Applications of Mathematics

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The second order optimality conditions for nonlinear mathematical programming with $C^{1,1}$ data

THE SECOND ORDER OPTIMALITY CONDITIONS FOR NONLINEAR MATHEMATICAL PROGRAMMING WITH $C^{1,1}$ DATA

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Abstract. To find nonlinear minimization problems are considered and standard C^2-regularity assumptions on the criterion function and constrained functions are reduced to $C^{1,1}$-regularity. With the aid of the generalized second order directional derivative for $C^{1,1}$ real-valued functions, a new second order necessary optimality condition and a new second order sufficient optimality condition for these problems are derived.

Keywords: nonlinear programming, constrained problems, $C^{1,1}$ functions, second order conditions

MSC 2000: 90C30

1. Introduction

Characterizing the optimal solution by means of second order conditions is a problem of continuous interest in the theory of mathematical programming constrained problems with twice continuously differentiable data. Many valuable results on second-order conditions have been established in [1–3, 10, 12, 16]. Recently, more attention has been paid to problems which are not with C^2 data (see [4–6, 8, 9, 13–15]).

The aim of this paper is to reduce C^2-regularity assumptions on constraints from Liu [13] to $C^{1,1}$-regularity. With the aid of the generalized second order directional derivative for $C^{1,1}$ real-valued functions (see Section 2), a nontraditional second order necessary optimality condition and a nontraditional second order sufficient optimality condition for $C^{1,1}$ constrained nonlinear minimization problems are derived (see Section 3). In these problems, all constrained functions are assumed to be $C^{1,1}$ functions and the criterion function f is also only a $C^{1,1}$ function. This has several

practical applications (cf. [8, 11]), e.g, if f is a C^2 function, then the penalty function $(f^+)^2$, where f^+ is a positive part of f, is $C^{1,1}$ but not C^2, in general.

Let E_n stand for the n-dimensional Euclidean vector space. Recall that $\overline{x} \in U \subset E_n$ is a local (strict local) minimum for the problem

$$\text{minimize } f(x) \qquad \text{over all } x \in U,$$

if there exists a neighbourhood $N(\overline{x})$ such that

$$f(\overline{x}) \leqslant f(x) \quad (f(\overline{x}) < f(x)) \qquad \forall x \in N(\overline{x}) \cap U \setminus \{\overline{x}\}.$$

Hiriart-Urruty et al in [8] defined the generalized Hessian matrix of f at $\overline{x} \in S$ as follows:

$$\partial^2 f(\overline{x}) = \mathrm{Co}\{A \mid \exists \{x_i\}_{i=1}^{\infty}\colon\ x_i \to \overline{x}, \text{with } f \text{ twice differentiable at } x_i$$
$$\text{and } \nabla^2 f(x_i) \to A\}$$

where "Co" stands for the convex hull and S is a nonempty open subset of E_n, f is a $C^{1,1}$ real-valued function on S and $\nabla^2 f(x)$ is the standard Hessian matrix of the second derivatives of f at x. If f is twice differentiable at $\overline{x}$ then

$$\partial^2 f(\overline{x}) = \{\nabla^2 f(\overline{x})\}.$$

Hiriart-Urruty concluded that: if $\overline{x} \in S$ is a local minimum for problem

(1.1) $$\text{minimize } f(x) \qquad \text{over all } x \in S,$$

then for each direction $d \in E_n$ there exists a matrix $A \in \partial^2 f(\overline{x})$ such that $(Ad, d) \geqslant 0$. However, by the following example, it is not true that $(Ad, d) \geqslant 0$ for all $A \in \partial^2 f(\overline{x})$.

E x a m p l e 1.1. Set $S = (-\infty, \infty)$ and

$$f(x) = \int_0^{|x|} \varphi(t)\,\mathrm{d}t \qquad \text{for } x \in S,$$

where
$$\varphi(t) = \left\{ \begin{array}{ll} 2t^2 + t^2 \sin(1/t) & \text{if } t > 0, \\ 0 & \text{if } t = 0. \end{array} \right.$$

Since $3t^2 \geqslant \varphi(t) \geqslant t^2 \geqslant 0$, we have $f(x) \geqslant |x|^3/3 \geqslant 0$ for all $x \in E_1$, and since $f(0) = 0$, it is clear that $\overline{x} = 0$ is a local (and also global) minimum point of f. Obviously (cf. [8]), $\partial^2 f(0) = [-1, 1]$ and consequently $(Ad, d) \geqslant 0$ for $A \in [0, 1] \subset \partial^2 f(0)$ and all $d \in E_1$, but $(Ad, d) < 0$ for $A \in [-1, 0) \subset \partial^2 f(0)$ and all $d \in E_1$.

To overcome the above disadvantage, the first author, in [13], defined the generalized second order directional derivative for $C^{1,1}$ vector functions, investigated the relation between it and the generalized Hessian matrix and derived a second order necessary optimality condition and a second order sufficient optimality condition for unconstrained nonlinear programming problems with a $C^{1,1}$ criterion function.

2. THE GENERALIZED SECOND-ORDER DIRECTIONAL DERIVATIVE FOR $C^{1,1}$ FUNCTIONS

Let $S \subset E_n$ be a nonempty open set and let $\|.\|$ stand for the Euclidean norm. We denote by $C^{1,1}(S)$ the class of all real-valued functions f which are differentiable on S and whose gradient ∇f is locally Lipschitz continuous on S, i.e.,

$$\forall z \in S \quad \exists C > 0 \quad \exists \varepsilon > 0 \quad \forall x, y \in z + \varepsilon B_n \quad \|\nabla f(x) - \nabla f(y)\| \leqslant C\|x - y\|,$$

where B_n is the unit open ball in E_n. The gradient ∇f is therefore differentiable almost everywhere on S by Rademacher's theorem (see [17]).

In Liu [13], the generalized second order directional derivative of $f \in C^{1,1}(S)$ at $\overline{x} \in S$ in the direction $d \in E_n$ was defined as the set

$$\partial_*^2 f(\overline{x})(d, d) = \{F \mid \exists \{t_i\}_{i=1}^\infty : t_i \to 0^+ \Longrightarrow 2t_i^{-2}(f(\overline{x} + t_i d) - f(\overline{x}) - t_i \nabla f(\overline{x})d) \to F\}.$$

Since $f \in C^{1,1}(S)$, there exists $C > 0$ such that for any $i \in \{1, 2, \ldots\}$ there exists $\tilde{x}_i \in [\overline{x}, \overline{x} + t_i d] \subset E_n$ such that

$$(2.1) \quad \|t_i^{-2}(f(\overline{x} + t_i d) - f(\overline{x}) - t_i \nabla f(\overline{x})d)\| = t_i^{-2}\|t_i \nabla f(\tilde{x}_i)d - t_i \nabla f(\overline{x})d\|$$
$$\leqslant t_i^{-1}\|\nabla f(\tilde{x}_i) - \nabla f(\overline{x})\|\|d\| \leqslant Ct_i^{-1}\|\tilde{x}_i - \overline{x}\|\|d\| \leqslant C\|d\|^2.$$

Hence,
$$\{t_i^{-2}(f(\overline{x} + t_i d) - f(\overline{x}) - t_i \nabla f(\overline{x})d)\}_{i=1}^\infty$$

is bounded for any $\overline{x} \in S$ and $d \in E_n$, and thus this sequence has at least one accumulation point. This means that $\partial_*^2 f(\overline{x})(d, d)$ is well-defined and nonempty.

In Example 1.1, $\overline{x} = 0$, $f(0) = 0$ and $f'(0) = \varphi(0) = 0$. Thus for any fixed $d \in E_1$ we obtain

$$|2t_i^{-2}(f(\overline{x} + t_i d) - f(\overline{x}) - t_i f'(\overline{x})d)| = |2t_i^{-2}f(t_i d)| = 2t_i^{-2}\left|\int_0^{|t_i d|} \varphi(t)\, \mathrm{d}t\right|$$

$$\leqslant 2t_i^{-2}\left|\int_0^{|t_i d|} 3t^2\, \mathrm{d}t\right| = 2t_i^{-2}[t^3]_0^{|t_i d|} \to 0 \qquad \text{as} \ \ t_i \to 0^+.$$

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Hence,

$$(2.2) \qquad \partial_*^2 f(0)(d, d) = \{0\} \qquad \forall d \in E_1.$$

Obviously, if f is twice differentiable at $\overline{x}$ then

$$(2.3) \qquad \partial_*^2 f(\overline{x})(d, d) = \{d^T \nabla^2 f(\overline{x}) d\} = \partial^2 f(\overline{x})(d, d) \qquad \forall d \in E_n,$$

where

$$\partial^2 f(\overline{x})(d, d) = \{d^T M d \mid M \in \partial^2 f(\overline{x})\}, \qquad d \in E_n.$$

We know from Liu [13] that if $f \in C^{1,1}(S)$ and $\overline{x} \in S$ then for each $d \in E_n$ we have

$$(2.4) \qquad \partial_*^2 f(\overline{x})(d, d) \subset \partial^2 f(\overline{x})(d, d).$$

From Example 1.1 and (2.2) we get that for all $d \in E_1$, $d \neq 0$,

$$\partial_*^2 f(0)(d, d) = \{0\} \subsetneqq [-d^2, d^2] = \partial^2 f(0)(d, d).$$

This shows that (2.4) is not the equality, in general.

3. The second order optimality conditions for $C^{1,1}$ problem

First we introduce a second order necessary optimality condition and a second order sufficient optimality condition for the unconstrained problem (1.1).

Theorem 3.1. Let $f \in C^{1,1}(S)$. If $\overline{x} \in S$ is a local minimum for problem (1.1) then $\nabla f(\overline{x}) = 0$ and for all $d \in E_n$ and for all $F \in \partial_*^2 f(\overline{x})(d, d)$, we have $F \geqslant 0$.

P r o o f   (see [13] or [14]). $\square$

As a consequence we get the well-known result:

Corollary 3.2. Let $f \in C^2(S)$. If $\overline{x} \in S$ is a local minimum for problem (1.1) then for all $d \in E_n$ we have

$$d^T \nabla^2 f(\overline{x}) d \geqslant 0.$$

R e m a r k 3.3. From (2.4) and Theorem 3.1, we can easily obtain Hiriart-Urruty's result which is introduced below (1.1), where only the existence of an element from the generalized Hessian matrix is guaranteed. On the other hand, the second order necessary condition from Theorem 3.1 holds for all $d \in E_n$ and for all $F \in \partial_*^2 f(\overline{x})(d, d)$.

Theorem 3.4. *Let $f \in C^{1,1}(S)$ and $\overline{x} \in S$. If $\nabla f(\overline{x}) = 0$ and if for all $d \in E_n$, $d \neq 0$, and for all $F \in \partial_*^2 f(\overline{x})(d,d)$ we have $F > 0$, then $\overline{x}$ is a strict local minimum for problem (1.1).*

The p r o o f can be obtained from [13] or from the proof of Theorem 3.9. $\qquad \square$

E x a m p l e 3.5. Set
$$\overline{f}(x) = f(x) + \tfrac{1}{2}x^2,$$
where f is from Example 1.1. Since $f(0) = 0$, it is obvious that $\overline{x} = 0$ is a local (and also global) minimum point of $\overline{f}$. For all $d \in E_n$ and $F \in \partial_*^2 \overline{f}(0)(d,d) = \{d^2\}$, we have $F \geqslant 0$ (cf. Theorem 3.1) and if $d \neq 0$ then $F > 0$ (cf. Theorem 3.4).

As a consequence of Theorem 3.4 we get the well-known result:

Corollary 3.6. *If $f \in C^2(S)$, $\overline{x} \in S$, $\nabla f(\overline{x}) = 0$ and if for any $d \in E_n$, $d \neq 0$, we have*
$$d^T \nabla^2 f(\overline{x})d > 0,$$
then $\overline{x}$ is a strict local minimum for problem (1.1).

Second we will generalize the above two theorems to the following inequality and equality constrained minimization problem:

$$(3.1) \qquad\qquad \text{minimize} \quad f(x)$$
$$(3.2) \qquad\qquad \text{subject to} \quad g_j(x) \leqslant 0, \quad j = 1, \ldots, \ell,$$
$$(3.3) \qquad\qquad\qquad\qquad\quad h_k(x) = 0, \quad k = 1, \ldots, m,$$

where f, g_j, $j = 1, \ldots, \ell$, and h_k, $k = 1, \ldots, m$, are $C^{1,1}$ functions on

$$R = \{x \in E_n \mid g_j(x) \leqslant 0, \ j = 1, \ldots, \ell; \ h_k(x) = 0, \ k = 1, \ldots, m\}.$$

Suppose that R is nonempty and let $\overline{x} \in R$ be a local minimum for problem (3.1)–(3.3). Moreover, assume the following constraint qualification:

$(H):$ $\qquad \nabla g_j(\overline{x})$, $j \in J(\overline{x})$, and $\nabla h_k(\overline{x})$, $k = 1, \ldots, m$, are linearly independent,

where $J(\overline{x}) = \{j \mid g_j(\overline{x}) = 0\}$, is satisfied. Then there exists (cf. [16]) a vector $(\lambda_1, \ldots, \lambda_\ell, \mu_1, \ldots, \mu_m) \in E_{\ell+m}$, such that the Kuhn-Tucker Optimality Conditions

$$(3.4) \qquad\qquad \nabla f(\overline{x}) + \sum_{j=1}^{\ell} \lambda_j \nabla g_j(\overline{x}) + \sum_{k=1}^{m} \mu_k \nabla h_k(\overline{x}) = 0,$$
$$(3.5) \qquad\qquad \lambda_j \geqslant 0 \quad \text{and} \quad \lambda_j g_j(\overline{x}) = 0 \quad \forall j = 1, \ldots, \ell,$$

are satisfied.

To get the second order conditions, we associate with each multiplier $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ a set $G(\lambda)$ defined as follows:

$$G(\lambda) = \left\{ \quad x \in E_n \quad \left| \quad \begin{array}{ll} g_j(x) = 0 & \text{for } j \text{ such that } \lambda_j > 0 \\ g_j(x) \leqslant 0 & \text{for } j \text{ such that } \lambda_j = 0 \\ h_k(x) = 0 & \text{for } k = 1, \ldots, m \end{array} \right. \right\}$$

and denote the cone of feasible directions to $G(\lambda)$ at $\overline{x}$ by

$$(3.6) \qquad D(\overline{x}; \lambda) = \{d \mid \exists \delta > 0 \quad \forall \theta \in (0, \delta] \quad \overline{x} + \theta d \in G(\lambda)\}.$$

If we express the usual Lagrangian function by

$$(3.7) \qquad \mathcal{L}(x; \lambda, \mu) = f(x) + \sum_{j=1}^{\ell} \lambda_j g_j(x) + \sum_{k=1}^{m} \mu_k h_k(x),$$

where $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ and $\mu = (\mu_1, \ldots, \mu_m)$, and denote the generalized second order directional derivative of $\mathcal{L}(.; \lambda, \mu)$ at $\overline{x}$ by $\partial_*^2 \mathcal{L}(\overline{x}; \lambda, \mu)(d, d)$, then the second order necessary condition can be formulated as follows:

Theorem 3.7. *Let $\overline{x}$ be a local minimum of problem (3.1)–(3.3) and let (H) hold. Then for each Kuhn-Tucker multiplier vector (λ, μ) satisfying (3.4) and (3.5) at $\overline{x}$, for each $d \in D(\overline{x}; \lambda)$ and for each $L \in \partial_*^2 \mathcal{L}(\overline{x}; \lambda, \mu)(d, d)$ we have $L \geqslant 0$.*

P r o o f. On the contrary assume that there exist (λ, μ), $d \in D(\overline{x}; \lambda)$ and $L \in \partial_*^2 \mathcal{L}(\overline{x}; \lambda, \mu)(d, d)$ such that $L < 0$. Then there exists a sequence $\{t_i\}_{i=1}^{\infty}$, $t_i \to 0^+$ as $i \to \infty$, such that

$$(3.8) \qquad t_i^{-2}(\mathcal{L}(\overline{x} + t_i d; \lambda, \mu) - \mathcal{L}(\overline{x}; \lambda, \mu) - t_i \nabla \mathcal{L}(\overline{x}; \lambda, \mu)d) < 0.$$

Since $d \in D(\overline{x}; \lambda)$ and $t_i \to 0^+$ as $i \to \infty$, we see from (3.6) that there exists an integer i_0 such that $\overline{x} + t_i d \in G(\lambda)$ for all $i > i_0$, and thereby $\mathcal{L}(\overline{x} + t_i d; \lambda, \mu) = f(\overline{x} + t_i d)$. But this together with (3.4), (3.5), (3.7) and (3.8) implies $f(\overline{x} + t_i d) < f(\overline{x})$ for all $i > i_0$, which contradicts the fact that $\overline{x}$ is a local minimum for problem (3.1)–(3.3). $\square$

From this theorem and (2.3) we can easily get the following result:

Corollary 3.8. *Let f, g_j, $j = 1, \ldots, \ell$, and h_k, $k = 1, \ldots, m$, be C^2 functions at $\overline{x} \in R$ and let (H) be assumed. If $\overline{x}$ is a local minimum for problem (3.1)–(3.3)*

then there exists a Kuhn-Tucker multiplier vector (λ, μ) satisfying (3.4) and (3.5) at $\overline{x}$ and for any $d \in D(\overline{x}; \lambda)$ we have

$$d^T \nabla^2 \mathcal{L}(\overline{x}; \lambda, \mu)d \geqslant 0.$$

If we define the tangent cone to R at $\overline{x}$ by

(3.9) $T(\overline{x}) = \{d \mid \exists\{t_i\}_{i=1}^{\infty}, \ t_i \to 0^+, \exists\{d_i\}_{i=1}^{\infty}, \ d_i \to d : \overline{x} + t_i d_i \in R \ \forall i = 1, 2, \ldots\}$

then we have the second order sufficient condition for the problem (3.1)–(3.3):

Theorem 3.9. *Let f, g_j, $j = 1, \ldots, \ell$, and h_k, $k = 1, \ldots, m$, be $C^{1,1}$ functions at $\overline{x} \in R$. If there exists a Kuhn-Tucker multiplier vector (λ, μ) satisfying (3.4) and (3.5) at $\overline{x}$ and if for each $d \in T(\overline{x})$, $d \neq 0$, and for each $L \in \partial_*^2 \mathcal{L}(\overline{x}; \lambda, \mu)(d, d)$ we have $L > 0$, then $\overline{x}$ is a strict local minimum of problem (3.1)–(3.3).*

P r o o f.   Suppose $\overline{x} \in R$ is not a strict local minimum for problem (3.1)–(3.3). Then there exists a sequence $\{x_i\}_{i=1}^{\infty} \subset R$, $x_i \to \overline{x}$ as $i \to \infty$, such that $x_i \neq \overline{x}$ and

(3.10) $f(x_i) \leqslant f(\overline{x}) \qquad \forall i = 1, 2, \ldots .$

We may suppose $x_i = \overline{x} + t_i d_i$, where $t_i \to 0^+$ as $i \to \infty$ and $\|d_i\| = 1$.

Since $\mathcal{L}(x; \lambda, \mu)$ is given by (3.7), we immediately see that $\mathcal{L}(.; \lambda, \mu) \in C^{1,1}(R)$. Hence, we can prove like in (2.1) that the sequence

(3.11) $\{2t_i^{-2}(\mathcal{L}(\overline{x} + t_i d_i; \lambda, \mu) - \mathcal{L}(\overline{x}; \lambda, \mu) - t_i \nabla \mathcal{L}(\overline{x}; \lambda, \mu)d_i)\}_{i=1}^{\infty}$

is bounded. So, there exists a convergent subsequence and we might assume (3.11) is convergent. Denote its limit by L. Since $\|d_i\| = 1$, we can select a converging subsequence of $\{d_i\}$, which converges to $d \neq 0$ and which is for simplicity denoted again by $\{d_i\}$, i.e., $d_i \to d$ as $i \to \infty$. Recalling that $t_i \to 0^+$ as $i \to \infty$ and $\overline{x} + t_i d_i \in R$ for all i, we conclude by (3.9) that $d \in T(\overline{x})$. Now we check whether

(3.12) $L \in \partial_*^2 \mathcal{L}(\overline{x}; \lambda, \mu)(d, d).$

By (3.11), the definition of L and the mean-value theorem, we see that there exists a sequence $\varepsilon_i \to 0$ as $i \to \infty$ such that

$$L = 2t_i^{-2}(\mathcal{L}(\overline{x} + t_i d_i; \lambda, \mu) - \mathcal{L}(\overline{x}; \lambda, \mu) - t_i \nabla \mathcal{L}(\overline{x}; \lambda, \mu)d_i) + \varepsilon_i$$

$$= 2t_i^{-2} \int_0^1 (t_i \nabla \mathcal{L}(\overline{x} + st_i d_i; \lambda, \mu)d_i - t_i \nabla \mathcal{L}(\overline{x}; \lambda, \mu)d_i)\,\mathrm{d}s + \varepsilon_i$$

$$(3.13) \qquad = 2t_i^{-1} \int_0^1 (\nabla \mathcal{L}(\overline{x} + st_i d_i; \lambda, \mu) - \nabla \mathcal{L}(\overline{x}; \lambda, \mu))(d_i - d)\,\mathrm{d}s$$

$$+ 2t_i^{-1} \int_0^1 (\nabla \mathcal{L}(\overline{x} + st_i d_i; \lambda, \mu) - \nabla \mathcal{L}(\overline{x} + st_i d; \lambda, \mu))d\,\mathrm{d}s$$

$$+ 2t_i^{-1} \int_0^1 (\nabla \mathcal{L}(\overline{x} + st_i d; \lambda, \mu) - \nabla \mathcal{L}(\overline{x}; \lambda, \mu))d\,\mathrm{d}s + \varepsilon_i.$$

Recall that ∇L is locally Lipschitz continuous and $\|d_i\| = \|d\| = 1$. Consequently, there exist $C_1 > 0$ and $C_2 > 0$ such that

$$|2t_i^{-1} \int_0^1 (\nabla \mathcal{L}(\overline{x} + st_i d_i; \lambda, \mu) - \nabla \mathcal{L}(\overline{x}; \lambda, \mu))(d_i - d)\,\mathrm{d}s|$$

$$(3.14) \qquad \leqslant 2t_i^{-1} \int_0^1 \|\nabla \mathcal{L}(\overline{x} + st_i d_i; \lambda, \mu) - \nabla \mathcal{L}(\overline{x}; \lambda, \mu)\| \|d_i - d\|\,\mathrm{d}s$$

$$\leqslant 2t_i^{-1} \int_0^1 C_1 st_i \|d_i\| \|d_i - d\|\,\mathrm{d}s = C_1 \|d_i - d\| \to 0 \quad \text{as} \quad i \to \infty$$

and

$$|2t_i^{-1} \int_0^1 (\nabla \mathcal{L}(\overline{x} + st_i d_i; \lambda, \mu) - \nabla \mathcal{L}(\overline{x} + st_i d; \lambda, \mu))d\,\mathrm{d}s|$$

$$(3.15) \qquad \leqslant 2t_i^{-1} \int_0^1 \|\nabla \mathcal{L}(\overline{x} + st_i d_i; \lambda, \mu) - \nabla \mathcal{L}(\overline{x} + st_i d; \lambda, \mu)\| \|d\|\,\mathrm{d}s$$

$$\leqslant C_2 \|d_i - d\| \to 0 \qquad \text{as} \quad i \to \infty.$$

By (3.13)–(3.15) and the mean-value theorem we have

$$(3.16) \qquad L = 2t_i^{-2}(\mathcal{L}(\overline{x} + t_i d; \lambda, \mu) - \mathcal{L}(\overline{x}; \lambda, \mu) - t_i \nabla \mathcal{L}(\overline{x}; \lambda, \mu)d) + \varepsilon_i',$$

where $\varepsilon_i' \to 0$ as $i \to \infty$. Letting $i \to \infty$, we get from (3.16) that (3.12) holds.

Recalling that $x_i = \overline{x} + t_i d_i \in R$, we obtain

$$(3.17) \qquad\qquad g_j(\overline{x} + t_i d_i) \leqslant 0, \qquad j = 1, \ldots, \ell,$$

$$(3.18) \qquad\qquad h_k(\overline{x} + t_i d_i) = 0, \qquad k = 1, \ldots, m.$$

Hence, from the first equality in (3.13) and from (3.7), (3.18), (3.4), (3.5), (3.17) and (3.10) we have

$$(3.19) \qquad\qquad\qquad L \leqslant 0.$$

Note that the relations (3.12) and (3.19) contradict the assumption $L > 0$ for all $L \in \partial_*^2 \mathcal{L}(\overline{x}; \lambda, \mu)(d, d)$ and all $d \in E_n$, $d \neq 0$. $\qquad\qquad\square$

The following corollary can be obtained directly from this theorem and (2.3).

Corollary 3.10. Let f, g_j, $j = 1, \ldots, \ell$, and h_k, $k = 1, \ldots, m$, be C^2 functions at $\overline{x} \in R$. If there exists a Kuhn-Tucker multiplier vector (λ, μ) satisfying (3.4) and (3.5) at $\overline{x}$, and if for any $d \in T(\overline{x})$, $d \neq 0$, we have

$$d^T \nabla^2 \mathcal{L}(\overline{x}; \lambda, \mu) d > 0,$$

then $\overline{x}$ is a strict local minimum for problem (3.1)–(3.3).

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