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NEWS AND NOTICES

IN MEMORY OF JOSEF KRÁL

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On May 24, 2006 a farewell ceremony for our teacher, colleague and friend Josef Král was held in the Church of St. Wenceslas in Pečky. He died on May 13, 2006, before his seventy-fifth birthday. Josef Král was an outstanding mathematician, exceptional teacher, model husband and father, and above all, a man of extraordinary human qualities. The results of his research place him among the most important Czech mathematicians of the second half of the twentieth century. His name is associated with original results in mathematical analysis and, in particular, in potential theory.

In 1967 Josef Král founded a seminar in Prague on mathematical analysis, with a particular emphasis on potential theory. He supervised a number of students and created a research group, which has been called *The Prague Harmonic Group* by friends and colleagues. At first lectures were organised, somewhat irregularly, at the Mathematical Institute of the Czechoslovak Academy of Sciences, Krakovská 10. The name *Seminar on Mathematical Analysis* was chosen and the meeting time was fixed for Monday afternoons. From the beginning it was agreed to devote the seminar mainly to potential theory, but this did not exclude other parts of Analysis which would be of interest to members. Before long the venue changed to the Faculty of Mathematics and Physics of Charles University, at the building on Malostranské nám. 25. The activities of the Seminar continue until the present and, for about thirty years now, have been based at the Faculty building at Karlín, Sokolovská 83.

The results of the group soon attracted international interest, and contacts were established with many world-famous specialists in potential theory. Among those who came to Prague were leading figures such as M. Brelot, H. Bauer, A. Cornea,

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The article has appeared also in Czech. Math. J. with the approval of the authors.

G. Choquet and B. Fuglede. Many others came to Prague in 1987 and to Kouty in 1994 when international conferences devoted specially to potential theory were held in our country. On the occasion of the thirtieth anniversary of the Seminar on Mathematical Analysis an international *Workshop on Potential Theory* was organized in Prague in 1996. Another international conference was organized in Hejnice in 2004.

The oldest writings about potential theory in Bohemia of which we are aware can be found in a three volume book *Foundations of Theoretical Physics* written in Czech (Základové theoretické fysiky) by August Seydler (1849–1891), a professor of Mathematical Physics and Theoretical Astronomy in the Czech part of Charles-Ferdinand University (nowadays Charles University). In the second volume, published in Prague in 1885 and called *Potential Theory. Theory of gravitational, magnetic and electric phenomena* (Theorie potenciálu. Theorie úkazů gravitačních, magnetických a elektrických) potential theory is treated from the point of view of physics. It was the first Czech book devoted to the field. He also wrote an article on logarithmic potentials.

Some years later František Graf published the article *On some properties of Newton and logarithmic potential and its first derivatives at simple singularities of mass surfaces and curves* (*O vlastnostech Newtonova logarithmického potenciálu i jeho prvních derivací v některých jednoduchých singularitách hmotných ploch a křivek*), in Cas. Pest. Mat. Fyz. 34 (1905), 5–19 and 130–147. Also, Karel Petr (1868–1950), a professor of Charles University during the period 1903–1938, wrote a note on potential theory: the article *Poisson integral as a direct consequence of Cauchy integral* written in Czech (*Poissonův integrál jako přímý důsledek integrálu Cauchyova*) appeared in Cas. Pest. Mat. Fyz. 42 (1913), 556–558.

In 1911, under K. Petr, Viktor Trkal (1888–1956) wrote his thesis *On the Dirichlet and Neumann problems from the integral equations viewpoint* (*O problému Dirichletově a Neumannově s hlediska rovnic integrálních*). V. Trkal later became a professor of theoretical physics at Charles University. George Pick (1859–1942) got his Habilitation from the Prague German University in 1882 and, from 1888, was a professor of this university. His main fields were Analysis and Geometry. Among his papers, which numbered more than 50, were at least two dealing with potential theory: *Ein Abschätzungssatz für positive Newtonsche Potentiale*, Jahresber. Dtsch. Math.-Ver. 24 (1915), 329–332, and *Über positive harmonische Funktionen*, Math. Z. 1 (1918), 44–51.

Karl Löwner (1893–1968) studied at the Prague German University where he also became a professor in 1930. Before his emigration in 1939, he was an adviser of Lipman Bers' (1914–1993) thesis *Über das harmonische Mass in Raume.*

Wolfgang Sternberg (1887–1953) belonged to the faculty at Prague German University during the period 1935–1939. He is known as the author of a two volume book on potential theory *Potentialtheorie I, II* published by Walter de Gruyter in the series Sammlung Göschen as well as of the book *The Theory of Potential and Spherical Harmonics*, later published in the U.S. His well-known work on the Perron method for the heat equation, *Über die Gleichung der Wärmeleitung*, Math. Ann. 101 (1929), 394–398, is often quoted in treatises on modern potential theory.

In the fifties, several papers on potential theory were published by Czech mathematicians who were interested in PDE's. Ivo Babuška (*1926) wrote several articles on the Dirichlet problem for domains with non-smooth boundaries and also papers on biharmonic problems. Rudolf Výborný (*1928) contributed to the study of maximum principles in several articles, especially for the heat equation. These mathematicians also wrote two papers jointly (*Die Existenz und Eindeutigkeit der Dirichletschen Aufgabe auf allgemeinen Gebieten*, Czech. Math. J. 9 (1959), 130–153, and *Reguläre und stabile Randpunkte für das Problem der Wärmeleitungsgleichung*, Ann. Polon. Math. 12 (1962), 91–104.)

An elementary approach to the Perron method for the Dirichlet problem was published in Czech by Jan Mařík (1920–1994) in the article *The Dirichlet problem* (Dirichletova úloha), Cas. Pest. Mat. 82 (1957), 257–282.

Following this a substantial period of development of potential theory in Czechoslovakia and later in Czech Republic is associated with Josef Král. He was born on December 23, 1931 in a village Dolní Bučice near Čáslav and graduated from the Faculty of Mathematics and Physics of Charles University in 1954. He became an Assistant in its Department of Mathematics and soon also a research student (*aspirant*). Under the supervision of J. Mařík he completed his thesis *On Lebesgue area of closed surfaces* and was granted (the equivalent of) a Ph.D. in 1960. In 1965 he joined the Mathematical Institute of the Czechoslovak Academy of Sciences as a researcher in the Department of Partial Differential Equations, and in the period 1980–1990 he served as the Head of the Department of Mathematical Physics. Meanwhile, in 1967, he defended his thesis *Fredholm method in potential theory* to obtain a DrSc., the highest scientific degree available in Czechoslovakia. Around the same time he also submitted his habilitation thesis *Heat flows and the Fourier problem.* In view of the extraordinarily high quality of the thesis, as well as the prominence both of his other research work and his teaching activities at the Faculty, the Scientific Board of the Faculty proposed to appoint J. Král professor in 1969. However, it took twenty years (sic!) before the changes in the country made it possible for J. Král to be actually appointed professor of mathematical analysis in 1990.

Although J. Král was affiliated to the Mathematical Institute for more than 30 years, he never broke his links with the Faculty. His teaching activities were remarkable in their extent. He continued to lecture courses—both elementary and advanced—in the theory of integral and differential equations, measure theory and potential theory. He supervised a number of diploma theses as well as Ph.D. theses, and was author and co-author of a four-volume lecture notes on potential theory ([72], [80], [88], [90]). He was frequently invited to give talks at conferences and universities abroad, and spent longer periods as visiting professor at Brown University in Providence, U.S.A. (1965–66), University Paris VI, France (1974), and University in Campinas, Brazil (1978). After retirement Josef Král lived in Pečky, a town about fifty kilometers east of Prague. Even though he was no longer able to participate in the seminar he founded, he maintained contact with its members and former students. He passed away in the hospital at Kolín.

Let us now review in more detail the research activities and scientific results of Josef Král. They principally relate to mathematical analysis, in particular to measure theory and integration, and to potential theory. The early papers of Josef Král appear in the scientific context of the late fifties, being strongly influenced by prominent mathematicians of the time, especially J. Mařík, V. Jarník and E. Čech. These papers primarily concern geometric measure theory, see [120].

Measure and integral

In papers [1], [2] [5], [7], [66], [13], and [87], J. Král studied curvilinear and surface integrals. As an illustration let us present a result following from [2], which was included in the lecture notes [72]: Let $f\colon [a,b] \to \mathbb{R}^2$ be a continuous closed parametric curve of finite length, let $f([a,b]) = K$, and let $\operatorname{ind}_f z$ denote the index of a point $z \in \mathbb{R}^2 \setminus K$ with respect to the curve. For any positive integer p set $G_p := \{z \in \mathbb{R}^2 \setminus K;\ \operatorname{ind}_f z = p\}$, $G := \bigcup\limits_{p \neq 0} G_p$. Let $\omega\colon G \to \overline{\mathbb{R}}$ be a locally integrable function and $v = (v_1, v_2)\colon K \cup G \to \mathbb{R}^2$ a continuous vector function. If

$$\int_{\partial R} (v_1 \,\mathrm{d}x + v_2 \,\mathrm{d}y) = \int_R \omega \,\mathrm{d}x \,\mathrm{d}y$$

for every closed square $R \subset G$ with positively oriented boundary ∂R, then for every $p \neq 0$ there is an appropriately defined improper integral $\iint_{G_p} \omega \,\mathrm{d}x \,\mathrm{d}y$, and the series

$$\sum_{p=1}^{\infty} p \left(\iint_{G_p} \omega \,\mathrm{d}x \,\mathrm{d}y - \iint_{G_{-p}} \omega \,\mathrm{d}x \,\mathrm{d}y \right)$$

(which need not converge) is summable by Cesàro's method of arithmetic means to the sum $\int_f (v_1 \,\mathrm{d}x + v_2 \,\mathrm{d}y)$.

Transformation of integrals was studied in [64], [3] and [70]. The last paper deals with the transformation of the integral with respect of the k-dimensional Hausdorff measure on a smooth k-dimensional surface in $\mathbb{R}^m$ to the Lebesgue integral in $\mathbb{R}^k$ (in particular, it implies the Change of Variables Theorem for Lebesgue integration in $\mathbb{R}^m$). A Change of Variables Theorem for one-dimensional Lebesgue-Stieltjes integrals is proved in [3]. As a special case one obtains a Banach-type theorem on the variation of a composed function which, as S. Marcus pointed out [Zentralblatt Math. 80 (1959), p. 271, Zbl 080.27101], implies a negative answer to a problem of H. Steinhaus from The New Scottish Book. To this category also belongs [6], where Král constructed an example of a mapping $T\colon\ D \to \mathbb{R}^2$, absolutely continuous in the Banach sense on a plane domain $D \subset \mathbb{R}^2$, for which the Banach indicatrix $N(\cdot, T)$ on $\mathbb{R}^2$ has an integral strictly greater than the integral over D of the absolute value of Schauder's generalized Jacobian $J_s(\cdot, T)$. In this way Král solved the problem posed by T. Radó in his monograph Length and Area [Amer. Math. Soc. 1948, (i) on p. 419]. The papers [65], [9], [10], [11], [12], [15] deal with surface measures; [9] and [10] are in fact parts of the above mentioned Ph.D. dissertation, in which Král (independently of W. Fleming) solved the problem on the relation between the Lebesgue area and perimeter in three-dimensional space, proposed by H. Federer [Proc. Amer. Math. Soc. 9 (1958), 447–451]. In [11] a question of E. Čech from The New Scottish Book, concerning the area of a convex surface in the sense of A. D. Alexandrov, was answered.

Papers [14] and [43] are from the theory of integration. The former yields a certain generalization of Fatou's lemma: *If* $\{f_n\}$ *is a sequence of integrable functions on a space* X *with a* σ*-finite measure* μ *such that, for each measurable set* $M \subset X$*, the sequence* $\{\int_M f_n \,\mathrm{d}\mu\}$ *is bounded from above, then the function* $\liminf f_n^+$ *is* μ*-integrable* (although the sequence $\{\int_X f_n^+ \,\mathrm{d}\mu\}$ need not be bounded). In the latter paper Král proved a theorem on dominated convergence for nonabsolutely convergent GP-integrals, answering a question of J. Mawhin [Czech. Math. J. 106 (1981), 614–632].

In [16] J. Král studied the relation between the length of a generally discontinuous mapping $f\colon\ [a, b] \to P$, with values in a metric space P, and the integral of the Banach indicatrix with respect to the linear measure on $f([a, b])$. For continuous mappings f the result gives an affirmative answer to a question formulated by G. Nöbeling in 1949.

In [27] it is proved that functions satisfying the integral Lipschitz condition coincide with functions of bounded variation in the sense of Tonelli-Cesari. The paper [87] presents a counterexample to the converse of the Green theorem. Finally, [52] provides an elementary characterization of harmonic functions in a disc representable by the Poisson integral of a Riemann-integrable function.

Still another paper from measure and integration theory is [33], in which Král gives an interesting solution of the mathematical problem on hair (formulated by

L. Zajíček): *For every open set $G \subset \mathbb{R}^2$ there is a set $H \subset G$ of full measure and a mapping assigning to each point $x \in H$ an arc $A(x) \subset G$ with the end point x such that $A(x) \cap A(y) = \emptyset$ provided $x \neq y$.*

The method of integral equations in potential theory

In [67] Král began to study the method of integral equations and its application to the solution of the boundary-value problems of potential theory. The roots of the method reach back into the 19^{th} century and are connected with, among others, the names of C. Neumann, H. Poincaré, A. M. Lyapunov, I. Fredholm and J. Plemelj. The generally accepted view, expressed, for example, in the monographs of F. Riesz and B. Sz.-Nagy, R. Courant and D. Hilbert, and B. Epstein, restrictive assumptions on the smoothness of the boundary were essential for this approach. This led to the belief that, for the planar case, this method had reached the natural limits of its applicability in the results of J. Radon, and was unsuitable for domains with nonsmooth boundaries. Let us note that, nonetheless, the method itself offers some advantages: when used, it beautifully exhibits the duality of the Dirichlet and the Neumann problem, provides an integral representation of the solution and—as was shown recently—is suitable also for numerical calculations.

In order to describe Král's results it is suitable to define an extremely useful quantity introduced by him, the so called cyclic variation. If $G \subset \mathbb{R}^m$ is an arbitrary open set with a compact boundary and $z \in \mathbb{R}^m$, let us denote by $p(z;\theta)$ the halfline with initial point z having direction $\theta \in \Gamma := \{\theta \in \mathbb{R}^m\,;\ |\theta| = 1\}$. For every $p(z;\theta)$ we calculate the number of points that are hits of $p(z;\theta)$ at ∂G; these are the points from $p(z;\theta) \cap \partial G$ in each neighborhood U of which, on this halfline, there are sufficiently many (in the sense of one-dimensional Hausdorff measure $\mathscr{H}_1$) points from both G and $\mathbb{R}^m \setminus G$, that is

$$\mathscr{H}_1\big(U \cap p(z;\theta) \cap G\big) > 0, \qquad \mathscr{H}_1\big(U \cap p(z;\theta) \cap (\mathbb{R}^m \setminus G)\big) > 0.$$

Let us denote by $n_r(z,\theta)$ the number of the hits of $p(z;\theta)$ at ∂G whose distance from z is at most $r > 0$, and define $v_r^G(z)$ to be the average number of hits $n_r(z,\theta)$ with respect to all possible halflines originating from z, that is

$$v_r^G(z) := \int_\Gamma n_r(z,\theta)\,\mathrm{d}\sigma(\theta),$$

the integral being taken with respect to the (normalized) surface measure σ on Γ. For $r = +\infty$ we write briefly $v^G(z) := v_\infty^G(z)$. From the viewpoint of application of the method of integral equations it is appropriate to consider the following questions:

(1) how general are the sets for which it is possible to introduce in a reasonable way the double-layer potential (the kernel is derived from the fundamental solution of the Laplace equation) or, as the case may be, the normal derivative of the single-layer potential defined by a mass distribution on the boundary ∂G;
(2) under what conditions is it possible to extend this potential (continuously) from the domain onto its boundary;
(3) when is it possible to solve operator equations defined by this extension?

The answers to the first two questions are in the form of necessary and sufficient conditions formulated in terms of the function v^G. In [67] Král definitively solved problem (2) using also the so-called radial variation; both quantities have their inspiration in the Banach indicatrix. We note here that the Dirichlet problem is easily formulated even for domains with nonsmooth boundaries, while attempts to formulate the Neumann problem for such sets encounter major obstacles from the very beginning, regardless of the method used. Therefore it was necessary to pass, in the formulation, from the description in terms of a point function in the boundary condition to a description using the potential flow induced by the signed measure on the boundary.

By the method of integral equations, the Dirichlet and Neumann problems are solved indirectly: the solution is sought in the form of a double-layer and a single-layer potential, respectively. These problems are reduced to the solution of the dual operator equations

$$T^G f = g \qquad \text{and} \qquad N^G U\mu = \nu$$

where f, g are respectively the sought and the given functions, and μ and ν are respectively the sought and given signed measures on the boundary ∂G. Here the operator T^G is connected with the jump formula for the double layer potential whereas N^G is the operator of the generalized normal derivative. Let us consider three quantities of the same nature which are connected with the solvability of problems (1)–(3) and which are all derived from the cyclic variation introduced above:

(a) $v^G(x)$,
(b) $V^G := \sup\{v^G(y);\ y \in \partial G\}$,
(c) $v_0^G := \lim\limits_{r\to 0+} \sup\{v_r^G(y);\ y \in \partial G\}$.

While, in [67], the starting point is the set $G \subset \mathbb{R}^2$ bounded by a curve K of finite length, the subsequent papers [22], [73] consider, from the outset, an arbitrary open set G with compact boundary ∂G. In [67] Král solved problem (2), which opened the way to a generalization of Radon's results established for curves of bounded rotation. The radial variation of a curve is also introduced here, and both variations are used in [69], [19] for studying angular limits of the double-layer potential. The results explicitly determine the value of the limit and give geometrically visualizable criteria which are necessary and sufficient conditions for the existence of these limits. The mutual relation of the two quantities and their relation to the length and boundary rotation of curves is studied in [68] and [17]. For the plane case the results are collected in [18], [20], and [21], where the interrelations of the results are explained and conditions of solvability of the resulting operator equations are given. For the case of $\mathbb{R}^m$, $m \geqslant 2$, angular limits of the double layer potentials are studied in [56].

Let us present these conditions explicitly for the dimension $m \geqslant 3$. If $G \subset \mathbb{R}^m$ is a set with a smooth boundary ∂G, then the double-layer potential Wf with a continuous moment f on ∂G is defined by the formula

$$Wf(x) := \int_{\partial G} f(y) \frac{(y-x)\cdot n(y)}{|x-y|^m} \,\mathrm{d}\sigma(y), \qquad x \in \mathbb{R}^m \setminus \partial G,$$

where $n(y)$ is the vector of the (outer) normal to G at the point $y \in \partial G$ and σ is the surface measure on ∂G. For $x \notin \partial G$ the value $W\varphi(x)$ can be defined distributively for an arbitrary open G with compact boundary and for every smooth function φ; this value is the integral with respect to a certain measure (dependent on x) if and only if the quantity (a) is finite. Then $Wf(x)$ can naturally be defined for a sufficiently general f by the integral of f with respect to this measure.

Consequently, if we wish to define a generalized double-layer potential on G, the value of (a) must be finite for all $x \in G$. In fact, it suffices that $v^G(x)$ be finite on a finite set of points x from G which, however, must not lie in a single hyperplane; then the set G already has a finite perimeter. On its essential boundary, a certain essential part of boundary, the (Federer) normal can be defined in an approximative sense. This fact proves useful: the formula for calculation of Wf remains valid if the classical normal occurring in it is replaced by the Federer normal. If the quantity V^G from (b) is finite, then v^G is finite everywhere in G, and Wf can be continuously extended from G to $\overline{G}$ for every f continuous on ∂G. This is again a necessary and sufficient condition; hence the solution of the Dirichlet problem can be obtained by solving the first of the above mentioned operator equations. A similar situation which we will not describe in detail occurs for the dual equation with the operator N^G.

These results (generalizing the previous ones to the multidimensional case) can be found in [22], [73], where, in addition, the solvability of the equations in question is studied; see also [49]. Here Král deduced a sufficient condition of solvability depending on the magnitude of the quantity in (c), by means of which he explicitly expressed the so called essential norm of certain operators related to those appearing in the equations considered. It is worth mentioning that the mere smoothness of the boundary does not guarantee the finiteness of the quantities in (b) or (a); see [23].

Considering numerous similar properties of the Laplace equation and the heat equation it is natural to ask whether Král's approach (fulfilling the plan traced out by Plemelj) can be used also for the latter. Replacing in the definition of v^G the pencil of halflines filling the whole space $\mathbb{R}^m$ by a pencil of parabolic arcs filling the halfspace of $\mathbb{R}^{m+1}$ that is in time "under" the considered point (x, t) of the timespace, we can arrive at analogous results also for the heat equation. Only a deeper insight into the relation and distinction of the equations enables us to realise that the procedure had to be essentially modified in order to obtain comparable results; see [74], [24]. It should be mentioned that the cyclic variation introduced by Král has proved to be a useful tool for the study of further problems, for instance those connected with the Cauchy integral; see [25], [28] and [59]. Angular limits of the integral with densities satisfying a Hölder-type condition were studied in [56]. We note that the cyclic variation was also used to solve mixed boundary value problems concerning analytic function by means of a reflection mapping; see [58].

In addition to the lecture notes mentioned above, Král later, in the monograph [38], presented a selfcontained survey of the results described above. This book provides the most accessible way for a reader to get acquainted with the results for the Laplace equation. It also includes some new results; for example, if the quantity in (c) is sufficiently small, then G has only a finite number of components—this is one of the consequences of the Fredholm method, cf. [82], [38]. Part of the publication is devoted to results of [35] concerning the contractivity of the Neumann operator, which is connected with the numerical solution of boundary value problems, a subject more than 100 years old. The solution is again definitive and depends on convexity properties of G. Related results in a more general context were obtained in [53], [60], [61], and [62].

The subject of the papers [85], [89], [95], [47] belongs to the field of application of the method of integral equations; they originated in connection with some invitations to deliver lectures at conferences and symposia. Král further developed the above methods and, for instance, in [95] indicated the applicability of the methods also to the "infinite-dimensional" Laplace equation.

The last period is characterized by Král's return to the original problems from a rather different viewpoint. The quantity in (c) may be relatively small for really complicated sets G, but can be unpleasantly large for some even very simple sets arising for example in $\mathbb{R}^m$ as finite unions of parallelepipeds. Even for this particular case the solution is already known. It turned out that an appropriate re-norming leads to a desirable reduction of the essential norm (the tool used here is a "weighted" cyclic variation); see [44], [48], and also [61], [122].

A characteristic feature of Král's results concerning the boundary value problems is that the analytical properties of the operators considered are expressed in visualizable geometrical terms. For the planar case see, in particular, [46].

The topics described above have also been investigated by V. G. Maz'ya whose results together with relevant references may be found in his treatise *Boundary Integral Equations*, Encyclopaedia of Mathematical Sciences 27, Analysis IV, Springer, 1991.

Removable singularities

Let us now pass to Král's contribution to the study of removable singularities of solutions of partial differential equations.

Let $P(D)$ be a partial differential operator with smooth coefficients defined in an open set $U \subset \mathbb{R}^m$ and let $L(U)$ be a set of locally integrable functions on U. A relatively closed set $F \subset U$ is said to be removable for $L(U)$ with respect to $P(D)$ if the following condition holds: for any $h \in L(U)$ such that $P(D)h = 0$ on $U \setminus F$ (in the sense of distributions), $P(D)h = 0$ on the whole set U.

As an example let us consider the case where $P(D)$ is the Laplace operator in $\mathbb{R}^m$, $m > 2$, and $L(U)$ is one of the following two sets of functions: (1) continuous functions on U; (2) functions satisfying the Hölder condition with an exponent $\gamma \in (0,1)$. It is known from classical potential theory that in the case (1) a set is removable for $L(U)$ if and only if it has zero Newtonian capacity. For the case (2) L. Carleson (1963) proved that a set is removable for $L(U)$ if and only if its Hausdorff measure of dimension $\gamma + m - 2$ is zero.

In [76] Král obtained a result of Carleson-type for solutions of the heat equation. Unlike the Laplace operator, the heat operator fails to be isotropic. Anisotropy enters Král's result in two ways: firstly, the Hölder condition is considered with the exponents γ and $\frac{1}{2}\gamma$ with respect to the spatial and the time variables, respectively, and secondly, anisotropic Hausdorff measure is used. Roughly speaking, the intervals used for covering have a length of edge s in the direction of the space coordinates, and s^2 along the time axis. The paper was the start of an extensive project, the aim of which was to master removable singularities for more general differential operators and wider scales of function spaces.

Let M be a finite set of multi-indices and suppose that the operator

$$P(D) := \sum_{\alpha \in M} a_\alpha D^\alpha$$

has infinitely differentiable complex-valued coefficients on an open set $U \subset \mathbb{R}^m$. Let us choose a fixed m-tuple $n = (n_1, n_2, \dots, n_m)$ of positive integers such that

$$|\alpha : n| := \sum_{k=1}^{m} \frac{\alpha_k}{n_k} \leqslant 1$$

for every multiindex $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m) \in M$.

We recall that an operator $P(D)$ with constant coefficients a_α is called semielliptic if the only real-valued solution of the equation

$$\sum_{|\alpha:n|=1} a_\alpha \xi^\alpha = 0$$

is $\xi = (\xi_1, \xi_2, \dots, \xi_m) = 0$. (Of course, for $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ we define here $\xi^\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \dots \xi_m^{\alpha_m}$.) The class of semielliptic operators includes, among others, the elliptic operators, the parabolic operators in the sense of Petrovskij (in particular, the heat operator), as well as the Cauchy-Riemann operator.

For n fixed and $\overline{n} := \max\{n_k;\ 1 \leqslant k \leqslant m\}$ the operator $P(D)$ is assigned the metric

$$\varrho(x,y) := \max\bigl\{|x_k - y_k|^{n_k/\overline{n}};\ 1 \leqslant k \leqslant m\bigr\}, \qquad x, y \in \mathbb{R}^m.$$

To each measure function f, a Hausdorff measure on the metric space $(\mathbb{R}^m, \varrho)$ is associated in the usual way. Roughly speaking, this measure reflects the possibly different behaviour of $P(D)$ with respect to the individual coordinates, and it was by measures of this type that J. Král succeeded in characterizing the removable singularities for a number of important and very general situations.

Removable singularities are studied in [30] (see also [81]) for anisotropic Hölder classes, and in [84] for classes with a certain anisotropic modulus of continuity; in the latter case the measure function for the corresponding Hausdorff measure is derived from the modulus of continuity. In [86] Hölder conditions of integral type (covering Morrey's and Campanato's spaces as well as the BMO) are studied.

The papers [39] and [42] go still further: spaces of functions are investigated whose prescribed derivatives satisfy conditions of the above mentioned types.

For general operators Král proved that the vanishing (or, as the case may be, the σ-finiteness) of an appropriate Hausdorff measure is a sufficient condition of removability for a given set of functions. (Let us point out that, when constructing the appropriate Hausdorff measure, the metric ϱ reflects the properties of the operator $P(D)$, while the measure function reflects the properties of the class of the functions considered.)

It is remarkable that, for semielliptic operators with constant coefficients, Král proved that the above sufficient conditions are also necessary. An additional restriction for the operators is used to determine precise growth conditions for the fundamental solution and its derivatives. The potential theoretic method (combined with a Frostman-type result on the distribution of measure), which is applied in the proof of necessary conditions, is very well explained in [109] and also in [55]. In the same work also the results on removable singularities for the wave operator are presented; see [54] and [50] dealing with related topics.

In the conclusion of this section let us demonstrate the completeness of Král's research by the following result for elliptic operators with constant coefficients, which is a consequence of the assertions proved in [42]: the removable singularities for functions that, together with certain of their derivatives, belong to a suitable Campanato space, are characterized by the vanishing of the classical Hausdorff measures, whose dimension (in dependence on the function space) fills in the whole interval between 0 and m. We note that J. Král lectured on removable singularities during the Spring school on abstract analysis (*Small and exceptional sets in analysis and potential theory*) organized at Paseky in 1992.

Potential theory

The theory of harmonic spaces started to develop in the sixties. Its aim was to build up an abstract potential theory that would include not only the classical potential theory but would also make it possible to study wide classes of partial differential equations of elliptic and parabolic types. Further development showed that the theory of harmonic spaces represents an appropriate link between partial differential equations and stochastic processes.

In the abstract theory the role of the Euclidean space is played by a locally compact topological space (this makes it possible to cover manifolds and Riemann surfaces and simultaneously to exploit the theory of Radon measures), while the solutions of a differential equation are replaced by a sheaf of vector spaces of continuous functions satisfying certain natural axioms. One of them, for example, is the axiom of basis, which guarantees the existence of basis of the topology consisting of sets regular for the Dirichlet problem, or the convergence axiom, which is a suitable analogue of the classical Harnack theorem.

While Král probably did not plan to work systematically on the theory of harmonic spaces, he realized that this modern and developing branch of potential theory must not be neglected. In his seminar he gave a thorough report on Bauer's monograph *Harmonische Räume und ihre Potentialtheorie*, and later on the monograph of C. Constantinescu and A. Cornea *Potential Theory on Harmonic Spaces.*

In Král's list of publications there are four papers dealing with harmonic spaces. In [32] an affirmative answer is given to the problem of J. Lukeš concerning the existence of a nondegenerate harmonic sheaf with Brelot's convergence property on a connected space which is not locally connected. The paper [26] provides a complete characterization of sets of ellipticity and absorbing sets on one-dimensional harmonic spaces. All noncompact connected one-dimensional Brelot harmonic spaces are described in [31]. In [29], harmonic spaces with the following continuation property are investigated: Each point is contained in a domain D such that every harmonic function defined on an arbitrary subdomain of D can be harmonically continued to the whole D. It is shown that a Brelot space X enjoys this property if and only if it has the following simple topological structure: for every $x \in X$ there exist arcs $C_1, C_2, \ldots, C_n$ such that $\bigcup\{C_j\,;\ 1 \leqslant j \leqslant n\}$ is a neighborhood of x and $C_j \cap C_k = \{x\}$ for $1 \leqslant j < k \leqslant n$.

The papers [41] and [37] are devoted to potentials of measures. In [41] it is shown that, for kernels K satisfying the domination principle, the following continuity principle is valid: If ν is a signed measure whose potential $K\nu$ is finite, and if the restriction of $K\nu$ to the support of ν is continuous, then the potential $K\nu$ is necessarily continuous on the whole space. In the case of a measure this is the classical Evans-Vasilesco theorem. However, this theorem does not yield (by passing to the positive and negative parts) the above assertion, since "cancellation of discontinuities" may occur.

In [37] a proof is given of a necessary and sufficient condition for measures ν on $\mathbb{R}^m$ to have the property that there exists a nontrivial measure ϱ on $\mathbb{R}$ such that the heat potential of the measure $\nu \otimes \varrho$ locally satisfies an anisotropic Hölder condition.

In [45] the size of the set of fine strict maxima of functions defined on $\mathbb{R}^m$ is studied. We recall that the fine topology in the space $\mathbb{R}^m$, $m > 2$, is defined as the coarsest topology for which all potentials are continuous. For $f\colon \mathbb{R}^m \to \mathbb{R}$ let us denote by $M(f)$ the set of all points $x \in \mathbb{R}^m$ which have a fine neighborhood V such that $f < f(x)$ on $V \setminus \{x\}$. It is shown in [45] that the set $M(f)$ has zero Newtonian capacity provided f is a Borel function.

In [40] Král proved the following theorem of Radó's type for harmonic functions (and in this way verified Greenfield's conjecture): If h is a continuously differentiable function on an open set $G \subset \mathbb{R}^m$ and h is harmonic on the set $G_h := \{x \in G;\ h(x) \neq 0\}$, then h is harmonic on the whole set G. In this case the set G_h on which h is harmonic, satisfies $h(G \setminus G_h) \subset \{0\}$. For various function spaces, Král characterized in [40], in terms of suitable Hausdorff measures, the sets $E \subset \mathbb{R}$ for which the condition $h(G \setminus G_h) \subset E$ guarantees that h is harmonic on the whole set G.

An analogue of Radó's theorem for differential forms and for solutions of elliptic differential equations is proved in [51].

The papers [91], [36] do not directly belong to potential theory, being only loosely connected with it. They are devoted to the estimation of the analytic capacity by means of the linear measure. For a compact set $Q \subset \mathbb{C}$ and for $z \in \mathbb{C}$ let us denote by $v^Q(z)$ the average number of points of intersection of the halflines originating at z with Q and set $V(Q) := \sup\{v^Q(z);\ z \in Q\}$. The main result of [36] is as follows: If $Q \subset \mathbb{C}$ is a continuum and $K \subset Q$ is compact, then the following inequality holds for the analytic capacity $\gamma(K)$ and the linear measure $m(K)$:

$$\gamma(K) \geqslant \frac{1}{2\pi}\,\frac{1}{2V(Q)+1}\,m(K).$$

Josef Král liked to solve problems; he published solutions of some problems which he found interesting; see, for example, [8], [77]. A search of *MathSciNet* reveals that he wrote more than 180 reviews for Mathematical Reviews.

We do hope that we have succeeded in, at least, indicating the depth and elegance of Král's mathematical results. Many of them are of definitive character and thus provide final and elegant solution of important problems. The way in which Král presented his results shows his conception of mathematical exactness, perfection and beauty.

His results, and their international impact, together with his extraordinarily successful activities in mathematical education, have placed Josef Král among the most prominent Czechoslovak mathematicians of the post-war period. His modesty, devotion and humble respect in the face of the immensity of Mathematics made him an exceptional person.

Publications containing new results with complete proofs

[1] Der Greensche Satz. Czech. Math. J. *7* (1957), 235–247 (with J. Mařík).
[2] On curvilinear integrals in the plane. Czech. Math. J. *7* (1957), 584–598. (In Russian.)
[3] Transformation of the Stieltjes-Lebesgue integral. Czech. Math. J. *8* (1958), 86–93 (with J. Marek). (In Russian.)
[4] On Lipschitzian mappings from the plane into Euclidean spaces. Czech. Math. J. *8* (1958), 257–266.
[5] Note on the Gauss-Ostrogradskij formula. Cas. Pest. Mat. *84* (1959), 283–292. (In Czech.)
[6] Note on strong generalized jacobians. Czech. Math. J. *9* (1959), 429–439.
[7] Closed systems of mappings and the surface integral. Czech. Math. J. *10* (1960), 27–67. (In Russian.)
[8] To the problem No. 2 proposed by J. Mařík. Cas. Pest. Mat. *85* (1960), 205. (In Czech.)
[9] Note on the sets whose characteristic function has a generalized measure for its partial derivative. Cas. Pest. Mat. *86* (1961), 178–194. (In Czech.)
[10] Note on perimeter of the Cartesian product of two sets. Cas. Pest. Mat. *86* (1961), 261–268. (In Czech.)
[11] On a problem concerning area of a convex surface. Cas. Pest. Mat. *86* (1961), 277–287. (In Czech.)

[12] On Lebesgue area of simple closed surfaces. Czech. Math. J. *12* (1962), 44–68. (In Russian.)
[13] Note on the Stokes formula for 2-dimensional integrals in n-space. Mat.-fyz. Cas. Slov. Akad. Vied *12* (1962), 280–292 (with B. Riečan).
[14] Note on sequences of integrable functions. Czech. Math. J. *13* (1963), 114–126 (with J. Jelínek).
[15] A note on perimeter and measure. Czech. Math. J. *13* (1963), 139–147.
[16] Note on linear measure and the length of path in a metric space. Acta Univ. Carol., Math. Phys. *1* (1963), 1–10. (In Czech.)
[17] Some inequalities concerning the cyclic and radial variations of a plane path-curve. Czech. Math. J. *14* (1964), 271–280.
[18] On the logarithmic potential of the double distribution. Czech. Math. J. *14* (1964), 306–321.
[19] Non-tangential limits of the logarithmic potential. Czech. Math. J. *14* (1964), 455–482.
[20] The Fredholm radius of an operator in potential theory I. Czech. Math. J. *15* (1965), 454–473.
[21] The Fredholm radius of an operator in potential theory II. Czech. Math. J. *15* (1965), 565–588.
[22] The Fredholm method in potential theory. Trans. Amer. Math. Soc. *125* (1966), 511–547.
[23] Smooth functions with infinite cyclic variation. Cas. Pest. Mat. *93* (1968), 178–185. (In Czech.)
[24] Flows of heat and the Fourier problem. Czech. Math. J. *20* (1970), 556–598.
[25] On the modified logarithmic potential. Czech. Math. J. *21* (1971), 76–98 (with J. Lukeš).
[26] Elliptic points in one-dimensional harmonic spaces. Commentat. Math. Univ. Carol. *12* (1971), 453–483 (with J. Lukeš, I. Netuka).
[27] Functions satisfying an integral Lipschitz condition. Acta Univ. Carol., Math. Phys. *12* (1971), 51–54.
[28] Integrals of the Cauchy type. Czech. Math. J. *22* (1972), 663–682 (with J. Lukeš).
[29] Indefinite harmonic continuation. Cas. Pest. Mat. *98* (1973), 87–94 (with J. Lukeš).
[30] Removable singularities of solutions of semielliptic equations. Rend. Mat. *6* (1973), 763–783.
[31] Brelot spaces on one-dimensional manifolds. Cas. Pest. Mat. *99* (1974), 179–185 (with J. Lukeš). (In Czech.)
[32] An example of a harmonic sheaf. Cas. Pest. Mat. *101* (1976), 98–99. (In Czech.)
[33] To a mathematical problem on hair. Cas. Pest. Mat. *101* (1976), 305–307. (In Czech.)
[34] On boundary behaviour of double-layer potentials. Trudy Seminara S. L. Soboleva, Akad. Nauk Novosibirsk, Novosibirsk, 1976, pp. 19–34. (In Russian.)
[35] Contractivity of C. Neumann's operator in potential theory. J. Math. Anal. Appl. *61* (1977), 607–619 (with I. Netuka).
[36] Analytic capacity and linear measure. Czech. Math. J. *28* (1978), 445–461 (with J. Fuka).
[37] Heat sources and heat potentials. Cas. Pest. Mat. *105* (1980), 184–191 (with S. Mrzena).
[38] Integral Operators in Potential Theory. Lect. Notes Math., vol. 823, Springer, 1980, pp. 171.
[39] Singularities of solutions of partial differential equations. Trudy Seminara S. L. Soboleva, Akad. Nauk Novosibirsk, Novosibirsk, 1983, pp. 78–89. (In Russian.)

[40] Some extension results concerning harmonic functions. J. London Math. Soc. *28* (1983), 62–70.
[41] A note on continuity principle in potential theory. Commentat. Math. Univ. Carol. *25* (1984), 149–157.
[42] Semielliptic singularities. Cas. Pest. Mat. *109* (1984), 304–322.
[43] Note on generalized multiple Perron integral. Cas. Pest. Mat. *110* (1985), 371–374.
[44] Some examples concerning applicability of the Fredholm-Radon method in potential theory. Aplikace matematiky *31* (1986), 293–308 (with W. Wendland).
[45] Fine topology in potential theory and strict maxima of functions. Exposition. Math. *5* (1987), 185–191 (with I. Netuka).
[46] Boundary regularity and normal derivatives of logarithmic potentials. Proc. Roy. Soc. Edinburgh, Sect. A *106* (1987), 241–258.
[47] On the applicability of the Fredholm-Radon method in potential theory and the panel method. Proceedings of the Third GAMM-Seminar Kiel 1987, Vieweg, Braunschweig, 1987, pp. 120–136 (with W. Wendland).
[48] Layer potentials on boundaries with corners and edges. Cas. Pest. Mat. *113* (1988), 387–402 (with T. Angell, R. E. Kleinman).
[49] Problème de Neumann faible avec condition frontière dans L^1. Séminaire de Théorie du Potentiel, Paris, no. 9, Lect. Notes Math., vol. 1393, Springer, 1989, pp. 145–160.
[50] Removable singularities of the functional wave equation. Z. Anal. Anwend. *8* (1989), 495–500 (with M. Chlebík).
[51] Extension results of the Radó type. Rev. Roumaine Math. Pures Appl. *36* (1991), 71–76.
[52] Poisson integrals of Riemann integrable functions. Amer. Math. Monthly *98* (1991), 929–931 (with W. F. Pfeffer).
[53] On the Neumann operator of the arithmetical mean. Acta. Math. Univ. Comenian. (N.S.) *61* (1992), 143–165 (with D. Medková).
[54] Removability of the wave singularities in the plane. Aequationes Math. *47* (1994), 123–142 (with M. Chlebík).
[55] Removability of singularities in potential theory. Potential Anal. *3* (1994), 119–131; also in: Proceedings of the International Conference on Potential Theory, Amersfoort, 1991, Kluwer, Dordrecht, 1994.
[56] Angular limits of double layer potentials. Czech. Math. J. *45* (1995), 267–292 (with D. Medková).
[57] Note on functions satisfying the integral Hölder condition. Math. Bohem. *121* (1996), 263–268.
[58] Reflection and a mixed boundary value problem concerning analytic functions. Math. Bohem. *122* (1997), 317–336 (with E. Dontová, M. Dont).
[59] Angular limits of the integrals of the Cauchy type. Czech. Math. J. *47* (1997), 593–617 (with D. Medková).
[60] On the Neumann-Poincaré operator. Czech. Math. J. *48* (1998), 653–668 (with D. Medková).
[61] Essential norms of a potential-theoretic boundary integral operator in L^1. Math. Bohem. *123* (1998), 419–436 (with D. Medková).
[62] Essential norms of the Neumann operator of the arithmetical mean. Math. Bohem. *126* (2001), 669–690 (with D. Medková).

Other publications

[63] Green's theorem. Cas. Pest. Mat. *81* (1956), 476–479 (with M. Neubauer). (In Czech.)

[64] Transformations of multidimensional integrals. Cas. Pest. Mat. *83* (1958), 365–368. (In Czech.)

[65] On Lebesgue area of closed surfaces. Czech. Math. J. *9* (1959), 470–471. (In Russian.)

[66] On r-dimensional integrals in $(r+1)$-space. Czech. Math. J. *11* (1961), 626–628.

[67] On the logarithmic potential. Commentat. Math. Univ. Carol. *3* (1962), 3–10.

[68] On cyclic and radial variations of a plane path. Commentat. Math. Univ. Carol. *4* (1963), 3–9.

[69] On angular limit values of Cauchy-type integrals. Dokl. Akad. Nauk SSSR *155* (1964), 32–34. (In Russian.)

[70] Integration with respect to Hausdorff measure on a smooth surface. Cas. Pest. Mat. *89* (1964), 433–447 (with J. Mařík). (In Czech.)

[71] On double-layer potential in multidimensional space. Dokl. Akad. Nauk SSSR *159* (1964), 1218–1220. (In Russian.)

[72] Potential Theory I. Státní pedagogické nakladatelství, Praha, 1965, pp. 159 (Lecture Notes). (In Czech.)

[73] On the Neumann problem in potential theory. Commentat. Math. Univ. Carol. *7* (1966), 485–493.

[74] Flows of heat. Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. *46* (1969), 140–143.

[75] Limits of double layer potentials. Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. *48* (1970), 39–42.

[76] Hölder-continuous heat potentials. Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. *51* (1971), 17–19.

[77] To the problem No. 4 proposed by I. Babuška. Cas. Pest. Mat. *97* (1972), 207–208. (In Czech.)

[78] Problem No. 1 (on semiregular sets). Cas. Pest. Mat. *97* (1972), 334. (In Czech.)

[79] On removable singularities of solutions of the heat equation. Primenenie funkcionalnych metodov k krajevym zadacham matematicheskoj fiziki, Akad. Nauk Novosibirsk, Novosibirsk, 1972, pp. 103–106. (In Russian.)

[80] Potential Theory II. Státní pedagogické nakladatelství, Praha, 1972, pp. 348 (with I. Netuka, J. Veselý—Lecture Notes). (In Czech.)

[81] Regularity of potentials and removability of singularities of partial differential equations. Proceedings of the Conference Equadiff 3, J. E. Purkyně University, Brno, 1972, pp. 179–185.

[82] A note on the Robin problem in potential theory. Commentat. Math. Univ. Carol. *14* (1973), 767–771.

[83] Nonlinear evolution equations and potential theory. Proceedings of a Summer School, Academia, Praha, 1975, pp. 145, editor.

[84] Potentials and removability of singularities. Nonlinear evolution equations and potential theory, Academia, Praha, 1975, pp. 95–106.

[85] Potentials and boundary value problems. Proceedings of the Conference 5. Tagung über Probleme der Mathematischen Physik, Karl-Marx-Stadt, 1975, pp. 484–500; cf. Correction of misprints, Commentat. Math. Univ. Carol. *17* (1976), 205–206.

[86] Singularités non essentielles des solutions des équations aux dérivées partielles. Séminare de Théorie du Potentiel, Paris (1972–74), Lect. Notes Math., vol. 518, Springer, 1976, pp. 95–106.

[87] An example to a converse of Green's theorem. Cas. Pest. Mat. *101* (1976), 205–206. (In Czech.)

[88] Potential Theory III. Státní pedagogické nakladatelství, Praha, 1976, pp. 216 (with I. Netuka, J. Veselý—Lecture Notes). (In Czech.)

[89] Potential theory and Neumann's method. Mitt. Math. Ges. DDR *3–4* (1976), 71–79.
[90] Potential Theory IV. Státní pedagogické nakladatelství, Praha, 1977, pp. 304 (with I. Netuka, J. Veselý—Lecture Notes). (In Czech.)
[91] Analytic capacity. Proceedings of the Conference Elliptische Differentialgleichungen, Rostock, 1977, pp. 133–142.
[92] On removability of singularities in solving differential equations and on Neumann's principle. Materialy soveshchaniya po primeneniyu metodov teorii funkcij i funkcionalnogo analiza k zadacham matematicheskoj fiziki (Alma-Ata 1976), Akad. Nauk Novosibirsk, Novosibirsk, 1978, pp. 132–140. (In Russian.)
[93] Estimates of analytic capacity. Zapiski nauchnych seminarov LOMI *81* (1978), 212–217.
[94] Boundary behavior of potentials. Proceedings of the Conference Equadiff 4, Lect. Notes Math., vol. 703, Springer, 1979, pp. 205–212.
[95] Potential flows. Proceedings of the Conference Equadiff 5, Teubner-Texte zur Mathematik, vol. 47, Teubner, 1981, pp. 198–204.
[96] Negligible values of harmonic functions. Proceedings Primenenie metodov teorii funkcij i funkcionalnogo analiza k zadacahm matematicheskoj fiziki, Jerevan, 1982, pp. 168–173. (In Russian.)
[97] Some extension problems. Linear and Complex Analysis Problem Book, Lect. Notes Math., vol. 1043, Springer, 1984, pp. 639–640.
[98] A conjecture concerning subharmonic functions. Cas. Pest. Mat. *110* (1985), 415. (In Czech.)
[99] Double layer potentials on boundaries with corners and edges. Commentat. Math. Univ. Carol. *27* (1986), 419 (with T. Angell, R. E. Kleinman).
[100] In memory of Professor Marcel Brelot. Pokroky Mat. Fyz. Astronom. *33* (1988), 170–173 (with J. Lukeš, I. Netuka, J. Veselý). (In Czech.)
[101] On weak normal derivative of the single-layer potential with integrable density. Proceedings Funkcionalnye i chislennye metody matematicheskoj fiziki, Kijev, 1988, pp. 107–110. (In Russian.)
[102] Potential Theory—Surveys and Problems. Lect. Notes Math., vol. 1344, Springer, 1988, pp. 270 (co-editor with J. Lukeš, I. Netuka, J. Veselý).
[103] Proceedings of the Conference on Potential Theory. Plenum Publishing Company, New York, 1988, pp. 366 (co-editor with J. Lukeš, I. Netuka, J. Veselý).
[104] Contractivity of the operator of the arithmetical mean. Proceedings Potential Theory—Surveys and Problems, Lect. Notes Math., vol. 1344, Springer, 1988, pp. 223–225 (with D. Medková).
[105] Boundary regularity and potential-theoretic operators. Proceedings Potential Theory—Surveys and Problems, Lect. Notes Math., vol. 1344, Springer, 1988, pp. 220–222.
[106] Fine maxima. Proceedings Potential Theory—Surveys and Problems, Lect. Notes Math., vol. 1344, Springer, 1988, pp. 226–228 (with I. Netuka).
[107] International Conference on Potential Theory. Pokroky Mat. Fyz. Astronom. *33* (1988), 108–110 (with J. Lukeš, I. Netuka, J. Veselý). (In Czech.)
[108] To sixtieth anniversary of birthday of RNDr. Jaroslav Fuka, CSc. Cas. Pest. Mat. *114* (1989), 315–317. (In Czech.)
[109] Hausdorff measures and removable singularities of solutions of partial differential equations. Pokroky Mat. Fyz. Astronom. *35* (1990), 319–330. (In Czech.)
[110] To seventieth anniversary of birthday of Professor Jan Mařík. Cas. Pest. Mat. *115* (1990), 433–436 (with K. Karták, J. Matyska). (In Czech.)
[111] Seventy years of Professor Jan Mařík. Czech. Math. J. *41* (1991), 180–183 (with K. Karták, J. Matyska).

[112] The Fredholm-Radon method in potential theory. Collection: Continuum mechanics and related problem of analysis, Tbilisi, 1991 Metsniereba, Tbilisi, 1993, pp. 390–397.
[113] On the awarding of an honorary doctorate from Charles University to Heinz Bauer. Pokroky Mat. Fyz. Astronom. *38* (1993), 95–101 (with J. Lukeš, I. Netuka, J. Veselý). (In Czech.)
[114] In memoriam Professor Jan Mařík (1920–1994). Czech. Math. J. *44* (1994), 190–192 (with J. Kurzweil, I. Netuka, Š. Schwabik).
[115] Professor Jan Mařík (obituary). Math. Bohem. *119* (1994), 213–215 (with J. Kurzweil, I. Netuka, Š. Schwabik). (In Czech.)
[116] In memory of Professor Mařík. Pokroky Mat. Fyz. Astronom. *39* (1994), 177–179. (In Czech.)
[117] Estimates of analytic capacity. Linear and Complex Analysis Problem Book 3. Part 1, (eds. V. P. Havin, N. K. Nikolski), Lect. Notes Math., vol. 1573, Springer, 1994, pp. 154–156.
[118] Some extension problems. Linear and Complex Analysis Problem Book 3. Part 1, (eds. V. P. Havin, N. K. Nikolski), Lect. Notes Math., vol. 1573, Springer, 1994, pp. 328–329.
[119] Limits of integrals of the Cauchy type. Linear and Complex Analysis Problem Book 3. Part 1, (eds. V. P. Havin, N. K. Nikolski), Lect. Notes Math., vol. 1573, Springer, 1994, pp. 415–417.
[120] The divergence theorem. Math. Bohem. *121* (1996), 349–356.
[121] Potential theory—ICPT 94. Proceedings of the International Conference held in Kouty, August 13–20, 1994, Walter de Gruyter & Co., Berlin, 1996, pp. 499 (co-editor with J. Lukeš, I. Netuka, J. Veselý).
[122] Essential norms of the integral operator corresponding to the Neumann problem for the Laplace equation. Mathematical aspects of boundary element methods, Palaiseau, 1998, Chapman & Hall/CRC, Res. Notes Math., 414, Chapman & Hall/CRC, Boca Raton, FL, 2000, pp. 215–226 (with D. Medková).
[123] Sixty years of Ivan Netuka. Math. Bohem. *129* (2004), 91–107 (with J. Veselý).

Diploma theses supervised by Josef Král

1962 J. Lukeš: *On the Lebesgue integral*

1969 J. Štulc: *On the length of curves and continua*

1970 M. Dont: *Limits of potentials in higher dimensional spaces*

M. Řezníček: *Convergence in length and variation*

1973 J. Beránek: *Solutions of the Dirichlet problem by means of logarithmic double layer potentials*

P. Veselý: *Some properties of functions analogous to double layer potentials*

S. Mrzena: *Heat potentials*

1975 Chu Viet Dai: *Heat potentials*

1977 V. Pelikán: *Boundary value problems with a transition condition*

1978 I. Fusek: *Green's potentials and boundary value problems*

1978 L. Šolc: *Semielliptic differential operators*

1982 D. Křivánková: *Function kernels in potential theory*

1984 H. Zlonická: *Singularities of solutions of PDE's*
1989 J. Vaněk: *Double layer potential*
1992 M. Havel: *Generalized heat potentials*
1993 M. Pištěk: *Removable singularities for solutions of PDE's*

Ph.D. theses supervised by Josef Král

1968 J. Lukeš: *Estimates of Cauchy integrals on the Carathéodory's boundary*
1970 J. Veselý: *On some properties of double layer potentials*
1972 I. Netuka: *The third boundary value problem in potential theory*
M. Dont: *The Fredholm method and the heat equation*
1986 D. Medková: *The Neumann operator in potential theory*
1988 M. Chlebík: *Tricomi's potentials*
1990 E. Dontová: *Reflexion and the Dirichlet and Neumann problems*

Further generations of Ph.D. students

- J. Lukeš (1968)
  - E. Čermáková (1984)
  - J. Malý (1985)
    - R. Černý (2003)
    - S. Hencl (2003)
  - P. Pyrih (1991)
  - J. Kolář (1999)
  - J. Spurný (2001)
  - M. Smrčka (2004)
  - T. Mocek (2005)
- J. Veselý (1970)
  - P. Trojovský (2000)
- I. Netuka (1972)
  - M. Brzezina (1991)
    - M. Šimůnková (2002)
  - J. Ranošová (1996)
  - E. Cator (1997)
  - R. Lávička (1998)
  - L. Štěpničková (2001)
- M. Dont (1972)
  - J. Král Jr. (1998)

Personal recollections

The idea of including a course of potential theory in the curriculum of students specializing in mathematics in the Faculty of Mathematics and Physics of the Charles University is due to Professor Jan Mařík, who was concerned with teaching surface integration in the mid fifties; he believed that potential theory would offer many opportunities of illustrating importance of integral formulae such as the divergence theorem and the Green identities.

One of the tasks which I received from him during my postgraduate studies (1956–1959) was to prepare such a course which I started about 1960 as of 4 hours of lectures and 2 hours of exercises per week during one semester. The extent of the course was consecutively changing. In the academic year 1965/66, when I was abroad, the course was taken over by Professor Mařík for approximately 3 years as 4 hours of lectures during one semester. During the period 1970/71–1975/76 the time devoted to this course was reduced to 3 hours per week for one semester and later the obligatory course of potential theory was abolished altogether.

Starting from 1976/77 up to now a non-obligatory course of potential theory was regularly offered (with isolated exceptions, e.g. in 1987 or 1992) as 2 hours of lectures per week in both semesters. Also the content of the lecture was continuously changing. The time available was not sufficient to allow for the teaching of potential theory itself as well as the original intention of using the course as a rigorous introduction of surface integrals and for training in the techniques of surface integration. I have also soon abandoned the exposition of special properties of planar logarithmic potentials (for which only knowledge of curvilinear integrals was sufficient) with their applications to boundary value problems. Mostly the necessary integral formulae were only formulated with reference to courses of integral calculus and analysis on manifolds which were later offered in mathematical curricula, and proper lecture on potential theory was regularly devoted to basic properties of harmonic functions and the Perron method of the generalized Dirichlet problem in Euclidean spaces. Sometimes also potentials derived from Riesz's kernels and their generalizations were treated and notions of energy and capacity together with their applications were discussed. On the whole the exposition was oriented towards classical potential theory on Euclidean spaces.

I was aware of the gap between the content of the course and the research articles in potential theory (represented e.g. by recent issues of the Paris seminar "Séminaire Brelot-Choquet-Deny" which I could buy during my first stay abroad) appearing

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in the contemporary literature. In an attempt to bridge this gap I invited several friends and my former students to establish a seminar on mathematical analysis which would pay attention to the development of potential theory. We studied consecutively several texts which were accessible to us such as Brelot's "Lectures on Potential Theory" (Tata Institute of Fundamental Research, 1960) and Bauer's treatise "Harmonische Räume und ihre Potentialtheorie" (Springer, 1966).

The atmosphere towards the end of the sixties was to a certain extent relatively convenient for establishing international contacts. When I learned about preparation of a summer school devoted to potential theory in Italy, I encouraged younger participants of our seminar to attempt to participate. The result of their attempt was unexpectedly favourable: a whole group of young Czech mathematicians was able to travel to Stresa in 1969. They returned full of enthusiasm—they were able to meet many of distinguished specialists and their students and to establish their first international scientific contacts. Thanks to these contacts many important studies in the field of potential theory (including e.g. the monograph "Potential Theory on Harmonic Spaces" prepared by C. Constantinescu and A. Cornea and published by Springer in 1972) became accessible to our seminar even before their publication.

In spite of problems caused by political development in the seventies and the eighties the scientific contacts were not interrupted and continued to develop. Many outstanding experts visited Bohemia and many participants of our seminar had opportunity to get acquainted with activities of centers of potential-theoretic research abroad; it was significant that some members of the seminar were able to participate in long-term stays in renowned institutions abroad.

Nowadays the possibilities of studying potential theory in Bohemia are good. During the past 30 years a number of gifted mathematicians have grown up who were able to achieve a number of remarkable results; I hope that they will continue in activities of the seminar. I believe that prospects of the future development of research and teaching in the field of potential theory in this country are very promising.