Mathematica Bohemica

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Mathematica Bohemica, Vol. 126 (2001), No. 2, 265–274

Persistent URL: http://dml.cz/dmlcz/134013

126 (2001) MATHEMATICA BOHEMICA No. 2, 265–274

LAPLACE EQUATION IN THE HALF-SPACE WITH A NONHOMOGENEOUS DIRICHLET BOUNDARY CONDITION

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Dedicated to Prof. J. Nečas on the occasion of his 70th birthday

Abstract. We deal with the Laplace equation in the half space. The use of a special family of weigted Sobolev spaces as a framework is at the heart of our approach. A complete class of existence, uniqueness and regularity results is obtained for inhomogeneous Dirichlet problem.

Keywords: the Laplace equation, weighted Sobolev spaces, the half space, existence, uniqueness, regularity

MSC 2000: 35J05, 58J10

1. Introduction

The purpose of this paper is to solve the problem

$$\text{(P)} \qquad \begin{cases} -\Delta u = f & \text{in} \quad \mathbb{R}^N_+, \\ \quad u = g & \text{on} \quad \Gamma = \mathbb{R}^{N-1}, \end{cases}$$

with the Dirichlet boundary condition on Γ. The approach is based on the use of a special class of weighted Sobolev spaces for describing the behavior at infinity. Many authors have studied the Laplace equation in the whole space $\mathbb{R}^N$ or in an exterior domain. The main difference is due to the nature of the boundary and one of difficulties is to obtain the appropriate spaces of traces. However, the half-space has a useful symmetric property.

Second author would like to thank the Grant Agency of the Czech Republic No. 201/99/0267 and both authors to the Barrande project between the Czech Republic and France.

Problem (P) has been investigated in weighted Sobolev spaces by several authors, but only in the Hilbert cases ($p=2$) and without the critical cases corresponding to the logarithmic factor (cf. [2], [4]). We can also mention the book by Simader, Sohr [5] where the Dirichlet problem for the Laplacian is investigated.

Let Ω be an open subset of $\mathbb{R}^N$, $N \geqslant 2$. Let $x=(x_1,\dots,x_N)$ be a typical point of $\mathbb{R}^N$ and $r=|x|=(x_1^2+\dots+x_N^2)^{1/2}$. We use two basic weights:

$$\varrho=(1+r^2)^{1/2} \quad \text{and } \lg\varrho=\ln(2+r^2).$$

As usual, $\mathcal{D}(\mathbb{R}^N)$ denotes the spaces of indefinitely differentiable functions with a compact support and $\mathcal{D}'(\mathbb{R}^N)$ denotes its dual space, called the space of distributions. For any nonnegative integers n and m, real numbers $p>1$, α and β, setting

$$k=k(m,N,p,\alpha)=\begin{cases} -1 & \text{if } \frac{N}{p}+\alpha\notin\{1,\dots,m\},\\ m-\frac{N}{p}-\alpha & \text{if } \frac{N}{p}+\alpha\in\{1,\dots,m\},\end{cases}$$

we define the following space:

$$\begin{aligned} W^{m,p}_{\alpha,\beta}(\Omega)=\{u\in\mathcal{D}'(\Omega);\ 0\leqslant|\lambda|\leqslant k,\ \varrho^{\alpha-m+|\lambda|}(\lg\varrho)^{\beta-1}D^\lambda u\in L^p(\Omega);\\ k+1\leqslant|\lambda|\leqslant m,\ \varrho^{\alpha-m+|\lambda|}(\lg\varrho)^{\beta}D^\lambda u\in L^p(\Omega)\}.\end{aligned} \tag{1.1}$$

In case $\beta=0$, we simply denote the space by $W^{m,p}_{\alpha}(\Omega)$. Note that $W^{m,p}_{\alpha,\beta}(\Omega)$ is a reflexive Banach space equipped with its natural norm

$$\begin{aligned}\|u\|_{W^{m,p}_{\alpha,\beta}(\Omega)}=\Big[\sum_{0\leqslant|\lambda|\leqslant k}\|\varrho^{\alpha-m+|\lambda|}(\lg\varrho)^{\beta-1}D^\lambda u\|^p_{L^p(\Omega)}\\ +\sum_{k+1\leqslant|\lambda|\leqslant m}\|\varrho^{\alpha-m+|\lambda|}(\lg\varrho)^{\beta}D^\lambda u\|^p_{L^p(\Omega)}\Big]^{1/p}.\end{aligned}$$

We also define the semi-norm

$$|u|_{W^{m,p}_{\alpha,\beta}(\Omega)}=\Big(\sum_{|\lambda|=m}\|\varrho^{\alpha}(\lg\varrho)^{\beta}D^\lambda u\|^p_{L^p(\Omega)}\Big)^{1/p},$$

and for any integer q, we denote by P_q the space of polynomials in N variables of a degree smaller than or equal to q, with the convention that P_q is reduced to $\{0\}$ when q is negative. The weights defined in (1.1) are chosen so that the corresponding space satisfies two properties:

$$\mathcal{D}(\overline{\mathbb{R}^N_+}) \quad \text{is dense in} \quad W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+), \tag{1.2}$$

and the following Poincaré-type inequality holds in $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$.

Theorem 1.1. *Let α and β be two real numbers and $m \geqslant 1$ an integer not satisfying simultaneously*

$$\frac{N}{p} + \alpha \in \{1, \dots, m\} \quad \text{and} \quad (\beta - 1)p = -1. \tag{1.3}$$

Then the semi-norm $|\cdot|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}$ defines on $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)/P_{q'}$ a norm which is equivalent to the quotient norm,

$$\forall u \in W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+),\ \|u\|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)/P_{q'}} \leqslant c|u|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)} \tag{1.4}$$

with $q' = \inf(q, m-1)$, where q is the highest degree of the polynomials contained in $W^{m,p}_{\alpha}(\mathbb{R}^N_+)$,

Proof. First, we construct a linear continuous extension operator such that

$$P\colon W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) \to W^{m,p}_{\alpha,\beta}(\mathbb{R}^N) \tag{1.5}$$

satisfying

$$\|Pu\|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N)} \leqslant \|u\|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}. \tag{1.6}$$

Since

$$\forall u \in W^{m,p}_{\alpha,\beta}(\mathbb{R}^N),\ \|u\|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N)/P_{q'}} \leqslant c|u|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N)} \tag{1.6}$$

holds [cf. 1], it automatically implies the statement of our theorem. □

Now, we define the space

$$\overset{\circ}{W}{}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) = \overline{\mathcal{D}(\mathbb{R}^N_+)}^{\|\cdot\|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}};$$

the dual space of $\overset{\circ}{W}{}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ is denoted by $W^{-m,p'}_{-\alpha,-\beta}(\mathbb{R}^N_+)$, where p' is the conjugate of p: $\frac{1}{p} + \frac{1}{p'} = 1$.

Theorem 1.2. *Under the assumptions of Theorem 1.1, the semi-norm $|\cdot|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}$ is a norm on $\overset{\circ}{W}{}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ such that it is equivalent to the full norm $\|\cdot\|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}$.*

We recall now some properties of weighted Sobolev spaces $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$. We have the algebraic and topological imbeddings

$$W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) \subset W^{m-1,p}_{\alpha-1,\beta}(\mathbb{R}^N_+) \subset \ldots \subset W^{0,p}_{\alpha-m,\beta}(\mathbb{R}^N_+)$$

if $\frac{N}{p}+\alpha \notin \{1,\dots,m\}$. When $\frac{N}{p}+\alpha = j \in \{1,\dots,m\}$, then we have:

$$W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) \subset \ldots \subset W^{m-j+1,p}_{\alpha-j+1,\beta}(\mathbb{R}^N_+) \subset W^{m-j,p}_{\alpha-j,\beta-1}(\mathbb{R}^N_+) \subset \ldots \subset W^{0,p}_{\alpha-m,\beta-1}(\mathbb{R}^N_+).$$

Note that in the first case, the mapping $u \to \varrho^\gamma u$ is an isomorphism from $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ onto $W^{m,p}_{\alpha-\gamma,\beta}(\mathbb{R}^N_+)$ for any integer m. Moreover, in both cases and for any multi-index $\lambda \in \mathbb{N}^N$, the mapping

$$u \in W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) \to D^\lambda u \in W^{m-|\lambda|,p}_{\alpha,\beta}(\mathbb{R}^N_+)$$

is continuous.

Finally, it can be readily checked that the highest degree q of the polynomials contained in $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ is given by

$$q = \begin{cases} m-(\frac{N}{p}+\alpha)-1 & \text{if} \begin{cases} \frac{N}{p}+\alpha \in \{1,\dots,m\} \text{ and } (\beta-1)p \geqslant -1 \\ \frac{N}{p}+\alpha \in \{j \in Z; j \leqslant 0\} \text{ and } \beta p \geqslant -1 \end{cases} \\ [m-(\frac{N}{p}+\alpha)] & \text{otherwise,} \end{cases}$$

where $[s]$ denotes the integer part of s.

In the sequel, for any integer $q \geqslant 0$, we will use the following polynomial spaces:

— P_q (P^Δ_q) is the space of polynomials (respectively, harmonic polynomials) of degree $\leqslant q$,

— P'_q is the subspace of polynomials in P_q depending only on the $N-1$ first variables, $x' = (x_1,\dots,x_{N-1})$,

— A^Δ_q (N^Δ_q) is the subspace of polynomials P^Δ_q satisfying the condition $p(x',0)=0$ (respectively, $\frac{\partial p}{\partial x_N}(x',0)=0$) or equivalently odd with respect to x_N (even with respect to x_N), with the convention that $P_q, P^\Delta_q, P'_q, \ldots$ are reduced to $\{0\}$ when q is negative.

2. The spaces of traces

In order to define the traces of functions of $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$, we introduce for any $\sigma \in]0,1[$ the space

(2.1) $$W^{\sigma,p}_0(\mathbb{R}^N) = \Big\{u \in \mathcal{D}'(\mathbb{R}^N);\ w^{-\sigma}u \in L^p(\mathbb{R}^N), \int_0^{+\infty} t^{-1-\sigma p}\,\mathrm{d}t \int_{\mathbb{R}^N} |u(x+te_i)-u(x)|^p\,\mathrm{d}x < \infty\Big\},$$

where

$$w = \begin{cases} \varrho & \text{if } \frac{N}{p} \neq \sigma, \\ \varrho(\lg\varrho)^{1/\sigma} & \text{if } \frac{N}{p} = \sigma, \end{cases}$$

and $e_1, \dots, e_N$ is a canonical basis of $\mathbb{R}^N$. It is a reflexive Banach space equipped with its natural norm

$$\|u\|_{W_0^{\sigma,p}(\mathbb{R}^N)} = \Big(\Big\|\frac{u}{w^\sigma}\Big\|^p_{L^p(\mathbb{R}^N)} + \sum_{i=1}^N \int_0^\infty t^{-1-\sigma p}\,\mathrm{d}t \int_{\mathbb{R}^N} |u(x+te_i) - u(x)|^p\,\mathrm{d}x\Big)^{1/p}$$

which is equivalent to the norm

$$\Big(\Big\|\frac{u}{w^\sigma}\Big\|^p_{L^p(\mathbb{R}^N)} + \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+\sigma p}}\,\mathrm{d}x\,\mathrm{d}y\Big)^{1/p}.$$

For any $s \in \mathbb{R}^+$, we set

$$W_0^{s,p}(\mathbb{R}^N) = \Big\{u \in W^{[s],p}_{[s]-s}(\mathbb{R}^N);\ \forall |\lambda| = [s],\ D^\lambda u \in W_0^{s-[s],p}(\mathbb{R}^N)\Big\}. \tag{2.2}$$

It is a reflexive Banach space equipped with the norm

$$\|u\|_{W_0^{s,p}(\mathbb{R}^N)} = \|u\|_{W^{[s],p}_{[s]-s}(\mathbb{R}^N)} + \sum_{|\lambda|=s} \|D^\lambda u\|_{W_0^{s-[s],p}(\mathbb{R}^N)}.$$

We notice that this definition and the next one coincide with the definition in the first section when $s = m$ is a nonnegative integer. For any $s \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}$, we then set

$$W_\alpha^{s,p}(\mathbb{R}^N) = \Big\{u \in W^{[s],p}_{[s]+\alpha-s}(\mathbb{R}^N),\ \forall |\lambda| = [s],\ \varrho^\alpha D^\lambda u \in W_0^{s-[s],p}(\mathbb{R}^N)\Big\}. \tag{2.3}$$

Finally, for any integer $m \geqslant 1$, we define the space

$$\begin{aligned} X_0^{m,p}(\mathbb{R}^N_+) = \Big\{u \in \mathcal{D}'(\mathbb{R}^N_+);\ 0 \leqslant |\lambda| \leqslant k,\ \varrho'^{|\lambda|-m}(\lg \varrho')^{-1} D^\lambda u \in L^p(\mathbb{R}^N_+),\\ k+1 \leqslant |\lambda| \leqslant m,\ \varrho'^{|\lambda|-m} D^\lambda u \in L^p(\mathbb{R}^N_+)\Big\} \end{aligned} \tag{2.4}$$

with $\varrho' = (1+|x'|^2)^{1/2}$ and $\lg \varrho' = \ln(2+|x'|^2)$. It is a reflexive Banach space. We can prove that

$$\mathcal{D}(\overline{\mathbb{R}^N_+}) \quad \text{is dense in} \quad X_0^{m,p}(\mathbb{R}^N_+).$$

We observe that the functions from $X_0^{m,p}(\mathbb{R}^N_+)$ and $W_0^{m,p}(\mathbb{R}^N_+)$ have the same traces on $\Gamma = \mathbb{R}^{N-1}$ (see below). If u is a function, we denote its traces on $\Gamma = \mathbb{R}^{N-1}$ by $x' \in \mathbb{R}^{N-1}$, $\gamma_0 u(x') = u(x',0), \dots, \gamma_j u(x') = \frac{\partial^j u}{\partial x_N^j}(x',0)$.

As in [3], we can prove the following trace lemma:

Lemma 2.1. *For any integer $m \geqslant 1$ and real number α, the mapping*

$$\gamma\colon \mathcal{D}(\overline{\mathbb{R}^N_+}) \to \prod_{j=0}^{m-1} \mathcal{D}(\mathbb{R}^{N-1})$$
$$u \mapsto (\gamma_0 u, \ldots, \gamma_{m-1} u)$$

can be extended by continuity to a linear and continuous mapping still denoted by γ from $W^{m,p}_\alpha(\mathbb{R}^N_+)$ to $\prod_{j=0}^{m-1} W_\alpha^{m-j-\frac{1}{p},p}(\mathbb{R}^{N-1})$. Moreover, γ is onto and

$$\operatorname{Ker} \gamma = \mathring{W}^{m,p}_\alpha(\mathbb{R}^N_+).$$

3. The Laplace equation

The aim of this section is to study the problem (P):

$$\text{(P)} \qquad \begin{cases} -\Delta u = f & \text{in } \mathbb{R}^N_+, \\ u = g & \text{in } \Gamma = \mathbb{R}^{N-1}. \end{cases}$$

Theorem 3.1. *Let $\ell \geqslant 0$ be an integer and assume that*

$$\text{(3.1)} \qquad \frac{N}{p'} \notin \{1, \ldots, \ell\}$$

with the convention that this set is empty if $\ell = 0$. For any f in $W^{-1,p}_\ell(\mathbb{R}^N_+)$ and g in $W_\ell^{\frac{1}{p'},p}(\Gamma)$ satisfying the compatibility condition

$$\text{(3.2)} \qquad \forall \varphi \in A^{\Delta}{}_{[\ell+1-\frac{N}{p'}]}, \langle f, \varphi \rangle_{W^{-1,p}_\ell \times W^{1,p'}_{-\ell}} = \Big\langle g, \frac{\partial \varphi}{\partial \gamma_N} \Big\rangle_\Gamma$$

where $\langle \cdot, \cdot \rangle_\Gamma$ denotes the duality between $W_\ell^{\frac{1}{p'},p}(\Gamma)$ and $W_{-\ell}^{-\frac{1}{p'},p'}(\Gamma)$, problem (P) has a unique solution $u \in W^{1,p}_\ell(\mathbb{R}^N_+)$ and there exists a constant C independent of u, f and g such that

$$\text{(3.3)} \qquad \|u\|_{W^{1,p}_\ell(\mathbb{R}^N_+)} \leqslant C(\|f\|_{W^{-1,p}_\ell(\mathbb{R}^N_+)} + \|g\|_{W_\ell^{\frac{1}{p'},p}(\Gamma)}).$$

P r o o f. First, the kernel of the operator

$$(-\Delta, \gamma_0)\colon W^{1,p}_\ell(\mathbb{R}^N_+) \to W^{-1,p}_\ell(\mathbb{R}^N_+) \times W_\ell^{\frac{1}{p'},p}(\Gamma)$$

is precisely the space $A^{\Delta}_{[\ell+1-N/p']}$ for any integer ℓ and $A^{\Delta}_{[\ell+1-\frac{N}{p'}]}$ is reduced to $\{0\}$ when $\ell \geqslant 0$. Thanks to Lemma 2.1, let $u_g \in W^{1,p}_{\ell}(\mathbb{R}^N_+)$ be the lifting function of g such that

$$u_g = g \text{ on } \Gamma \text{ and } \|u_g\|_{W^{1,p}_{\ell}(\mathbb{R}^N_+)} \leqslant C_1 \|g\|_{W^{\frac{1}{p'},p}_{\ell}(\Gamma)}.$$

Then problem (P) is equivalent to

$$\text{(3.4)} \qquad \begin{cases} -\Delta v = f + \Delta u_g & \text{in } \mathbb{R}^N_+, \\ v = 0 & \text{on } \Gamma. \end{cases}$$

Set $h = f + \Delta u_g$. For any $\varphi \in W^{1,p'}_{-\ell}(\mathbb{R}^N)$ set

$$\sqcap\varphi(x', x_N) = \varphi(x', x_N) - \varphi(x', -x_N) \quad \text{if} \quad x_N > 0.$$

It is clear that $\sqcap\varphi \in \overset{\circ}{W}{}^{1,p'}_{-\ell}(\mathbb{R}^N_+)$. Then h can be extended to $h_\pi \in W^{-1,p}_{\ell}(\mathbb{R}^N)$ defined by

$$\varphi \in W^{1,p'}_{-\ell}(\mathbb{R}^N),\ h_\pi(\varphi) = \langle h, \sqcap\varphi \rangle_{W^{-1,p}_{\ell}(\mathbb{R}^N_+) \times W^{1,p'}_{-\ell}(\mathbb{R}^N_+)}.$$

Moreover,

$$\|h_\pi\|_{W^{-1,p}_{\ell}(\mathbb{R}^N)} = \|h\|_{W^{-1,p}_{\ell}(\mathbb{R}^N_+)}.$$

Let q be a polynomial in $P^{\Delta}_{[\ell+1-N/p']}$. We can write it in the form

$$q = r + s,\ r \in A^{\Delta}_{[\ell+1-N/p']} \text{ and } s \in N^{\Delta}_{[\ell+1-N/p]}.$$

Then,

$$\langle h_\pi, q \rangle = \langle f + \Delta u_g, r \rangle_{W^{-1,p}_{\ell}(\mathbb{R}^N_+) \times W^{1,p'}_{-\ell}(\mathbb{R}^N_+)}$$

and applying the Green formula we get

$$\begin{aligned} \langle \Delta u_g, r \rangle &= -\int_{\mathbb{R}^N_+} \nabla u_g \cdot \nabla r \,\mathrm{d}x \\ &= -\Big\langle g, \frac{\partial r}{\partial x_N} \Big\rangle_{W^{\frac{1}{p'},p}_{\ell}(\Gamma) \times W^{-\frac{1}{p'},p'}_{-\ell}(\Gamma)} \end{aligned}$$

(note that $\Delta r = 0$ in $\mathbb{R}^N_+$ and $r = 0$ on Γ). Thus, $h_\pi \in W^{-1,p}_{\ell}(\mathbb{R}^N)$ and it satisfies

$$\forall q \in P^{\Delta}_{[\ell+1-N/p']},\ \langle h_\pi, q \rangle = 0.$$

Recall that (cf. [1]) since (3.1) holds, the operators

$$\Delta\colon\ W^{1,p}_{\ell}(\mathbb{R}^N) \to W^{-1,p}_{\ell} \perp P^{\Delta}_{[\ell+1-\frac{N}{p'}]} \text{ if } \ell \geqslant 1,$$

$$\Delta\colon\ W^{1,p}_{0}(\mathbb{R}^N)/P_{[1-\frac{N}{p}]} \to W^{-1,p}_{0}(\mathbb{R}^N) \perp P_{[1-\frac{N}{p'}]} \text{ if } \ell = 0$$

are isomorphisms. Hence, there exists $\tilde{v}$ in $W^{1,p}_{\ell}(\mathbb{R}^N)$ such that $-\Delta\tilde{v} = h_\pi$. Now we remark that the function $w = \frac{1}{2} \sqcap \tilde{v}$ belongs to $W^{1,p}_{\ell}(\mathbb{R}^N_+)$ and

$$-\Delta w = h \quad \text{in} \quad \mathbb{R}^N_+ \quad \text{and} \quad w = 0 \quad \text{on} \quad \Gamma,$$

i.e. w is a solution of (3.4). □

Remark. The kernel $A^{\Delta}_{[-\ell+1-N/p]}$ is reduced to $\{0\}$ if $\ell \geqslant 0$ and to $P_{[1-N/p]}$ if $\ell = 0$.

With similar arguments, we can prove the following theorem:

Theorem 3.2. *Let $\ell \geqslant 1$ be an integer and assume that*

$$\frac{N}{p} \notin \{1, \dots, -\ell\}. \tag{3.5}$$

Then for any f in $W^{-1,p}_{-\ell}(\mathbb{R}^N_+)$ and g in $W^{\frac{1}{p'},p}_{-\ell}(\Gamma)$, problem (P) has a unique solution $u \in W^{1,p}_{-\ell}(\mathbb{R}^N_+)/A^{\Delta}_{[\ell+1-N/p]}$ and there exists a constant C independent of u, f and g such that

$$\inf_{q \in A^{\Delta}_{[\ell+1-\frac{N}{p}]}} \|u + q\|_{W^{1,p}_{-\ell}(\mathbb{R}^N_+)} \leqslant C(\|f\|_{W^{-1,p}_{-\ell}(\mathbb{R}^N_+)} + \|g\|_{W^{\frac{1}{p'},p}_{-\ell}(\Gamma)}).$$

Theorem 3.3. *Let m be a nonnegative integer, let g belong to $W^{\frac{1}{p'}+m,p}_{m}(\Gamma)$ and assume that*

$$f \in W^{-1+m,p}_{m}(\mathbb{R}^N_+) \text{ if } \frac{N}{p'} \neq 1 \text{ or } m = 0, \tag{3.6}$$

or

$$f \in W^{-1+m,p}_{m}(\mathbb{R}^N_+) \cap W^{-1,p}_{0}(\mathbb{R}^N_+) \text{ if } \frac{N}{p'} = 1 \text{ and } m \neq 0. \tag{3.7}$$

Then problem (P) has a unique solution $u \in W^{1+m,p}_{m}(\mathbb{R}^N_+)$ and u satisfies

$$\|u\|_{W^{m+1,p}_{m}(\mathbb{R}^N_+)} \leqslant C(\|f\|_{W^{-1+m,p}_{m}(\mathbb{R}^N_+)} + \|g\|_{W^{\frac{1}{p'}+m,p}_{m}(\Gamma)}) \text{ if } \frac{N}{p'} \neq 1 \text{ or } m = 0$$

and

$$\|u\|_{W^{m+1,p}_{m}(\mathbb{R}^N_+)} \leqslant C(\|f\|_{W^{1,p}_{0}(\mathbb{R}^N_+)} + \|f\|_{W^{-1+m,p}_{m}(\mathbb{R}^N_+)} + \|g\|_{W^{\frac{1}{p'}+m,p}_{m}(\Gamma)}) \quad \text{if } \frac{N}{p'} = 1 \text{ and } m \neq 0.$$

P r o o f. First, we observe that for any integer $m \geqslant 0$ we have the inclusion

$$W_m^{-1+m,p}(\mathbb{R}^N_+) \subset W_0^{-1,p}(\mathbb{R}^N_+)$$

if $\frac{N}{p'} \neq 1$ or $m = 0$. Thus, under the assumptions (3.6) or (3.7) and thanks to Theorem 3.1, there exists a unique solution $u \in W_0^{1,p}(\mathbb{R}^N_+)$ of problem (P). Let us prove by induction that

$$g \in W_m^{\frac{1}{p'}+m,p}(\Gamma) \text{ and } f \text{ satisfies (3.6) or (3.7)} \Longrightarrow u \in W_m^{m+1,p}(\mathbb{R}^N_+). \tag{3.8}$$

For $m = 0$, (3.8) is valid. Assume that (3.8) is valid for $0, 1, \ldots, m$ and suppose that $g \in W_{m+1}^{\frac{1}{p'}+m+1,p}(\Gamma)$ and $f \in W_{m+1}^{m,p}(\mathbb{R}^N_+)$ with $\frac{N}{p'} \neq 1$ (a similar argument can be used for f satisfying (3.7)). Let us prove that $u \in W_{m+1}^{m+2,p}(\mathbb{R}^N_+)$. We observe first that

$$W_{m+1}^{m,p}(\mathbb{R}^N_+) \subset W_m^{m-1,p}(\mathbb{R}^N_+) \text{ and } W_{m+1}^{\frac{1}{p'}+m+1,p}(\Gamma) \subset W_m^{\frac{1}{p'}+m,p}(\Gamma),$$

hence u belongs to $W_m^{m+1,p}(\mathbb{R}^N_+)$ thanks to the induction hypothesis. Now, for $i = 1, \ldots, N-1$,

$$\Delta(\varrho\partial_i u) = \varrho\partial_i f + \frac{2}{\varrho} r \cdot \nabla(\partial_i u) + \Big(\frac{2}{\varrho} + \frac{1}{\varrho^3}\Big)\partial_i u.$$

Thus, $\Delta(\varrho\partial_i u) \in W_m^{m-1,p}(\mathbb{R}^N_+)$ and $\gamma_0(\varrho\partial_i u) \in W_m^{m+1,p}(\mathbb{R}^{N-1})$. Applying the induction hypothesis, we can deduce that

$$\partial_i u \in W_{m+1}^{m+1,p}(\mathbb{R}^N_+) \text{ for } i = 1, \ldots, N-1.$$

It remains to prove that $v = \partial_N u \in W_{m+1}^{m+1,p}(\mathbb{R}^N_+)$. This is a consequence of the fact that v belongs to $W_m^{m,p}(\mathbb{R}^N_+)$ and

$$\partial_i\partial_N u = \partial_N\partial_i u \in W_{m+1}^{m,p}(\mathbb{R}^N_+), \; i = 1, \ldots, N-1,$$
$$\partial_N(\partial_N u) = \Delta u - \sum_{i=1}^{N-1} \partial_i^2 u \in W_{m+1}^{m,p}(\mathbb{R}^N_+).$$

We can conlude that $u \in W_{m+1}^{m+2,p}(\mathbb{R}^N_+)$. □

Corollary 3.4. *Let $\ell \geqslant 1$ and $m \geqslant 1$ be two integers.*
(i) *Under the assumption*

$$\frac{N}{p'} \notin \{1, \ldots, \ell+1\},$$

for any $f \in W^{m-1,p}_{m+\ell}(\mathbb{R}^N_+)$ *and* $g \in W^{\frac{1}{p'}+m,p}_{m+\ell}(\Gamma)$ *satisfying the compatibility condition* (3.2) *there exists a unique solution* $u \in W^{m+1,p}_{m+\ell}(\mathbb{R}^N_+)$ *of* (P) *and* u *satisfies*

$$\|u\|_{W^{m+1,p}_{m+\ell}(\mathbb{R}^N_+)} \leqslant C(\|f\|_{W^{m-1,p}_{m+\ell}(\mathbb{R}^N_+)} + \|g\|_{W^{\frac{1}{p'}+m,p}_{m+\ell}(\Gamma)})$$

where $C = C(m,p,\ell,N)$ *is a constant independent of* u, f *and* g.

(ii) *Under the assumption*

$$m \geqslant \ell \quad \text{or} \quad \frac{N}{p} \notin \{1,\ldots,\ell-m\},$$

for any $f \in W^{m-1,p}_{m-\ell}(\mathbb{R}^N_+)$ *and* $g \in W^{\frac{1}{p'}+m,p}_{m-\ell}(\Gamma)$ *there exists a unique solution* $u \in W^{m+1,p}_{m-\ell}(\mathbb{R}^N_+)/A^{\Delta}_{[1+\ell-N/p]}$ *of* (P) *and* u *satisfies*

$$\inf_{q\in A^{\Delta}_{[1+\ell-N/p]}} \|u+q\|_{W^{m+1,p}_{m-\ell}(\mathbb{R}^N_+)} \leqslant C(\|f\|_{W^{m-1,p}_{m-\ell}(\mathbb{R}^N_+)} + \|g\|_{W^{\frac{1}{p'}+m,p}_{m-\ell}(\Gamma)}).$$

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