Applications of Mathematics

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Applications of Mathematics, Vol. 41 (1996), No. 6, 467–478

Persistent URL: http://dml.cz/dmlcz/134338

HIGHER ORDER FINITE ELEMENT APPROXIMATION OF A QUASILINEAR ELLIPTIC BOUNDARY VALUE PROBLEM OF A NON-MONOTONE TYPE

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(Received May 3, 1996)

Summary. A nonlinear elliptic partial differential equation with homogeneous Dirichlet boundary conditions is examined. The problem describes for instance a stationary heat conduction in nonlinear inhomogeneous and anisotropic media. For finite elements of degree $k \geqslant 1$ we prove the optimal rates of convergence $\mathcal{O}(h^k)$ in the H^1-norm and $\mathcal{O}(h^{k+1})$ in the L^2-norm provided the true solution is sufficiently smooth. Considerations are restricted to domains with polyhedral boundaries. Numerical integration is not taken into account.

Keywords: nonlinear boundary value problem, finite elements, rate of convergence, anisotropic heat conduction

AMS classification: 65N30

1. INTRODUCTION

In this paper we deal with a quasilinear elliptic problem whose classical formulation reads:

Find $u \in C(\overline{\Omega})$ such that $u|_{\Omega} \in C^2(\Omega)$ and

$$-\operatorname{div}(A(x,u)\operatorname{grad} u) = f \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^d$, $d \in \{1,2,3\}$, is a bounded domain with a Lipschitz boundary, $f \in L^2(\Omega)$, $A = (a_{ij})_{i,j=1}^d$ is a bounded uniformly positive definite matrix, i.e.,

$$\max_{x\in\Omega}\max_{\xi\in\mathbb{R}} |a_{ij}(x,\xi)| \leqslant C \qquad \forall i,j \in \{1,\dots,d\}, \tag{1.3}$$

$$C_0\eta^T\eta \leqslant \eta^T A(x,\xi)\eta \quad \forall \eta \in \mathbb{R}^d \quad \forall x \in \Omega \quad \forall \xi \in \mathbb{R}, \tag{1.4}$$

where $C_0 > 0$ and, moreover, we assume that the derivatives $\partial a_{ij}/\partial\xi$ and $\partial^2 a_{ij}/\partial\xi^2$ are bounded and continuous on $\overline{\Omega} \times \mathbb{R}$. The matrix A need not be symmetric.

The problem (1.1)–(1.2) for $d > 1$ cannot be converted, in general, by the well-known Kirchhoff transformation to a linear problem even if A is independent of x, since A is a matrix function.

The existence of a weak solution u is obtained as a weak limit of Galerkin approximations. The uniqueness of the classical and weak solutions is proved in [13] and [14], respectively. Several uniqueness and comparison theorems for similar problems can be found in [1, 5, 11, 16]. The existence of the weak solution for various kinds of boundary conditions (including (1.2)) is studied in [9, 11, 14, 21].

In [4], Douglas and Dupont derived an optimal rate of convergence of the finite element method for the problem (1.1)–(1.2) in the case that

$$A(x,u) = \lambda(x,u)\, I, \tag{1.5}$$

where I is the identity matrix and λ is a smooth scalar function. The main aim of this paper (see Theorem 4.1) is to generalize the result of [4] to any smooth uniformly positive definite matrix $A(x,u)$ satisfying (1.3) and (1.4). This represents a practically interesting case, since the problem (1.1)–(1.2) describes a steady-state heat conduction in nonlinear inhomogeneous anisotropic media (e.g., in magnetic cores of large transformers, see [17]). The unknown function u represents the temperature, A is the matrix of heat conductivities and f is the density of volume heat sources. In this case A is symmetric.

The finite element method for the case (1.5) has been considered by many other authors. For instance, in [22], the method of Douglas and Dupont from [4] is generalized to obtain an asymptotic error estimate in the L^∞-norm. An optimal rate of convergence in the L^p-norm is proved in [19] for a mixed finite element method. Similar results were also obtained in the paper [2].

Note that an analogue of the well-known Céa's lemma holds for those nonlinear elliptic problems whose associated operators are strongly monotone and Lipschitz continuous (see [3, 17]). Hence, in this case it is easy to derive the rate of convergence $\mathcal{O}(h^k)$ in the H^1-norm for the Lagrange elements of degree k. However, the papers [9, 14] contain one-dimensional examples which illustrate that the problem (1.1)–(1.2) is of a non-monotone and non-potential type.

Finite element approximations of nonlinear elliptic problems of strongly monotone and also pseudomonotone type are profoundly studied in [7, 8, 27]. The authors consider the numerical integration as well as the approximation of a curved boundary. They obtain a linear rate of convergence in the H^1-norm for linear finite elements provided the true solution is sufficiently smooth. In [27], the rate of convergence

$\mathcal{O}(h^\varepsilon)$ is proved for $u \in H^{1+\varepsilon}(\Omega)$. However, the papers [7, 8, 27] do not deal with higher order elements and the optimal error estimates in the $L^2(\Omega)$-norm.

2. Weak formulation and finite element approximation

Throughout the paper we shall employ the standard Sobolev space notation (see [3, 20]). The norm in the product Sobolev space $(W_p^k(\Omega))^n$, $k \in \{0, 1, \dots\}$, $p \in [1, \infty]$, $n \in \{1, 2, \dots\}$, is denoted by $\|\cdot\|_{k,p}$. In particular, if $p = 2$ then we set $H^k(\Omega) = W_2^k(\Omega)$ and $\|\cdot\|_k = \|\cdot\|_{k,2}$. By $H_0^1(\Omega)$ we mean the space of functions from $H^1(\Omega)$ whose traces vanish on $\partial\Omega$. The symbol $(\cdot,\cdot)_0$ stands for the usual scalar product in $L^2(\Omega)$.

According to the Cauchy-Schwarz and Hölder inequalities, we get

$$\|v\|_{0,3}^3 \leqslant \|v\|_0 \|v^2\|_0 \leqslant \|v\|_0 \|v\|_{0,3} \|v\|_{0,6} \quad \forall v \in L^6(\Omega).$$

From here and the imbedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ for $d \leqslant 3$ (see [3, p. 114]) we find the inequality which will be used later:

$$\|v\|_{0,3} \leqslant C(\|v\|_0 \|v\|_1)^{1/2} \quad \forall v \in H^1(\Omega). \tag{2.1}$$

The weak formulation of problem (1.1)–(1.2) consists in finding $u \in H_0^1(\Omega)$ such that

$$a(u; u, v) = (f, v)_0 \quad \forall v \in H_0^1(\Omega), \tag{2.2}$$

where

$$a(z; w, v) = \int_\Omega (\operatorname{grad} w)^T A(x, z) \operatorname{grad} v \, dx, \quad v,\, w \in H^1(\Omega),\ z \in L^2(\Omega).$$

The argument x will be sometimes omitted in what follows. From (1.4), (1.3) and Friedrichs' inequality we see that there exist positive constants C_0 and C_1 such that

$$a(z; v, v) \geqslant C_0 \|v\|_1^2 \quad \forall z \in L^2(\Omega) \quad \forall v \in H_0^1(\Omega)$$

and

$$|a(z; w, v)| \leqslant C_1 \|v\|_1 \|w\|_1 \quad \forall z \in L^2(\Omega) \quad \forall w, v \in H^1(\Omega).$$

This means that $a(\cdot;\cdot,\cdot)$ is uniformly $H_0^1(\Omega)$-elliptic and continuous.

Theorem 2.1. *The weak solution of* (2.2) *exists and is unique.*

The proof is given in [14]. □

From now on assume that $\Omega \subset \mathbb{R}^d$, $d \in \{1,2,3\}$, has a polyhedral boundary and let $\mathcal{T}_h$ be a standard triangulation of $\overline{\Omega}$ into polyhedral elements (see [3]). Let us introduce the approximate problem: Find $u_h \in V_h$ such that

$$a(u_h; u_h, v_h) = (f, v_h)_0 \quad \forall v_h \in V_h, \tag{2.3}$$

where

$$V_h = \{v_h \in H_0^1(\Omega) \mid v_h\big|_K \in P_K \quad \forall K \in \mathcal{T}_h\}$$

is the finite element space, P_K is a finite dimensional space such that $P_K \supseteq P_k(K)$, $k \geqslant 1$ is an integer and $P_k(K)$ is the space of all polynomials of degree at most k defined on K. The space V_h can be generated by the Lagrange elements (or Hermite elements for $k \geqslant 3$).

Remark 2.2. The existence of at least one solution u_h of (2.3) can be proved by the Brouwer fixed-point theorem (see [14, p. 174]). Some special sufficient conditions guaranteeing the uniqueness of u_h are given in [12, 14]. Nevertheless, the uniqueness of u_h, in general, has remained an open problem until now.

Remark 2.3. In [6], the existence of a discrete solution is proved in the case of linear elements, numerical integration and approximation of a piecewise curved boundary by a polygonal one. The proof is based on some results of [7, 8, 26]. A discrete maximum principle for the problem (1.1)–(1.2) in the case (1.5) is derived in [15]. The publications [17, 18, 24] are devoted to numerical calculation of real-life technical problems which are described by the equation (1.1).

Remark 2.4. The convergence of approximate solutions u_h to the weak solution u of (1.1)–(1.2) in the $H^1(\Omega)$-norm was proved in [14]. However, no attempt to derive any rate of convergence was made there.

Finally, we introduce an auxiliary lemma which will be used in Section 4.

Lemma 2.5. *Let α, β and γ be arbitrary real nonnegative numbers such that*

$$\alpha \leqslant C(\beta + \sqrt{\alpha\gamma}). \tag{2.4}$$

Then there exists a constant $C' > 0$ independent of α, β, γ such that

$$\alpha \leqslant C'(\beta + \gamma). \tag{2.5}$$

Proof. If $\alpha = 0$ then (2.5) holds. So let $\alpha \neq 0$. Then by (2.4)

$$C^2\gamma \geqslant \frac{(\alpha - C\beta)^2}{\alpha} \geqslant \alpha - 2C\beta.$$

□

3. Adjoint problem

In the next section we derive the optimal a priori asymptotic error estimate in the $H^1(\Omega)$-norm and also in the $L^2(\Omega)$-norm. In the latter case, we will employ the Aubin-Nitsche trick. To this end we shall utilize the weak solution φ of the linear problem

$$
\begin{aligned}
L^*\varphi \equiv -\operatorname{div}(A^T(x,u)\ \operatorname{grad}\varphi) + (\operatorname{grad} u)^T A_u^T(x,u)\ \operatorname{grad}\varphi &= \zeta \quad \text{in } \Omega,\\
\varphi &= 0 \quad \text{on } \partial\Omega,
\end{aligned}
\tag{3.1}
$$

where u is the unique solution of (2.2), $\zeta \in L^2(\Omega)$, $A_u = ((a_{ij})_u)_{i,j=1}^d$ and the subscript u means the differentiation with respect to the last variable, i.e., $(a_{ij})_u = \partial a_{ij}(x,u)/\partial u$. In Theorem 3.1, we give a sufficient condition guaranteeing the existence and uniqueness of the weak (generalized) solution of the problem (3.1).

First we show how the above problem (3.1) can formally be obtained. Set

$$\mathcal{L}(u) = -\operatorname{div}(A(u)\ \operatorname{grad} u)$$

and choose $v \in H_0^1(\Omega) \cap H^2(\Omega)$ arbitrarily. Then for any real $t \neq 0$ we have

$$
\begin{aligned}
\frac{1}{t}(\mathcal{L}(u+tv) - \mathcal{L}(u)) &= -\frac{1}{t}\operatorname{div}(A(u+tv)\ \operatorname{grad}(u+tv) - A(u)\ \operatorname{grad} u\\
&\quad - A(u)\ \operatorname{grad}(tv) + tA(u)\ \operatorname{grad} v)\\
&= -\operatorname{div}\Big(\frac{A(u+tv) - A(u)}{t}\ \operatorname{grad}(u+tv) + A(u)\ \operatorname{grad} v\Big).
\end{aligned}
$$

Letting $t \to 0$, we obtain the Gâteaux derivative of $\mathcal{L}$ at the point u and in the direction v

$$Lv \equiv D\mathcal{L}(u;v) = -\operatorname{div}(A(x,u)\ \operatorname{grad} v + vA_u(x,u)\ \operatorname{grad} u).$$

Notice that this operator is linear.

Now choose $\varphi \in H_0^1(\Omega) \cap H^2(\Omega)$ arbitrarily. Then, applying twice the Green theorem, we get

$$
\begin{aligned}
(Lv,\varphi)_0 &= -\int_\Omega \operatorname{div}(A(u)\ \operatorname{grad} v + vA_u(u)\ \operatorname{grad} u)\varphi\, dx\\
&= \int_\Omega (\operatorname{grad}\varphi)^T(A(u)\ \operatorname{grad} v + vA_u(u)\ \operatorname{grad} u)\, dx\\
&= \int_\Omega v(-\operatorname{div}(A^T(u)\ \operatorname{grad}\varphi) + (\operatorname{grad} u)^T A_u^T(u)\ \operatorname{grad}\varphi)\, dx\\
&= (v, L^*\varphi)_0,
\end{aligned}
$$

i.e., the linear operator L^* is adjoint to L. If $A(u)$ is independent of u then, of course, $A_u = 0$ and we get the standard adjoint problem like in [3, p. 138].

The weak formulation of (3.1) reads: Find $\varphi \in H_0^1(\Omega)$ such that

$$b(\varphi, v) = (\zeta, v)_0 \quad \forall v \in H_0^1(\Omega),$$

where

$$b(\varphi, v) = \int_\Omega [(\operatorname{grad} v)^T \mathcal{A}^T \operatorname{grad} \varphi + v c^T \operatorname{grad} \varphi] \, dx,$$

$$\mathcal{A}(x) = A(x, u(x)),$$
$$c(x) = A_u(x, u(x)) \operatorname{grad} u(x)$$

for $x \in \Omega$ and $u \in H_0^1(\Omega)$ is the unique weak solution of (1.1)–(1.2) (compare Theorem 2.1).

Theorem 3.1. *Let $c \in (L^\infty(\Omega))^d$ and let (1.3) and (1.4) hold. Then there exists precisely one weak solution $\varphi \in H_0^1(\Omega)$ of the classical problem (3.1).*

Proof. By (1.3) and (1.4), the matrix $\mathcal{A}$ is bounded and uniformly positive definite. Since c is also bounded, the bilinear form $b(\cdot,\cdot)$ is continuous and the theorem directly follows from [11, p. 170]. (The proof of uniqueness of φ is based on the weak maximum principle and the existence of φ is a consequence of the Gårding inequality, the Fredholm alternative and the uniqueness.) □

Remark 3.2. If the weak solution u of the problem (1.1)–(1.2) belongs to the space of Lipschitz continuous functions $W_\infty^1(\Omega)$, then the assumption $c \in (L^\infty(\Omega))^d$ of the above Theorem 3.1 is obviously satisfied.

Remark 3.3. If $c \in (C^1(\overline{\Omega}))^d$ and div $c \leqslant 0$ then for any $v \in H_0^1(\Omega)$ we get, by the Green theorem, that

$$(vc, \operatorname{grad} v)_0 = -(\operatorname{div}(cv), v)_0 = -(v \operatorname{div}\, c, v)_0 - (vc, \operatorname{grad} v)_0$$

and thus

$$(vc, \operatorname{grad} v)_0 = -\frac{1}{2}(\operatorname{div}\, c, v^2)_0 \geqslant 0.$$

Hence, the bilinear form is $H_0^1(\Omega)$-elliptic (see also [20, p. 44]),

$$b(v, v) \geqslant \int_\Omega (\operatorname{grad} v)^T \mathcal{A}^T \operatorname{grad} v \, dx \geqslant C_0 \|v\|_1^2 \quad \forall v \in H_0^1(\Omega)$$

by the Friedrichs inequality and thus the well-known Lax-Milgram lemma [3, 17, 20] can be applied.

In the next Section 4 we shall, moreover, require the regularity

$$\|\varphi\|_2 \leqslant C\|\zeta\|_0, \tag{3.2}$$

where $\varphi \in H_0^1(\Omega) \cap H^2(\Omega)$ is the weak solution of (3.1).

4. Rate of convergence

Throughout this section we assume that the family $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ of triangulations is regular, i.e., there exists a constant $\varkappa > 0$ such that for any triangulation $\mathcal{T}_h \in \mathcal{F}$ and any element $K \in \mathcal{T}_h$ there exists a ball B_K of radius ϱ_K such that $B_K \subset K$ and

$$\varkappa \operatorname{diam} K \leqslant \varrho_K. \tag{4.1}$$

Theorem 4.1. *Let $u \in H^{k+1}(\Omega)$ for $k \geqslant 1$ be the weak solution of* (1.1)–(1.2) *and let* (3.2) *hold. If u_h is a solution of* (2.3), *then there exist C, $h_0 > 0$ such that for any $h \in (0, h_0)$ we have*

$$\|u - u_h\|_0 + h\|u - u_h\|_1 \leqslant Ch^{k+1}, \tag{4.2}$$

where C depends on $\|u\|_{k+1}$.

Proof. Since $d \leqslant 3$, we have $H^2(\Omega) \hookrightarrow W_6^1(\Omega)$ (see [3, p. 114]) and thus $\|u\|_{1,6}$ is finite. According to [3, p. 123], for the solution $u \in H^{k+1}(\Omega)$ of (1.1)–(1.2) and sufficiently small h we obtain by the regularity of $\mathcal{F}$ (see (4.1)) that

$$\|u - \pi_h u\|_1 + h\|u - \pi_h u\|_{1,6} \leqslant Ch^k\|u\|_{k+1}, \tag{4.3}$$

where $\pi_h u \in V_h$ is the V_h-interpolant of u. In particular,

$$\|\pi_h u\|_{1,6} \leqslant \|u - \pi_h u\|_{1,6} + \|u\|_{1,6} \leqslant C\|u\|_{k+1}. \tag{4.4}$$

By the uniform $H_0^1(\Omega)$-ellipticity of $a(\cdot;\cdot,\cdot)$, (2.2), (2.3) and the Hölder inequality, we arrive at

$$\begin{aligned}
&C_0\|u_h - \pi_h u\|_1^2 \leqslant a(u_h; u_h - \pi_h u, u_h - \pi_h u)\\
&= a(u_h; u_h, u_h - \pi_h u) - a(u_h; \pi_h u, u_h - \pi_h u)\\
&= a(u; u, u_h - \pi_h u) - a(u_h; \pi_h u, u_h - \pi_h u)\\
&\leqslant |a(u; u - \pi_h u, u_h - \pi_h u)| + |a(u; \pi_h u, u_h - \pi_h u) - a(u_h; \pi_h u, u_h - \pi_h u)|\\
&\leqslant C_1\|u - \pi_h u\|_1\|u_h - \pi_h u\|_1 + C_2\|A(u) - A(u_h)\|_{0,3}\|\operatorname{grad} \pi_h u\|_{0,6}\|u_h - \pi_h u\|_1.
\end{aligned}$$

This, (4.4) and (4.3) imply that

$$(4.5)\qquad \begin{aligned} C_0\|u_h-\pi_h u\|_1 &\leqslant C_1\|u-\pi_h u\|_1 + C_3\|A(u)-A(u_h)\|_{0,3} \\ &\leqslant C(h^k\|u\|_{k+1} + \|A(u)-A(u_h)\|_{0,3}), \end{aligned}$$

where C_3 depends on u. Since the entries $a_{ij} = a_{ij}(x,\xi)$ are Lipschitz continuous with respect to the last variable ξ, we get by (2.1) that

$$\|A(u)-A(u_h)\|_{0,3} \leqslant C_1\|u-u_h\|_{0,3} \leqslant C_2(\|u-u_h\|_0\|u-u_h\|_1)^{1/2}.$$

From here, (4.3) and (4.5) it follows that

$$\|u-u_h\|_1 \leqslant \|u-\pi_h u\|_1 + \|u_h-\pi_h u\|_1 \leqslant C(h^k\|u\|_{k+1} + \|u-u_h\|_0^{1/2}\|u-u_h\|_1^{1/2}).$$

Setting

$$\zeta \equiv u-u_h \in L^2(\Omega),$$

we see by Lemma 2.5 that

$$(4.6)\qquad \|\zeta\|_1 \leqslant C(h^k\|u\|_{k+1} + \|\zeta\|_0).$$

In order to bound $\|\zeta\|_0 = \|u-u_h\|_0$, we use a duality argument (see [4, 23]) based on the Aubin-Nitsche trick. Let $\varphi \in H_0^1(\Omega)\cap H^2(\Omega)$ be the weak solution of the linear adjoint problem (3.1). Then, by the Green theorem,

$$(4.7)\qquad \begin{aligned} \|\zeta\|_0^2 &= \int_\Omega \zeta^2\,\mathrm{d}x = \int_\Omega \zeta(L^*\varphi)\,\mathrm{d}x \\ &= \int_\Omega [(\operatorname{grad}\zeta)^T A^T(u)\operatorname{grad}\varphi + \zeta\,(\operatorname{grad}u)^T A_u^T\operatorname{grad}\varphi]\,\mathrm{d}x \\ &= \int_\Omega [(\operatorname{grad}\varphi)^T A(u)\operatorname{grad}u - (\operatorname{grad}\varphi)^T A(u)\operatorname{grad}u_h \\ &\quad + \zeta\,(\operatorname{grad}u)^T A_u^T\operatorname{grad}\varphi]\,\mathrm{d}x \\ &= \int_\Omega [(\operatorname{grad}\varphi)^T A(u)\operatorname{grad}u - (\operatorname{grad}\varphi)^T A(u_h)\operatorname{grad}u_h \\ &\quad + (\operatorname{grad}\varphi)^T(A(u_h)-A(u))\operatorname{grad}u_h + \zeta\,(\operatorname{grad}\varphi)^T A_u\operatorname{grad}u]\,\mathrm{d}x. \end{aligned}$$

For any $x\in\Omega$ we have, by the mean value theorem,

$$\begin{aligned} A(x,u_h) - A(x,u) &= \int_0^1 A_u(x,u+t(u_h-u))(u_h-u)\,\mathrm{d}t \\ &= -\zeta\int_0^1 A_u(x,u-t\zeta)\,\mathrm{d}t = -\zeta\overline{A}_u(x), \end{aligned}$$

where $\overline{A}_u = ((\overline{a}_{ij})_u)_{i,j=1}^d$, and $(\overline{a}_{ij})_u = (a_{ij})_u(u - \theta_{ij}\zeta)$ for some $\theta_{ij} = \theta_{ij}^h(x) \in [0,1]$. Hence, for any $v_h \in V_h$ we obtain by (4.7), (2.2) and (2.3) that

$$
\begin{aligned}
\|\zeta\|_0^2 = & \int_\Omega [(\operatorname{grad}(\varphi - v_h))^T A(u) \operatorname{grad} u - (\operatorname{grad}(\varphi - v_h))^T A(u_h) \operatorname{grad} u_h \\
& + \zeta(\operatorname{grad}\varphi)^T \overline{A}_u \operatorname{grad}(\zeta - u) + \zeta(\operatorname{grad}\varphi)^T A_u \operatorname{grad} u] \, dx \\
= & \int_\Omega (\operatorname{grad}(\varphi - v_h))^T (A(u) - A(u_h)) \operatorname{grad} u \, dx \\
& + \int_\Omega (\operatorname{grad}(\varphi - v_h))^T A(u_h) \operatorname{grad}(u - u_h) \, dx \\
& + \int_\Omega \zeta \, (\operatorname{grad}\varphi)^T \overline{A}_u \operatorname{grad}\zeta \, dx + \int_\Omega \zeta \, (\operatorname{grad}\varphi)^T (A_u - \overline{A}_u) \operatorname{grad} u \, dx.
\end{aligned}
\tag{4.8}
$$

Using similar arguments as before, the differentiability of A and the substitution $z = st$, we find for any $x \in \Omega$ that

$$
\begin{aligned}
A_u(x,u) - \overline{A}_u(x) &= \int_0^1 [A_u(x,u) - A_u(x, u + t(u_h - u))] \, dt \\
&= \int_0^1 \Big(\int_0^1 A_{uu}(x, u + st(u_h - u)) t\zeta \, ds \Big) \, dt \\
&= -\zeta \int_0^1 \Big(\int_0^t A_{uu}(x, u - \zeta z) \, dz \Big) \, dt \\
&= -\zeta \int_0^1 \Big(\int_z^1 A_{uu}(x, u - \zeta z) \, dt \Big) \, dz \\
&= -\zeta \int_0^1 (1 - z) A_{uu}(x, u - \zeta z) \, dz = -\zeta \overline{A}_{uu}(x).
\end{aligned}
$$

Hence, since the derivatives of a_{ij} up to order two are bounded and since

$$\|\zeta\|_{0,3} \leqslant C\|\zeta\|_1$$

and $H^2(\Omega) \hookrightarrow W_6^1(\Omega)$ for $n \leqslant 3$, we have by (4.8) and the Hölder inequality that

$$
\begin{aligned}
\|\zeta\|_0^2 = & \int_\Omega \zeta (\operatorname{grad}(\varphi - v_h))^T \overline{A}_u \operatorname{grad} u \, dx \\
& + \int_\Omega (\operatorname{grad}(\varphi - v_h))^T A(u_h) \operatorname{grad}(u - u_h) \, dx \\
& + \int_\Omega \zeta \, (\operatorname{grad}\varphi)^T \overline{A}_u \operatorname{grad}\zeta \, dx - \int_\Omega \zeta^2 \, (\operatorname{grad}\varphi)^T \overline{A}_{uu} \operatorname{grad} u \, dx \\
\leqslant & \, C\|\zeta\|_{0,3} \|\operatorname{grad}(\varphi - v_h)\|_0 \|\operatorname{grad} u\|_{0,6} + C\|\varphi - v_h\|_1 \|\zeta\|_1 \\
& + C\|\zeta\|_{0,3} \|\operatorname{grad}\varphi\|_{0,6} \|\operatorname{grad}\zeta\|_0 + C\|\zeta\|_{0,3}^2 \|\operatorname{grad}\varphi\|_{0,6} \|\operatorname{grad} u\|_{0,6} \\
\leqslant & \, C(u)(\|\varphi - v_h\|_1 + \|\zeta\|_1 \|\varphi\|_2) \|\zeta\|_1
\end{aligned}
\tag{4.9}
$$

for any $v_h \in V_h$. Now choose $v_h \in V_h$ such that

$$\|\varphi - v_h\|_1 + h\|\varphi - v_h\|_{1,6} \leqslant Ch\|\varphi\|_2. \tag{4.10}$$

Then, by (4.9), we obtain

$$\|\zeta\|_0^2 \leqslant C(h + \|\zeta\|_1)\|\zeta\|_1\|\varphi\|_2,$$

where C depends on $\|u\|_2$. Therefore, the inequality (3.2) implies that

$$\|\zeta\|_0 \leqslant C(h\|\zeta\|_1 + \|\zeta\|_1^2).$$

Utilizing (4.6), we get

$$\|\zeta\|_0 \leqslant C(h^{k+1} + h\|\zeta\|_0 + h^{2k} + \|\zeta\|_0^2),$$

where C depends on $\|u\|_{k+1}$. Using (4.6) once again, we find that

$$\|\zeta\|_0 + h\|\zeta\|_1 \leqslant C(h^{k+1} + h\|\zeta\|_0 + h^{2k} + \|\zeta\|_0^2).$$

Consequently, for $k \geqslant 1$ and sufficiently small h we have

$$\|\zeta\|_0 + h\|\zeta\|_1 \leqslant C'(h^{k+1} + \|\zeta\|_0^2). \tag{4.11}$$

This inequality proves the theorem provided we can show that $\|\zeta\|_0 \to 0$ as $h \to 0$ (see also [14]).

From (4.3), (4.5) and the boundedness of A, we see that

$$\|u - u_h\|_1 \leqslant \|u - \pi_h u\|_1 + \|\pi_h u - u_h\|_1 \leqslant Ch^k\|u\|_{k+1} + C\alpha_1 \leqslant C.$$

Hence,

$$\|u_h\|_1 \leqslant C.$$

As a consequence of the Eberlein-Schmulyan theorem (see [25, Chap. V]) there exist an element $\omega \in H^1(\Omega)$ and a subsequence of $\{u_h\}$, denoted again by $\{u_h\}$, such that $u_h \rightharpoonup \omega$ in $H^1(\Omega)$. By the Rellich theorem (see [20, p. 17]), $u_h \to \omega$ in $L^2(\Omega)$. We wish to demonstrate that $\omega \equiv u$. To do that let $v \in C_0^\infty(\Omega)$. Then $\pi_h v \in V_h$ and we have $\|v - \pi_h v\|_1 \to 0$ as $h \to 0$. Therefore, by the relations (2.2), (2.3) and the Lipschitz continuity of a_{ij}, we get

$$\begin{aligned}|a(\omega;\omega,v) - (f,v)_0| &\leqslant |a(\omega;\omega - u_h, v)| + |a(\omega;u_h,v) - a(u_h;u_h,v)| \\ &\quad + |a(u_h;u_h,v-\pi_h v)| + |(f,\pi_h v - v)_0| \\ &\leqslant |a(\omega;\omega - u_h,v)| + C(v)\|\omega - u_h\|_0\|u_h\|_1 \\ &\quad + C_1\|u_h\|_1\|v - \pi_h v\|_1 + C_2\|v - \pi_h v\|_1 \to 0\end{aligned}$$

as $h \to 0$ due to the convergence of u_h to ω and $\pi_h v$ to v. By the density $\overline{C_0^\infty(\Omega)} = H_0^1(\Omega)$ we get that

$$a(\omega; \omega, v) = (f, v)_0 \quad \forall v \in H_0^1(\Omega).$$

Hence ω is the weak solution of (1.1)–(1.2). From the uniqueness of the weak solution of (1.1)–(1.2) it follows that $\omega \equiv u$ (see Theorem 2.1). It is easy to see that the whole original sequence $\{u_h\}$ converges to u. Hence, $\|\zeta\|_0 \to 0$ as $h \to 0$ and for h sufficiently small we obtain

$$C' \|\zeta\|_0^2 \leqslant \frac{1}{2} \|\zeta\|_0.$$

From (4.11) the inequality (4.2) follows. □

Remark 4.2. Asymptotic $L^\infty(\Omega)$-error estimates for quasilinear elliptic boundary value problems are established in [10, 22].

Acknowledgement. The authors wish to thank Prof. R. H. Nochetto from the University of Maryland for fruitful discussions. The work was supported by the COMAS graduate school at Jyväskylä and by grant No. A1019601 of the Grant Agency of the Academy of Sciences of the Czech Republic. These supports are gratefully acknowledged.

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