = \int_\Omega p_j^\tau(x, u(x))\varphi(x)\,\mathrm{d}x$ for all $u, \varphi \in \mathbb{V}$ where p_j^τ are suitable (real) functions.

R e m a r k 2.4. Let $\Psi\colon \widetilde{\mathbb{V}} \to (-\infty, +\infty]$ be a positive convex lower semicontinuous functional such that $\Psi \not\equiv +\infty$. Let us define the multivalued mapping $M = \partial\Psi$—the subdifferential of Ψ. Then (2.9) is equivalent to

$$\langle D(d)U - BAU + N(U), Z - U\rangle + \Psi(Z) - \Psi(U) \geqslant 0 \qquad \text{for any } Z \in \widetilde{\mathbb{V}} \tag{2.31}$$

(see [2]).

3. Main results

In the sequel, we will consider a general real Hilbert space $\mathbb{V}$, operators A, N satisfying (2.1), a multivalued mapping M for which there exists a corresponding multivalued positively homogeneous operator M_0 such that the conditions (2.14)–(2.18) are fulfilled and such that there exist operators P^τ satisfying (2.19)–(2.23).

Further, we will consider a differentiable curve $\sigma\colon (-\infty, +\infty) \to \mathbb{R}^2_+$, $\sigma = [\sigma_1, \sigma_2]$ such that

$$\begin{aligned} &\sigma(s) \in D_S \text{ for } s > s_0, \sigma(s) \in D_U \text{ for } s < s_0,\\ &\lim_{s\to+\infty} \sigma_1(s) = +\infty,\ \liminf_{s\to+\infty} \sigma_2(s) > 0. \end{aligned} \tag{3.1}$$

We will study the problems (2.2) and (2.9) only on the curve σ, i.e. the problems

$$D(\sigma(s))U - BAU + N(U) = 0 \tag{3.2}$$

and

$$D(\sigma(s))U - BAU + N(U) \in -M(U) \tag{3.3}$$

with a single bifurcation parameter s.

Proposition 3.1. *If* (1.5), (2.1) *and* (3.1) *hold then there is no bifurcation point of* (3.2) *greater than* s_0.

P r o o f follows directly from Proposition 2.1 and the well-known fact that any bifurcation point s of (3.2) is simultaneously a critical point, i.e. $E_B(\sigma(s)) \neq \{0\}$. $\square$

Notice that if, moreover, σ intersects C transversally at $\sigma(s_0)$ then s_0 really is a bifurcation point of (1.1), (1.4) under reasonable assumptions (see e.g. [15], [17]).

Theorem 3.1. *Let* (1.3), (1.5), (2.1) *hold, let* $M\colon \widetilde{\mathbb{V}} \to 2^{\widetilde{\mathbb{V}}}$ *be a multivalued mapping for which there exists a corresponding homogenization* $M_0\colon \widetilde{\mathbb{V}} \to 2^{\widetilde{\mathbb{V}}}$ *such that* (2.14)–(2.18) *hold. Suppose that there exist operators* $P^\tau\colon \widetilde{\mathbb{V}} \to \widetilde{\mathbb{V}}$ ($\tau \in [0, +\infty)$) *satisfying* (2.19)–(2.23). *Consider a differentiable curve* σ *satisfying* (3.1) *and intersecting* C *at* $d^0 = \sigma(s_0)$. *Suppose that* $d^0 \in C_p$, (2.25) *holds with some* $U_0 = U_p = [\alpha(d)e_0, e_0]$ *where* e_0 *is an eigenvector of* A *corresponding to* κ_p *and that* σ *intersects* C_p *transversally at* d^0. * *Then there exists a bifurcation point* $s_I > s_0$ *of* (3.3).

Corollary 3.1. *Let* (1.3), (1.5) *hold, let* σ *satisfy* (3.1) *and intersect* C *at* $d^0 = \sigma(s_0)$. *Suppose that* $d^0 \in C_p$ *and* σ *intersects* C_p *transversally. Let* m_1, m_2 *be from Example* 2.1 *or* 2.2 *or* 2.3 *and suppose that there exists an eigenfunction* e_p *of* $-\Delta$ *with* (1.4) *corresponding to* κ_p *such that* (2.24) *or* (2.26) *or* (2.30), *respectively, is fulfilled. Then stationary spatially nonconstant weak solutions (spatial patterns) of* (1.1) *with* (1.2) *or* (2.29) *with* (1.4), *respectively, bifurcate at some* $s_I > s_0$.

P r o o f follows from Theorem 3.1, Examples 2.1 or 2.2 or 2.3, respectively, and the fact that no nontrivial constant functions can satisfy (1.2) (with $\bar{u} = \bar{v} = 0$). $\square$

R e m a r k 3.1. Theorem 3.1 (particularly Corollary 3.1) asserts that a bifurcation of solutions to (3.3) (particularly a bifurcation of stationary spatially nonconstant solutions to (1.1) with (1.2)) occurs in the domain where a bifurcation for the corresponding equation (3.2) (particularly for (1.1), (1.4)) is excluded by Proposition 3.1.

---

* The case when d^0 is an intersection point of two different hyperbolas $C_p \neq C_q$ is not excluded but it is essential that σ intersects transversally the hyperbola with the index p corresponding to $U_0 = U_p$ from the assumption (2.25) of the form described—see also Proposition 2.1.

This can be understood as a destabilizing effect of unilateral conditions—cf. e.g. [12], [13]. The first result of this type was proved in [3] for a particular case of conditions given by variational inequalities with cones $K = \mathbb{V} \times K_2$ (i.e. for unilateral conditions only for the inhibitor v) with $\operatorname{int} K_2 \neq \emptyset$ and for the special curve $\sigma_2(s) = 1$. In this case, the condition (2.25) has the form $U_0 \in \operatorname{int} K$. A destabilizing effect of such unilateral conditions in terms of instability of the trivial solution of the corresponding linearized inequality was shown in [4] (already for cones with empty interior). The idea of the proofs in these papers (analogously to the proof of our Theorem 3.1) consists in a certain homotopical joining of the inequality with the corresponding linearized equation. This idea is taken from [9] where it was used for inequalities of a simpler type.

Another method for the investigation of bifurcations for inequalities was found by P. Quittner [19] who introduced (in a slightly different way than in our Notation 2.1) a pseudointerior and strengthened the results mentioned (see [20]). His method is based on a reformulation of the inequality as a (strongly nonlinear) equation with the projection onto K and a direct use of the Leray-Schauder degree (a jump of the degree implies a bifurcation). This method is simpler than the homotopy approach mentioned but, unfortunately, in some cases it is not clear how to use it, e.g. for the proof of Theorem 3.1 when nonstandard conditions are prescribed for both u and v.

A destabilizing effect of unilateral conditions for the inhibitor in the case of a general curve was proved for quasivariational inequalities in [12] by using Quittner's method and for inclusions in [13] by using the homotopy method (see also a forthcoming paper of J. Eisner [5]).

On the other hand, if unilateral conditions are prescribed only for the activator u then a bifurcation can occur only in the interior of the domain of instability of the corresponding equation under certain assumptions. This stabilizing effect of unilateral conditions is proved in [14] for inclusions and in [10] for quasivariational inequalities.

4. Proof of the main results

In the sequel, we will consider the assumptions of Theorem 3.1 with $d^0 \in C_p$.

Observation 4.1. (Cf. [4], Section 2.) Let us consider an eigenvalue problem

$$D^{-1}(d)BAU - U = \mu U. \tag{4.1}$$

We can write $\varphi = \sum\limits_{i=1}^{+\infty} \langle \varphi, e_i \rangle e_i$ for any $\varphi \in \mathbb{V}$ under the assumption (1.3), where e_i are eigenvectors of the operator A (see Notation 2.1). Writing (4.1) as two single equations and multiplying by e_i we have

$$\langle u, e_i\rangle(b_{11} - d_1\kappa_i - \mu d_1\kappa_i) + \langle v, e_i\rangle b_{12} = 0,$$
$$\langle u, e_i\rangle b_{21} + \langle v, e_i\rangle(b_{22} - d_2\kappa_i - \mu d_2\kappa_i) = 0,$$

$i = 1, 2, \ldots$. The couple of $\langle u, e_i\rangle$, $\langle v, e_i\rangle$ can be nontrivial for some i if and only if

$$(b_{11} - d_1\kappa_i - \mu d_1\kappa_i)(b_{22} - d_2\kappa_i - \mu d_2\kappa_i) - b_{12}b_{21} = 0.$$

Hence, μ is an eigenvalue of (4.1) if and only if μ is a root of

$$\mu^2 d_1 d_2 \kappa_i^2 - \mu\beta_i(d)\kappa_i + \gamma_i(d) = 0 \tag{4.2}$$

$$\text{with} \quad \beta_i(d) = d_1 b_{22} + d_2 b_{11} - 2d_1 d_2\kappa_i,$$
$$\gamma_i(d) = (b_{11} - d_1\kappa_i)(b_{22} - d_2\kappa_i) - b_{12}b_{21}$$

for at least one i. Let us denote by $\mu_i^{1,2}(d)$ the roots of (4.2), set $\omega(d) = \beta_i^2(d) - 4d_1d_2\gamma_i(d)$. Hence, $\mu_i^1(d)$, $\mu_i^2(d)$, $i = 1, 2, \ldots$ are all eigenvalues of (4.1). An elementary investigation gives the following information (see [5] for details). The number $\omega(d)$ does not depend on i and $\mathcal{T} = \{d;\ d_2 = d_1\frac{\det B - b_{12}b_{21} + 2\sqrt{-b_{12}b_{21}\det B}}{b_{11}^2}\} = \{d;\ \omega(d) = 0\}$ is the joint tangent to all hyperbolas C_j, $j = 1, 2, \ldots$ (Cf. also [15]). For d lying to the left from C_i we have $\mu_i^1(d) \neq \mu_i^2(d)$, one of them is positive, the second one is negative. Further, for $d \in C_i \setminus \mathcal{T}$, one of the roots $\mu_i^{1,2}(d)$ is zero, the second one has the same sign as $\beta_i(d) \neq 0$. For $d \in C_i \cap \mathcal{T}$ we have $\beta_i(d) = \gamma_i(d) = 0$ and therefore both $\mu_i^{1,2}(d) = 0$. For d lying strictly between C_i and $\mathcal{T}$ we have $\mu_i^1(d) \neq \mu_i^2(d)$ and $\operatorname{sign}\mu_i^{1,2}(d) = \operatorname{sign}\beta_i(d) \neq 0$. If d lies below $\mathcal{T}$ then the roots are complex (not real). We have $\mu_i^1(d) = \mu_i^2(d)$ if and only if $d \in \mathcal{T}$.

Further, we will suppose that the characteristic value κ_p of A (i.e. the eigenvalue κ_p of $-\Delta$ with (1.4) in the situation from Weak Formulation 2.1) has a multiplicity k, $\kappa_p = \ldots = \kappa_{p+k-1}$. Hence, $C_p = \ldots = C_{p+k-1}$ and Observation 4.1 yields $\mu_p^{1,2}(d) = \ldots = \mu_{p+k-1}^{1,2}(d)$ for any d.

N o t a t i o n 4.1. Let $\mathcal{U}$ be a neighbourhood of d^0 such that $\mathcal{U} \setminus \{d^0\}$ contains no intersection point of C_p with the other hyperbolas C_j, $j \notin \{p, \ldots, p+k-1\}$ and in the case $d^0 \in C_p \setminus \mathcal{T}$, moreover no point $d \in \mathcal{T}$. If $d^0 \in C_p \setminus \mathcal{T}$ then we will denote by $\mu_p(d) = \ldots = \mu_{p+k-1}(d)$, $d \in \mathcal{U}$, the root changing the sign for d crossing $C_p \cap \mathcal{U}$. If $d^0 \in C_p \cap \mathcal{T}$ then $\mu_p(d) = \ldots = \mu_{p+k-1}(d)$ will denote the positive root for $d \in \mathcal{U}$, d lying to the left from $C_p = \ldots = C_{p+k-1}$ and $\mu_p(d) = \ldots = \mu_{p+k-1}(d)$ will denote the real part of complex roots for d lying below $\mathcal{T}$. (In fact we shall consider our problem only on the curve $\sigma(s)$ intersecting C_p transversally at d^0 and therefore we need no definition for d lying between C_p and $\mathcal{T}$ in the case $d^0 \in C_p \cap \mathcal{T}$.) Further, define

$$
(4.3)\qquad
\begin{aligned}
U_i^{1,2}(d) &= \Big[\frac{d_2\kappa_i - b_{22} + \mu_i^{1,2}(d) d_2\kappa_i}{b_{21}} e_i, e_i\Big], \quad i \in \mathbb{N},\ d \in \mathcal{U}\\
U_i(d) &= \Big[\frac{d_2\kappa_i - b_{22} + \mu_i(d) d_2\kappa_i}{b_{21}} e_i, e_i\Big], \quad \begin{cases} i = p, \dots, p+k-1,\\ d \text{ such that } \mu_i(d) \text{ is defined.}\end{cases}
\end{aligned}
$$

Observation 4.2. For all $i \in \mathbb{N}$, $d \in \mathcal{U}$, $U_i^{1,2}(d)$ are eigenvectors of (4.1) corresponding to $\mu_i^{1,2}(d)$. Moreover, $\operatorname{Ker}(D^{-1}(d)BA - I - \mu_i^r(d)I) = \operatorname{Lin}\{U_j^r(d);\ j \in \mathbb{N},\ \mu_j^r(d) = \mu_i^r(d)\}$ for any $i = 1, 2, \dots$, $r = 1, 2$.
If $d^0 \in C_p \setminus \mathcal{T}$ then $U_i(d)$ are eigenvectors corresponding to $\mu_i(d) = \mu_p(d)$ for $i = p, \dots, p+k-1$, $d \in \mathcal{U}$. If $d^0 \in C_p \cap \mathcal{T}$ then $U_i(d)$ are eigenvectors corresponding to $\mu_i(d)$ for $i = p, \dots, p+k-1$, $d \in \mathcal{U}$, d lying to the left from C_p and they are real parts of complex eigenvectors corresponding to $\mu_i^{1,2}(d)$ for $d \in \mathcal{U}$, d lying below $\mathcal{T}$. (See Observation 4.1, Notation 4.1.) Note that the vectors $U_i(d)$ from (4.3) for $d \in C_i$, $i = p, \dots, p+k-1$ coincide with those introduced in Proposition 2.1.
If $\mu_q(d) \neq \mu_p(d)$ for all q satisfying $\kappa_q \neq \kappa_p$ then $\operatorname{Ker}(D^{-1}(d)BA - I - \mu_p(d)I) = \operatorname{Lin}\{U_i(d)\}_{i=p}^{p+k-1}$. Particularly, if $d \in C_p$ (i.e. $\mu_p(d) = 0$) and $d \notin C_q$ for all $C_q \neq C_p$, then

$$
(4.4)\qquad E_B(d) = \operatorname{Lin}\{U_i(d)\}_{i=p}^{p+k-1}.
$$

If $\mu_q(d) = \mu_p(d)$ for some q satisfying $\kappa_p \neq \kappa_q = \dots = \kappa_{q+l-1}$, where κ_q has a multiplicity l, then $\operatorname{Ker}(D^{-1}(d)BA - I - \mu_p(d)I) = \operatorname{Lin}\{U_i(d)\}_{i=p,\dots,p+k-1,q,\dots,q+l-1}$. Particularly, if $d \in C_p \cap C_q$ for some $C_q \neq C_p$, then

$$
(4.5)\qquad E_B(d) = \operatorname{Lin}\{U_i(d)\}_{i=p,\dots,p+k-1,q,\dots,q+l-1}.
$$

Remark 4.1. Let $d^0 = [d_1^0, d_2^0] \in C_p$. Let (2.25) hold with some $U_0 = [\alpha(d)e_0, e_0]$ where e_0 is an eigenvector of A corresponding to the characteristic value $\kappa_p = \dots = \kappa_{p+k-1}$ with a multiplicity k (see Observation 4.2). Then the system $\{e_i\}_{i=1}^{\infty}$ can be chosen such that $U_0 = U_p(d^0) = \Big[\frac{d_2^0\kappa_p - b_{22}}{b_{21}} e_p, e_p\Big]$ with $U_p^*(d^0) = \Big[\frac{d_2^0\kappa_p - b_{22}}{b_{12}} e_p, e_p\Big] \in K^-$.

Notation 4.2. Set $I(d^0) = \{i \in \mathbb{N} \setminus \{p\};\ d^0 \in C_i\}$. Let us choose $\eta > 0$ small enough. Let ξ be a continuous function such that $\xi(s_0) = 1$, $\xi(s) \in (0,1)$ for $s \in (s_0 - \eta, s_0) \cup (s_0, s_0 + \eta)$ and $\xi(s) = 0$ for $s \notin (s_0 - \eta, s_0 + \eta)$. For any $\delta > 0$ small, introduce the linear completely continuous operator $L_\delta(s) : \widetilde{\mathbb{V}} \to \widetilde{\mathbb{V}}$ (for any

$s \in \mathbb{R}$ fixed) by

$$\begin{aligned} L_\delta(s)U &= 0 \quad \text{for } s \notin (s_0 - \eta, s_0 + \eta) \\ &= \delta \cdot \sum_{i \in I(d^0)} \xi(s) \frac{\langle U_i(\sigma(s)), U\rangle}{\|U_i(\sigma(s))\|^2} \cdot U_i(\sigma(s)) \quad \text{for } s \in [s_0 - \eta, s_0 + \eta] \end{aligned}$$

(cf. [4]).

R e m a r k 4.2. We have $L_\delta(s) \equiv 0$ for $s \in \mathbb{R}$ if $I(d^0) = \emptyset$, i.e. if $\dim E_B(d^0) = 1$. Further, it follows from the form of $U_i^{1,2}(d)$, $U_i(d)$, $U_i^*(d)$ (see Notations 4.1 and 4.2, Remark 2.1) that $\langle U_i^{1,2}(d), U_j^{1,2}(d)\rangle = \langle U_i(d), U_j(d)\rangle = \langle U_i^*(d), U_j(d)\rangle = 0$ for all $j \neq i$ and

$$\begin{aligned} &L_\delta(s)U_p(\sigma(s)) = 0,\ L_\delta(s)U_i^{1,2}(\sigma(s)) = 0 \text{ for } i \notin I(d^0),\ s \in (s_0 - \eta, s_0 + \eta), \\ &\langle D(\sigma(s))L_\delta(s)U, U_p(\sigma(s))\rangle = \langle D(\sigma(s))L_\delta(s)U, U_p^*(\sigma(s))\rangle = 0 \\ &\qquad\qquad \text{for any } s \in (s_0 - \eta, s_0 + \eta),\ U \in \widetilde{\mathbb{V}}. \end{aligned} \tag{4.6}$$

Lemma 4.1. *There exist $\delta > 0$ and $\eta > 0$ such that the following assertions hold. (a) Let $d^0 \in C_p \setminus \mathcal{T}$. Then for all $(s_0 - \eta, s_0 + \eta)$, the eigenvalue $\mu_p(\sigma(s))$ from Notation 4.1 is simultaneously an algebraically simple eigenvalue of the operator $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ with the corresponding eigenvector $U_p(\sigma(s))$. It changes the sign as s crosses s_0. The other positive eigenvalues have constant signs and constant multiplicities on $(s_0 - \eta, s_0 + \eta)$.*
(b) Let $d^0 \in C_p \cap \mathcal{T}$. Then for $s \in (s_0 - \eta, s_0]$, $\mu_p(\sigma(s))$ is an eigenvalue of $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ with the only normed eigenvector $U_p(\sigma(s))$. For $s \in (s_0 - \eta, s_0)$, $\mu_p(\sigma(s))$ is positive and algebraically simple, $\mu_p(\sigma(s_0)) = 0$ is not algebraically simple. The sum of algebraic multiplicities of the other positive eigenvalues of $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ is even for all $s \in (s_0 - \eta, s_0)$. For $s \in (s_0, s_0 + \eta)$, all eigenvalues of this operator are complex.
In both cases (a), (b), $\operatorname{Ker}(D^{-1}(\sigma(s_0))BA - L_\delta(s_0) - I) = \operatorname{Lin}\{U_p(d^0)\}$ and the number $\nu(s_0 - \varepsilon) - \nu(s_0 + \varepsilon)$ is odd for all $\varepsilon \in (0, \eta)$ where $\nu(s)$ is the sum of algebraic multiplicities of all positive eigenvalues of the operator $D^{-1}(\sigma(s))BA - L_\delta(s) - I$.

P r o o f. Analogously as in Observation 4.1 we obtain that μ is an eigenvalue of the problem

$$D^{-1}(\sigma(s))BAU - L_\delta(s)U - U = \mu U$$

if and only if μ is a root of a quadratic equation

$$\mu^2 - \beta_i^\delta(s)\mu + \gamma_i^\delta(s) = 0 \tag{4.7}$$

with coefficients $\beta_i^\delta(s)$, $\gamma_i^\delta(s)$ depending continuously on s and δ.

Let $i \notin I(d^0)$. It follows from (4.6) that $\mu_i^{1,2}(\sigma(s))$ and $U_i^{1,2}(\sigma(s))$ from Observation 4.1 and Notation 4.1 are simultaneously eigenvalues and eigenvectors of $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ and (4.7) is equivalent to (4.2) for any $s \in \mathbb{R}$. Particularly, this means by the definition of $\mu_p(d)$, $U_p(d)$ that $\mu_p(\sigma(s))$ and $U_p(\sigma(s))$ is an eigenvalue and an eigenvector of $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ for any $s \in (s_0 - \eta, s_0 + \eta)$ or $s \in (s_0 - \eta, s_0]$ in the case $d^0 \in C_p \setminus \mathcal{T}$ or $d^0 \in C_p \cap \mathcal{T}$, respectively.

If $i \notin I(d^0) \cup \{p\}$ then d^0 and also $\sigma(s)$ for any $s \in (s_0 - \eta, s_0 + \eta)$ lie to the right from C_i. (Recall that $d^0 \in C$.) It follows from Observation 4.1 that if $d^0 \in C_p \setminus \mathcal{T}$, $i \notin I(d^0) \cup \{p\}$ then the sign of both $\mu_i^1(\sigma(s)) \neq \mu_i^2(\sigma(s))$ is constant on $(s_0 - \eta, s_0 + \eta)$, and if $d^0 \in C_p \cap \mathcal{T}$, $i \notin I(d^0) \cup \{p\}$ then $\mu_i^1(\sigma(s)) \neq \mu_i^2(\sigma(s))$ are both positive or negative on $(s_0 - \eta, s_0)$ and complex on $(s_0, s_0 + \eta)$. Further, for $i = p$, $\mu_p(\sigma(s))$ changes its sign at s_0 and the sign of the corresponding second root is constant on $(s_0 - \eta, s_0 + \eta)$ in the case $d^0 \in C_p \setminus \mathcal{T}$. In the case $d^0 \in C_p \cap \mathcal{T}$ we have $\mu_p(\sigma(s)) > 0$ and the second root is negative on $(s_0 - \eta, s_0)$, both roots are complex on $(s_0, s_0 + \eta)$.

Let $i \in I(d^0)$. Notations 4.1, 4.2 yield that $\mu_i(\sigma(s)) - \delta\xi(s) = \mu_p(\sigma(s)) - \delta\xi(s)$ is an eigenvalue of $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ and one of the roots of (4.7). It follows from Notation 4.1 and Observation 4.1 that we can choose $\delta > 0$ and $\eta > 0$ such that $\mu_i(\sigma(s)) - \delta\xi(s) < 0$ on $(s_0 - \eta, s_0 + \eta)$. The roots of (4.7) depend continuously on $s \in \mathbb{R}$, $\delta \geqslant 0$ and therefore the choice of $\delta > 0$ and $\eta > 0$ can be such that the second root has the constant sign on $(s_0 - \eta, s_0 + \eta)$ in the case $d^0 \in C_p \setminus \mathcal{T}$ and it is negative on $(s_0 - \eta, s_0]$ and complex on $(s_0, s_0 + \eta)$ in the case $d^0 \in C_p \cap \mathcal{T}$. (See Observation 4.1.)

It follows from the relation of eigenvalues of the operator $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ and the roots of (4.7) mentioned above that there are no further eigenvalues and eigenvectors besides those discussed.

Let us show that for $i \in \mathbb{N}$, $r = 1, 2$ and $s \in (s_0 - \eta, s_0 + \eta)$ or $s \in (s_0 - \eta, s_0)$ in the case $d^0 \in C_p \setminus \mathcal{T}$ or $d^0 \in C_p \cap \mathcal{T}$, respectively, the algebraic and geometric multiplicity of the eigenvalue $\mu_i^r(\sigma(s))$ coincide. (In fact, we need this information only for positive eigenvalues.) The adjoint equation to (4.1) is

$$B^* D^{-1}(d)AU - U = \mu U$$

and similar considerations as in Observation 4.1 imply that the eigenvectors of this equation corresponding to $\mu_i^{1,2}(d)$ are

$$\widetilde{U}_i^{1,2}(d) = \Big[\frac{d_1}{d_2}\frac{d_2\kappa_i - b_{22} + \mu_i^{1,2}(d)d_2\kappa_i}{b_{12}}e_i, e_i\Big] = \Big[\frac{d_1}{d_2}\frac{b_{21}}{b_{12}}\alpha_i^{1,2}(d)e_i, e_i\Big].$$

(Recall that $U_i^{1,2}(d) = \left[\alpha_i^{1,2}(d)e_i, e_i\right]$—see Observation 4.2.) We have

$$\langle L_\delta^*(s)\widetilde{U}_i^r(\sigma(s)), V\rangle = \langle \widetilde{U}_i^r(\sigma(s)), L_\delta(s)V\rangle = 0 \text{ for } i \notin I(d^0),\ r = 1,2,\ V \in \mathbb{V},$$

that means

$$L_\delta^*(s)\widetilde{U}_i^r(\sigma(s)) = 0 \text{ for all } i \notin I(d^0),\ r = 1,2,\ s \in (s_0 - \eta, s_0 + \eta).$$

It follows that $\widetilde{U}_i^r(\sigma(s))$ is simultaneously an eigenvector of the adjoint operator $\left(D^{-1}(\sigma(s))BA\right)^* - L_\delta^*(s) - I$ corresponding to $\mu_i^r(\sigma(s))$, $r = 1, 2$. An elementary calculation gives

$$|\langle U_i^r(d), \widetilde{U}_i^r(d)\rangle| = \frac{\sqrt{\omega(d)}}{d_2} \neq 0 \text{ for } i = 1,2,\dots,\ r = 1,2,\ d \in \mathcal{U} \setminus \mathcal{T},$$
$$\langle U_i^r(d), \widetilde{U}_j^r(d)\rangle = 0 \text{ for any } i \neq j,\ r = 1,2.$$

Hence, $\det\left(\langle \widetilde{U}_i^r(d), U_j^r(d)\rangle\right)_{i,j\in J} \neq 0$ for any $J \subset \mathbb{N}$, $r = 1, 2$, $d \in \mathcal{U} \setminus \mathcal{T}$. It follows that the algebraic and geometric multiplicity of $\mu_i^r(d)$ coincide for $i \in \mathbb{N}$, $r = 1, 2$, $d \in \mathcal{U} \setminus \mathcal{T}$ (see e.g. [18]). Particularly, this holds for $d = \sigma(s)$ with $s \in (s_0 - \eta, s_0 + \eta)$ or $s \in (s_0 - \eta, s_0)$ if $d^0 \in C_p \setminus \mathcal{T}$ or $d^0 \in C_p \cap \mathcal{T}$, respectively.

Our considerations lead to the following conclusion. If $d^0 \in C_p \setminus \mathcal{T}$ then $\mu_p(\sigma(s))$ is the only eigenvalue of the operator $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ changing its sign at s_0 and it is algebraically simple. The other eigenvalues have constant signs and multiplicities on $(s_0 - \eta, s_0 + \eta)$. If $d^0 \in C_p \cap \mathcal{T}$ then $\mu_p(\sigma(s))$ is a real positive algebraically simple eigenvalue on $(s_0 - \eta, s_0)$. The other possible positive eigenvalues (which can correspond only to $i \notin I(d^0)$) form pairs $\mu_i^{1,2}(\sigma(s))$, $\mu_i^1(\sigma(s)) \neq \mu_i^2(\sigma(s))$ where $\mu_i^1(\sigma(s))$, $\mu_i^2(\sigma(s))$ have the same algebraic multiplicity, i.e. the sum of algebraic multiplicities for any such pair is even. All eigenvalues are complex for $(s_0, s_0 + \eta)$. The assertion of Lemma 4.1 follows. □

Further, let $\delta > 0$ and $\eta > 0$ be from Lemma 4.1. Consider a penalty equation

$$D(\sigma(s))U - BAU + \frac{\tau}{1+\tau}N(U) + \frac{D(\sigma(s))}{1+\tau}L_\delta(s)U + P^\tau(U) = 0 \tag{4.8}$$

where τ is an additional parameter and the operator L_δ is from Notation 4.2. It can be understood as a homotopy joining the perturbed linearized equation (obtained for $\tau = 0$) with our inclusion (obtained for $\tau \to +\infty$—see Lemma 4.4).

The penalty equation (4.8) will be supplemented by the norm condition

$$\|U\|^2 = \frac{\varrho\tau}{1+\tau} \tag{4.9}$$

where $\varrho > 0$ is a given small number.

The proof of Theorem 3.1 is similar to that of Theorem 2.1 in [11]. The idea is to prove, for any $\varrho > 0$ small, the existence of a branch of triplets $[s, U, \tau]$ satisfying (4.8), (4.9) which starts at s_0 and is unbounded in τ. By the limiting process $\tau \to +\infty$ along this branch we obtain a solution U_ϱ, $\|U_\varrho\|^2 = \varrho$ of (3.3) with some s_ϱ. Any accumulation point of s_ϱ for $\varrho \to 0+$ is a bifurcation point of (3.3). In [11], the situation was simpler because variational inequalities (a special case of inclusions) were considered and therefore the penalty term was simpler. Moreover, it was supposed that $\dim E_B(d^0) = 1$ and no operator L_δ was necessary.

Lemma 4.2. *If there exists $U_0 \in E_B(d^0)$, $U_0^* \in K^-$ then $E_I(d^0) = E_B(d^0) \cap K$.*

Proof can be done analogously as the proof of Lemma 3.3 in [13]. □

Lemma 4.3. *If $[s_n, U_n, \tau_n] \in \mathbb{R} \times \widetilde{\mathbb{V}} \times \mathbb{R}^+$, $s_n \to s$, $U_n \rightharpoonup U$, $\tau_n \to \tau \in [0, +\infty]$,*

$$(4.10)\quad D(\sigma(s_n))U_n - BAU_n + \frac{\tau_n}{1+\tau_n} N(U_n) + \frac{D(\sigma(s_n))}{1+\tau_n} L_\delta(s_n)U_n + P^{\tau_n}(U_n) = 0$$

then $U_n \to U$. If, moreover, $\|U\| = 0$, $W_n = [w_n, z_n] = \frac{U_n}{\|U_n\|} \rightharpoonup W$ then $W_n \to W$.

Proof can be done similarly as in [13], Remark 3.1 on the basis of the assumption (2.20). (Note that L_δ is compact and the term containing L_δ vanishes for $\tau \to +\infty$.) □

Lemma 4.4. *Let $s_n \to s$, $U_n \rightharpoonup U$, $\tau_n \to +\infty$ and* (4.10) *hold. Then $U_n \to U$ and s, U satisfy the inclusion* (3.3)*. If, moreover, $\|U_n\| \to 0$, $\frac{U_n}{\|U_n\|} \rightharpoonup W$ then $\frac{U_n}{\|U_n\|} \to W$ and s, W satisfy* (2.12) *with $d = \sigma(s)$.*

Proof can be done similarly as that of Lemma 3.2 in [13] on the basis of the assumptions (2.20) and (2.21). □

Remark 4.3. It follows from the assumption (2.19) that if $\frac{P^{\tau_n}(U_n)}{\|U_n\|} \rightharpoonup Z$ then

$$(4.11)\qquad \langle Z, V\rangle = \lim \frac{\langle P^{\tau_n}(U_n), V\rangle}{\|U_n\|} \leqslant 0 \text{ for any } V \in K.$$

Further, if $Z \neq 0$ then $\langle Z, V\rangle < 0$ for all $V \in K^-$. Otherwise we would have $V \in K^-$ such that $\langle Z, V\rangle = 0$ and the definition of K^- would imply the existence of $F \in \widetilde{\mathbb{V}}$ such that $\langle Z, F\rangle > 0$, $V \pm F \in K$. This means $V + F \in K$, $\langle Z, V + F\rangle > 0$, which contradicts (4.11).

P r o o f o f T h e o r e m 3.1. Let $\varrho > 0$ be fixed. Let $U_p = U_p(d^0)$ be the element from Remark 4.1. The equations (4.8), (4.9) are equivalent to

$$x - T(s)x + G(s,x) = 0 \tag{4.12}$$

in the space $\mathbb{X} = \widetilde{\mathbb{V}} \times \mathbb{R}$ (with points $x = [U, \tau]$ and the norm $|||x||| = \|U\| + |\tau|$) where

$$T(s) = \Big[D^{-1}(\sigma(s))BAU - L_\delta(s)U, 0\Big] \text{ for all } s \in \mathbb{R}, x = [U,\tau] \in \mathbb{X}$$
$$G(s,x) = \Big[D^{-1}(\sigma(s))\Big(\frac{\tau}{1+\tau}N(U) + P^\tau U\Big) - \frac{\tau}{1+\tau}L_\delta(s)U, \frac{1+\tau}{\varrho}\|U\|^2\Big]$$
$$\text{for all } s \in \mathbb{R}, x = [U,\tau] \in \mathbb{X}$$

with $P^\tau U = P^{-\tau}U$ for $\tau < 0$. It follows from (2.1), (2.21), (2.22) that T, G satisfy the following conditions:

$$T, G\colon\ \mathbb{R} \times \mathbb{X} \to \mathbb{X} \text{ are completely continuous,} \tag{4.13}$$

$$T(s)\colon\ \mathbb{X} \to \mathbb{X} \text{ is linear for any fixed } s \in \mathbb{R}, \tag{4.14}$$

$$\lim_{|||x||| \to 0} \frac{|||G(s,x)|||}{|||x|||} = 0 \text{ uniformly on bounded } s\text{-intervals.} \tag{4.15}$$

It is easy to see that λ is an eigenvalue of $T(s) - I$ with the algebraic multiplicity k (for some $s \in \mathbb{R}$) and $x = [U, \tau]$ is a corresponding eigenvector if and only if λ is an eigenvalue of $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ with the algebraic multiplicity k and U is a corresponding eigenvector, $\tau = 0$. (The symbol I denotes the identity operators in the corresponding spaces.) Particularly, Lemma 4.1 implies that

$$\operatorname{Ker}(T(s_0) - I) = \operatorname{Lin}\{x_0\} \text{ with } x_0 = [U_p, 0]. \tag{4.16}$$

For any compact linear operator T in a Banach space we have $\operatorname{ind}(I - T) = (-1)^{\gamma(T)}$ where ind denotes the Leray-Schauder index, $\gamma(T)$ is the sum of the multiplicities of all positive eigenvalues of $T - I$ (see e.g. [18]). It follows from Lemma 4.1 that

$$\operatorname{ind}(I - T(s_0 - \varepsilon)) \neq \operatorname{ind}(I - T(s_0 + \varepsilon)) \text{ for all } \varepsilon \in (0, \eta). \tag{4.17}$$

Set

$$C = \overline{\{[s,x] \in \mathbb{R} \times \mathbb{X}; |||x||| \neq 0,\ (4.12) \text{ holds}\}} = \overline{\{[s,U,\tau]; \tau \neq 0,\ (4.8), (4.9) \text{ hold}\}}$$

and let C_0 be the component of C containing $[s_0, 0, 0]$. Analogously as in [1] we can define subcontinua C_0^+ and C_0^- of C_0 starting at $[s_0, 0, 0]$ in the direction of x_0 and

$-x_0$, respectively, with $x_0 = [U_p, 0]$. More precisely, let us choose $\vartheta \in (0,1)$ and set

$$\begin{aligned} K_\vartheta &= \{[s,x] \in \mathbb{R} \times \mathbb{X};\ \frac{|\langle x, x_0\rangle|}{\|x_0\|} > \vartheta \|x\|\} \\ &= \{[s,U,\tau] \in \mathbb{R} \times \widetilde{\mathbb{V}} \times \mathbb{R};\ \frac{|\langle U, U_p\rangle|}{\|U_p\|} > \vartheta \|U\|\}, \\ K_\vartheta^+ &= \{[s,x] \in K_\vartheta;\ \langle x, x_0\rangle > 0\} = \{[s,U,\tau] \in K_\vartheta;\ \langle U, U_p\rangle > 0\}, \\ K_\vartheta^- &= \{[s,x] \in K_\vartheta;\ \langle x, x_0\rangle < 0\} = \{[s,U,\tau] \in K_\vartheta;\ \langle U, U_p\rangle < 0\} \end{aligned}$$

(cf. [9]). As in [21], Lemma 1.24 there exists $R > 0$ such that

$$(C \setminus \{[s_0,0,0]\}) \cap B_R(s_0,0,0) \subset K_\vartheta,$$

where $B_R(s_0,0,0) = \{[s,U,\tau] \in \mathbb{R} \times \widetilde{\mathbb{V}} \times \mathbb{R};\ |s - s_0| + \|U\| + |\tau| \leqslant R\}$. For each $r \in (0,R]$ denote by $D_r^\pm$ the components of the sets $\overline{\{[s_0,0,0]\} \cup (C \cap B_r(s_0,0,0) \cap K_\vartheta^\pm)}$, respectively, containing $[s_0,0,0]$. Denote by $C_{0,r}^+$ and $C_{0,r}^-$ the components of $\overline{C_0 \setminus D_r^-}$ and $\overline{C_0 \setminus D_r^+}$, respectively, containing $[s_0,0,0]$. Set

$$C_0^+ = \overline{\bigcup_{0<r\leqslant R} C_{0,r}^+}, \qquad C_0^- = \overline{\bigcup_{0<r\leqslant R} C_{0,r}^-}.$$

The sets C_0^+ and C_0^- are independent of the choice of $\vartheta \in (0,1)$, both C_0^+ and C_0^- are connected and $C_0 = C_0^+ \cup C_0^-$ (see [1] and Lemma 1.24 in [21]). Under the assumptions (4.13)–(4.17), considerations from the proof of the global Dancer's bifurcation theorem ([1], Theorem 2) can be used and an analogue of this theorem for the equation (4.12) can be proved (cf. [9], Theorem 4.1). That means

$$\text{either } C_0^+ \cap C_0^- \neq \{[s_0,0,0]\} \text{ or both } C_0^+ \text{ and } C_0^- \text{ are unbounded} \tag{4.18}$$

(cf. [1], [9] for details). It follows from the definition that C_0^+ and C_0^- contain $[s_0,0,0]$ and

$$\text{there are } [s_n,U_n,\tau_n] \in C_0^+, [s_n,U_n,\tau_n] \to [s_0,0,0], \frac{U_n}{\|U_n\|} \to \frac{U_p}{\|U_p\|}, \tag{4.19}$$

$$\text{there are } [s_n,U_n,\tau_n] \in C_0^-, [s_n,U_n,\tau_n], \to [s_0,0,0], \frac{U_n}{\|U_n\|} \to -\frac{U_p}{\|U_p\|}. \tag{4.20}$$

We will write C_ϱ, $C_{\varrho,0}$, $C_{\varrho,0}^+$, $C_{\varrho,0}^-$ instead of C, C_0, C_0^+, C_0^- in order to emphasize the role of ϱ. We shall prove successively that the following statements hold for all $\varrho \in (0,\varrho_0)$ if ϱ_0 is small enough:

$$\left\{ \begin{array}{l} \text{if } [s_n,U_n,\tau_n] \in C_{\varrho,0}^+, [s_n,U_n,\tau_n] \to [s_0,0,0], \frac{U_n}{\|U_n\|} \to \frac{U_p}{\|U_p\|} \\ \qquad\qquad \text{then } \limsup \frac{s_n - s_0}{\tau_n} < 0, \end{array} \right. \tag{4.21}$$

$$(4.22) \qquad \begin{cases} \text{if } [s_n, U_n, \tau_n] \in C^-_{\varrho,0}, [s_n, U_n, \tau_n] \to [s_0, 0, 0], \frac{U_n}{\|U_n\|} \to -\frac{U_p}{\|U_p\|} \\ \qquad\qquad\qquad \text{then } \liminf \frac{s_n - s_0}{\tau_n} > 0 \end{cases}$$

(the branch $C^+_{\varrho,0}$ starts downwards from s_0, $C^-_{\varrho,0}$ starts upwards from s_0),

$$(4.23) \qquad [s_0, U, \tau] \in C_{\varrho,0} \Longrightarrow \tau = 0$$

($C_{\varrho,0}$ can intersect the level $s = s_0$ only at the initial point $[s_0, 0, 0]$),

$$(4.24) \qquad [s, U, \tau] \in C_{\varrho,0} \Longrightarrow s \leqslant c \text{ (with some } c > 0).$$

Suppose for a moment that (4.21)–(4.24) hold. It follows from (4.21), (4.22), (4.23) and the definition of $C^+_{\varrho,0}$ and $C^-_{\varrho,0}$ that $C^+_{\varrho,0}$ and $C^-_{\varrho,0}$ remain below and above s_0, respectively, with the exception of $[s_0, 0, 0]$ and therefore $C^+_{\varrho,0} \cap C^-_{\varrho,0} = \{[s_0, 0, 0]\}$. Hence (4.18) implies that $C^+_{\varrho,0}, C^-_{\varrho,0}$ are unbounded. But (4.9) together with (4.24) (and the fact that $C^-_{\varrho,0}$ lies above s_0) imply the boundedness of $C^-_{\varrho,0}$ in $\|U\|$ and s and therefore $C^-_{\varrho,0}$ is unbounded in τ. It follows from (4.9), the connectedness and the fact that $[s_0, 0, 0] \in C_{\varrho,0}$ that $\tau \geqslant 0$ for all $[s, U, \tau] \in C_{\varrho,0}$. Particularly, there exists a sequence $[s_n, U_n, \tau_n] \in C^-_{\varrho,0}$ with $s_n > s_0, \tau_n \to +\infty$. We have (4.10) and

$$(4.25) \qquad \|U_n\|^2 = \frac{\varrho \tau_n}{1 + \tau_n}.$$

We can suppose $s_n \to s_\varrho, U_n \rightharpoonup U_\varrho$ with some $s_\varrho \geqslant s_0$, U_ϱ and Lemma 4.4 implies that $U_n \to U_\varrho$, U_ϱ is a solution of (3.3) with $s = s_\varrho$. We would like to know that

$$(4.26) \qquad s_\varrho \geqslant s_0 + \varepsilon \text{ for all } \varrho > 0 \text{ small enough with some } \varepsilon > 0.$$

Suppose by contradiction that there are $\varrho_n \to 0$, $s_{\varrho_n} \to s_0$, $\|U_{\varrho_n}\| \to 0$, $\frac{U_{\varrho_n}}{\|U_{\varrho_n}\|} \rightharpoonup W$,

$$(4.27) \qquad D(\sigma(s_{\varrho_n}))U_{\varrho_n} - BAU_{\varrho_n} + N(U_{\varrho_n}) \in -M(U_{\varrho_n}).$$

Dividing (4.27) by $\|U_{\varrho_n}\|$ and using the assumptions (2.1) and (2.17) we obtain $\frac{U_{\varrho_n}}{\|U_{\varrho_n}\|} \to W$ and W is a solution of (2.12) with $d = d^0$. We get $W \in E_B(d^0) \cap K$ by the second part of the assumption (2.25) and Lemma 4.2 but this contradicts the first part of the assumption (2.25). Hence, (4.26) holds. Thus, any accumulation point s_I of s_ϱ for $\varrho \to 0_+$ is a bifurcation point of (3.3) and $s_I \in [s_0 + \varepsilon, c]$.

For the completeness of the proof it is sufficient to show that (4.21)–(4.24) hold.

Proof of (4.21): Let (4.10) hold, $[s_n, U_n, \tau_n] \to [s_0, 0, 0]$ and $\frac{U_n}{\|U_n\|} \to \frac{U_p}{\|U_p\|}$. Multiply (4.10) by $\frac{U_p^*}{\|U_n\|}$, and

$$(4.28) \qquad D(\sigma(s_0))U_p^* - B^* A U_p^* = 0$$

by $\frac{U_n}{\|U_n\|}$ and subtract. A simple calculation using (4.6) yields

$$R_n(s_n - s_0) + \Big\langle \frac{\tau_n}{1+\tau_n}\frac{N(U_n)}{\|U_n\|} + \frac{P^{\tau_n}(U_n)}{\|U_n\|}, U_p^*\Big\rangle = 0,$$
$$R_n = \sigma_1'(\bar{s}_n)\Big\langle \frac{u_n}{\|U_n\|}, u_p^*\Big\rangle + \sigma_2'(\widetilde{s}_n)\Big\langle \frac{v_n}{\|U_n\|}, v_p^*\Big\rangle$$

with some $\bar{s}_n, \widetilde{s}_n$ between s_0 and s_n where $U_n = [u_n, v_n], U_p = [u_p, v_p]$, $U_p^* = [u_p^*, v_p^*]$. Further, it follows from the formula for u_p, u_p^*, v_p, v_p^* (see Observation 4.2 and Remark 2.1) and the equations defining C_j, C (by using the transversality assumption and the orientation of σ given by (3.1)) that

$$R_n \to R = \sigma_1'(s_0)\frac{\langle u_p, u_p^*\rangle}{\|U_p\|} + \sigma_2'(s_0)\frac{\langle v_p, v_p^*\rangle}{\|U_p\|} = \frac{(\sigma_2(s_0)\kappa_p - b_{22})^2}{b_{12}b_{21}\|U_p\|}\sigma_1'(s_0) + \frac{\sigma_2'(s_0)}{\|U_p\|} < 0.$$

We have $U_p \in E_B(d^0)$, $U_p \notin K$, $U_0^* = U_p^* \in K^-$ by the assumption (2.25) and Remark 4.1. Hence, it follows by using (2.1) and (2.23) that

$$\limsup_{n\to\infty} \frac{s_n - s_0}{\tau_n} = -\frac{1}{R}\limsup_{n\to\infty}\Big\langle \frac{P^{\tau_n}(U_n)}{\tau_n\|U_n\|}, U_p^*\Big\rangle < 0.$$

Proof of (4.22) is the same but we have $\frac{U_n}{\|U_n\|} \to -\frac{U_p}{\|U_p\|}$ and $R > 0$.

Proof of (4.23): suppose by contradiction that there are $\varrho_n \to 0$ and $[s_0, U_n, \tau_n] \in C_{\varrho_n,0}$, $\tau_n > 0$. Then $\|U_n\| \to 0$,

$$D(\sigma(s_0))U_n - BAU_n + \frac{\tau_n}{1+\tau_n}N(U_n) + \frac{D(\sigma(s_0))}{1+\tau_n}L_\delta(s_0)U_n + P^{\tau_n}(U_n) = 0. \tag{4.29}$$

We can suppose without loss of generality that $\frac{U_n}{\|U_n\|} \rightharpoonup W$ and $\tau_n \to \tau \in [0, +\infty]$.

If $\tau < +\infty$ then we obtain from (4.29) by using Lemma 4.3 that $\frac{U_n}{\|U_n\|} \to W$. Hence, it follows from (4.29) divided by $\|U_n\|$ and (2.1) that $\frac{P^{\tau_n}U_n}{\|U_n\|} \to Z$,

$$D(\sigma(s_0))W - BAW + \frac{D(\sigma(s_0))}{1+\tau}L_\delta(s_0)W + Z = 0. \tag{4.30}$$

If $\tau = 0$ then $Z = 0$ by the assumption (2.22). If $\tau \in (0, +\infty)$, multiply (4.30) by U_p^*, (4.28) by W and subtract. We get $\langle Z, U_p^*\rangle = 0$ by using (4.6) and Remark 4.3 implies $Z = 0$ again. In both cases we obtain by using Lemma 4.1 (with δ replaced by $\frac{\delta}{1+\tau}$) and Proposition 2.1 that $W = \pm\frac{U_p}{\|U_p\|} \in E_B(d^0)$. The assumption (2.25) implies $W \notin K$. Multiply (4.29) by $\frac{U_p^*}{\|U_n\|}$, (4.28) by $\frac{U_n}{\|U_n\|}$ and subtract. We get

$$\Big\langle \frac{\tau_n}{1+\tau_n}\frac{N(U_n)}{\|U_n\|} + \frac{D(\sigma(s_0))}{1+\tau_n}L_\delta(s_0)W_n + \frac{P^{\tau_n}(U_n)}{\|U_n\|}, U_p^*\Big\rangle = 0. \tag{4.31}$$

By using the assumptions (2.1) and (4.6), dividing (4.31) by τ_n and letting $n \to \infty$ we obtain $\lim \langle \frac{P^{\tau_n}(U_n)}{\tau_n \|U_n\|}, U_p^* \rangle = 0$, which is excluded by (2.23) for $W \notin K$. This is a contradiction.

If $\tau = \infty$ then (4.29) and Lemma 4.4 imply that $\frac{U_n}{\|U_n\|} \to W$, W satisfies (2.12) with $d = d^0$. Lemma 4.2 gives $W \in E_B(d^0) \cap K$ and this is a contradiction with (2.25).

For the proof of (4.24) see [13], Lemma 3.4 or [5]. Note that no operator L_δ is considered in [13], but we have $L_\delta(s) \equiv 0$ for $s > s_0 + \eta$. □

5. Another example

In the situation from Examples 2.1–2.3, the homogeneous problem (2.12) is equivalent to the variational inequality

$$U \in K; \quad \langle D(d)U - BAU, V - U \rangle \geqslant 0 \text{ for any } V \in K, \tag{5.1}$$

where the cone K is defined by (2.15). In the following example we will show a boundary condition such that the corresponding homogeneous problem (2.12) is not equivalent to (5.1).

E x a m p l e 5.1. Let $\Omega = (0,1)$, $\mathbb{V} = \{\varphi \in W_2^1(0,1);\ \varphi(0) = 0\}$. Let $x_1, x_2 \in (0,1)$ be fixed. Let us consider the multivalued mappings $m_1, m_2\colon \mathbb{R} \to 2^{\mathbb{R}}$ defined by

$$\left.\begin{aligned} &m_1(\xi) = 0 \text{ for } \xi < 0,\\ &m_1(\xi) \geqslant 0 \text{ for } \xi > 0, \ \lim_{\xi \to 0_+} m_1(\xi) = m_1^0,\\ &m_1(0) = [0, m_1^0] \quad \text{with some } m_1^0 \in [0, +\infty] \end{aligned}\right\}$$

$$\left.\begin{aligned} &m_2(\xi) = 0 \text{ for } \xi > 0,\\ &m_2(\xi) \leqslant 0 \text{ for } \xi < 0, \ \lim_{\xi \to 0_-} m_2(\xi) = m_2^0,\\ &m_2(0) = [m_2^0, 0] \quad \text{with some } m_2^0 \in [-\infty, 0]. \end{aligned}\right\}$$

Set

$$\left.\begin{aligned} &\underline{m}_j(\xi) = \overline{m}_j(\xi) = m_j(\xi) \quad \text{for } \xi \neq 0\\ &\underline{m}_1(0) = 0, \quad \overline{m}_1(0) = m_1^0\\ &\underline{m}_2(0) = m_2^0, \quad \overline{m}_2(0) = 0. \end{aligned}\right\}$$

Define the corresponding mapping $M\colon \widetilde{\mathbb{V}} \to 2^{\widetilde{\mathbb{V}}}$, $M(U) = [M_1(u), M_2(v)]$ for $U = [u, v]$ by

$$\begin{aligned} M_j(\psi) = \{z \in \mathbb{V};\ \underline{m}_j(\psi(x_j))\varphi(1) \leqslant \langle z, \varphi \rangle \leqslant \overline{m}_j(\psi(x_j))\varphi(1)\\ \text{for all } \varphi \in \mathbb{V}, \varphi(1) \geqslant 0\} \end{aligned} \tag{5.2}$$

for any $\psi \in \mathbb{V}$, $j = 1, 2$. Then a solution of (2.9) is a weak solution of the stationary problem corresponding to (1.1) with the boundary conditions

$$u(0) = v(0) = 0, \; -d_1 u_x(1) \in m_1(u(x_1)), \; -d_2 v_x(1) \in m_2(v(x_2)). \tag{5.3}$$

The multivalued condition in (5.3) describes for example a semipermeable membrane on the boundary like in Example 2.2 but with sensors in the interior of the domain; particularly, the sensors are at different points than the source. In the situation from Example 2.2, we had $x_1 = x_2 = 1$ (for $\Omega = (0, 1)$), i.e. the sensors were at the same point as the source (membrane). From this point of view, the multivalued condition in Example 5.1 is more general.

Let us define convex cones $K_{x_1} = \{\varphi \in \mathbb{V};\ \varphi(x_1) \leqslant 0\}$, $K_{x_2} = \{\varphi \in \mathbb{V};\ \varphi(x_2) \geqslant 0\}$ and $K_1 = \{\varphi \in \mathbb{V};\ \varphi(1) \geqslant 0\}$. The corresponding homogeneous mapping M_0 is $M_0(U) = [M_{01}(u), M_{02}(v)]$ with

$$\begin{aligned}
&M_{01}(\psi) = \{0\} \quad \text{if } \psi(x_1) < 0,\\
&M_{01}(\psi) = \{z \in \mathbb{V};\ \langle z, \varphi\rangle \geqslant 0 \text{ for all } \varphi \in K_1\} \;\text{ if } \psi(x_1) = 0,\\
&M_{01}(\psi) = \emptyset \quad \text{if } \psi(x_1) > 0,\\
&M_{02}(\psi) = \{0\} \quad \text{if } \psi(x_2) > 0,\\
&M_{02}(\psi) = \{z \in \mathbb{V};\ \langle z, \varphi\rangle \leqslant 0 \text{ for all } \varphi \in K_1\} \;\text{ if } \psi(x_2) = 0,\\
&M_{02}(\psi) = \emptyset \quad \text{if } \psi(x_2) < 0.
\end{aligned}$$

Then the set K from (2.15) is $K_{x_1} \times K_{x_2}$. A solution of (2.12) is a weak solution of (2.5) with $\lambda = 0$ and with the boundary conditions

$$\begin{gathered}
u(0) = v(0) = 0,\\
u_x(1) \leqslant 0,\; u(x_1) \leqslant 0,\; u_x(1) \cdot u(x_1) = 0, \quad v_x(1) \geqslant 0,\; v(x_2) \geqslant 0,\; v_x(1) \cdot v(x_2) = 0.
\end{gathered}$$

A suitable penalty operator for M is $P^\tau U = [P_1^\tau u, P_2^\tau v]$ with

$$\langle P_j^\tau u, \varphi\rangle = p_j^\tau(u(x_j))\varphi(1)$$

for all u, $\varphi \in \mathbb{V}$, where p_j^τ are the same functions as in Example 2.2. Set $\mathcal{K} = ((-K_1) \cap K_{x_1}) \times (K_1 \cap K_{x_2})$ and consider the condition

$$E_B(d^0) \cap K = \{0\} \text{ and there exists } U_0 \in E_B(d^0),\; U_0^* \in \mathcal{K}^- \tag{5.4}$$

instead of (2.25). It is easy to see by using Proposition 2.1 that the condition (5.4) is fulfilled for $d^0 \in C_p$ if the eigenfunction e_p of $-u_{xx}$ with the boundary conditions

$u(0) = u_x(1) = 0$ corresponding to κ_p satisfies $e_p(x_1) > 0$, $e_p(x_2) > 0$ and $e_p(1) > 0$. (Note that the eigenvalues κ_j are simple in the one-dimensional case.) Replacing K by $\mathcal{K}$ in the appropriate places and (2.25) by (5.4), we can go through the whole procedure used in Section 4 and prove the assertion of Theorem 3.1 or Corollary 3.1, respectively, also in this situation. The proofs of all assertions from Section 4 can be done analogously as above with the exception of the proof of (4.24) where the condition $\langle P^\tau(U), U\rangle \geqslant 0$ from (2.19) is used. But now $\langle P^\tau(U), U\rangle \geqslant 0$ is not satisfied for all $U \in \widetilde{\mathbb{V}}$. We have to strengthen the condition (3.1) by

$$\lim_{s\to+\infty} \sigma_2(s) = +\infty$$

and prove (4.24) analogously as in [5].

References

[1] *E. N. Dancer*: On the structure of solutions of non-linear eigenvalue problem. Indiana Univ. Math. J. *23* (1974), 1069–1076.

[2] *G. Duvaut, J. L. Lions*: Les Inéquations en Mechanique et en Physique. Dunod, Paris, 1972.

[3] *P. Drábek, M. Kučera, M. Míková*: Bifurcation points of reaction-diffusion systems with unilateral conditions. Czech. Math. J. *35* (1985), 639–660.

[4] *P. Drábek, M. Kučera*: Reaction-diffusion systems: Destabilizing effect of unilateral conditions. Nonlinear Analysis, Theory, Methods, Applications *12* (1988), 1173-1192.

[5] *J. Eisner*: Reaction-diffusion systems: Destabilizing effect of conditions given by inclusions. To appearin Mathematica Bohemica.

[6] *S. Fučík, A. Kufner*: Nonlinear Differential Equations. Elsevier, Amsterdam, 1980.

[7] *A. Gierer, H. Meinhardt*: A theory of biological pattern formation. Kybernetik *12* (1972), 30–3.

[8] *H. Kielhöfer*: Stability and semilinear evolution equations in Hilbert space. Arch. Rat. Mech. Analysis *57* (1974), 150–165.

[9] *M. Kučera*: Bifurcation points of variational inequalities. Czechoslovak Math. J. *32* (1982), 208–226.

[10] *M. Kučera*: Reaction-diffusion systems: Stabilizing effect of conditions described by quasivariational inequalities. Czechoslovak Math. J. *47 (122)* (1997), 469–486.

[11] *M. Kučera*: Bifurcation of solutions to reaction-diffusion system with conditions described by inequalities and inclusions. Proceedings of the Second World Congress of Nonlinear Analysts. Athens, 1996.

[12] *M. Kučera, M. Bosák*: Bifurcation for quasi-variational inequalities of reaction-diffusion type. SAACM *3* (1993), 111–127.

[13] *M. Kučera*: Bifurcation of solutions to reaction-diffusion systems with unilateral conditions. Navier-Stokes Equations and related Topics (by A. Sequeira, eds.). Plenum Press, New York, 1995, pp. 307–322.

[14] *M. Kučera*: Reaction-diffusion systems: Bifurcation and stabilizing effect of conditions given by inclusions. Nonlin. Anal., T.M.A. *27* (1996), 249–260.

[15] *M. Mimura, Y. Nishiura, M. Yamaguti*: Some diffusive prey and predator systems and their bifurcation problems. Ann. N.Y. Acad. Sci. *316* (1979), 490–521.

[16] *J. D. Murray*: Mathematical Biology 19. Springer-Verlag, 1993.

[17] *Y. Nishiura*: Global structure of bifurcating solutions of some reaction-diffusion systems.. SIAM J. Math. Analysis *13* (1982), 555–593.

[18] *L. Nirenberg*: Topics in Nonlinear Functional Analysis. Academic Press, New York, 1974.

[19] *P. Quittner*: Spectral analysis of variational inequalities. Commentat. Math. Univ. Carol. *27* (1986), 605–629.

[20] *P. Quittner*: Bifurcation points and eigenvalues of inequalities of reaction-diffusion type. J. reine angew. Math. *380* (1987), 1–13.

[21] *P. H. Rabinowitz*: Some global results for nonlinear eigenvalue problems. J. Funct. Anal. *7* (1987), 487–513.

[22] *D. H. Sattinger*: Topics in Stability and Bifurcation Theory. Lecture Notes in Mathematics 309, Springer-Verlag, Berlin-Heidelberg-New York, 1973.

[23] *A. M. Turing*: The chemical basis of morphogenesis. Phil. Trans. Roy. Soc. (B) (1952), 37–72.

Authors' address: *Jan Eisner*, *Milan Kučera*, Mathematical Institute, Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Praha 1, Czech Republic, e-mail: `eisner@math.cas.cz`, `kucera@math.cas.cz`.