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ON A NEW METHOD IN THE ANALYSIS WITH APPLICATIONS.

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In what follows I shall develop how the systematic use of our two main theorems (theorems I. and II. below) leads to partly wellknown, partly new results in various chapters of analysis. A first outline of this method — with emphasis on applications to the theory of zeta-function of Riemann — I gave in the winterterm of 1947—48 in lectures delivered at the Mathematisk Institut in Copenhagen. In this lecture I shall indicate also some unpublished new applications of the method.

At first a few words on our two main theorems. Kronecker's classical theorem can be stated in an analytical form essentially due to H. Bohr, according which if $z_1, z_2, \ldots, z_n$ are complex numbers with linearly independent arguments, then the function

$$f(y) = \sum_{\nu=1}^{n} b_\nu z_\nu{}^y, \tag{1}$$

— the generalised power-sum of the z_ν's — assumes for an appropriate real y a value absolutely greater than $\sum_{\nu}^{n} |b_\nu|\ |z_\nu|^y - \varepsilon$. Or

$$\max_{y} \frac{\left|\sum_{\nu=1}^{n} b_\nu z_\nu{}^y\right|}{\sum_{\nu=1}^{n} |b_\nu|\ |z_\nu|^y} > 1 - \varepsilon, \tag{2}$$

if ε is a prescribed but arbitrary small positive number. Here no limitation can be given to that value y for which (2) is effected. *If the b_j numbers are positive* then Dirichlet's well-known theorem gives immediately that

$$\max_{1 \leqq y \leqq 5^n} \frac{|f(y)|}{\sum_{\nu=1}^{n} b_\nu |z_\nu|^y} \geqq \cos\frac{2\pi}{5}. \tag{3}$$

Though the limitation $1 \leqq y \leqq 5^n$ is rather rough, the inequality (3) enabled H. Bohr, Hardy and Littlewood to find a series of applications in the analysis and analytical theory of numbers. In many questions of these subjects however the rather sharp lower limitation is not as essential as to find narrower limits for the corresponding y-values. Now, if

$$|z_1| \leqq |z_2| \leqq \cdots \leqq |z_n|. \tag{4}$$

I observed in 1943 that investigating the expressions

$$\text{a)}\quad \max_{y} \frac{|f(y)|}{|z_1|^y}, \qquad \text{b)}\quad \max_{y} \frac{|f(y)|}{z_n^y},$$

instead of those in (2) and (3) one can find indeed inequalities in which the range of y is much narrower. As a matter of fact I proved then the following two theorems.

Theorem I. *If $m > n$, then we have*

$$\max_{\substack{m-n \leqq y \leqq m \\ y \text{ integer}}} \frac{\left|\sum_{\nu=1}^{n} b_\nu z_\nu^y\right|}{|z_1|^y} > \left(\frac{n}{6m}\right)^n |b_1 + \ldots + b_n|.$$

Theorem II. *If $m \geqq 28n$, then*

$$\max_{\substack{m-n \leqq y \leqq m \\ y \text{ integer}}} \frac{\left|\sum_{\nu=1}^{n} b_\nu z_\nu^y\right|}{|z_n|^y} > \left(\frac{n}{e^{22}m}\right)^n \min_{j=1,\ldots,n} \frac{|b_1 + \ldots + b_j|}{j}$$

Owing to the form (2) of Kronecker's theorem, the theorems I. and II. may be considered as results from the theory of diophantine approximations. Obviously theorem II. is the deeper; an important tool in its proof is the theorem of H. Cartan, according which for any given $H > 0$ and $g(z) = z^n + c_1 z^{n-1} + \ldots + c_n$ we have

$$|g(z)| \geqq \left(\frac{H}{e}\right)^n,$$

except a set which can be covered by n circles at most the sum of whose radii do not exceed $2H$. By some alterations of the proofs one can give also theorems, intermediary between our theorems I and II. resp. Kronecker's theorem (2), allowing larger intervals for y and giving sharper lower bounds. Needless to say, nothing is required in theorems I. and II. of linear independence of the quantities $\operatorname{arc} z_\nu$.

The first group of applications of our theorem I. refers to the theory of gap-theorems. A proof of the classical gap-theorem of Fabry, according which the function

$$U(z) = \sum_{\nu=1}^{\infty} a_\nu z^{\lambda_\nu},$$

with integral exponents λ_ν and

$$\frac{\lambda_\nu}{\nu} \to \infty$$

and with convergence-radius 1 is singular in every point of the unit-circle, can be reduced by suitable approximation to the proof of the inequality

$$\max \left| \sum_{\nu=1}^{n} b_\nu e^{i\lambda_\nu x} \right| \leqq \left(\frac{11\pi}{\delta}\right)^n \max_{a \leq x \leq a+\delta} \left| \sum_{\nu=1}^{n} b_\nu e^{i\lambda_\nu x} \right| \tag{6}$$

and this follows from theorem I. choosing the z_ν-numbers and m properly. For the sake of orientation I remark that inequality (6) with $\left(\frac{11\pi}{\delta}\right)^{\lambda_n}$ instead of $\left(\frac{11\pi}{\delta}\right)^n$ is quite simple to prove but would be insufficient to give a proof of Fabry's gap-theorem. In the special case $\lambda_{n+1} - \lambda_n \to \infty$ N. WIENER reduced previously the proof of Fabry's theorem to another inequality of type (6). If we drop the restriction the numbers λ_ν being integers, the corresponding inequality leads to a proof of a generalisation of Fabry's theorem to Dirichlet-series which is however weaker than Pólya's generalisation (though contains the well-known theorems of Carlson-Landau and O. Szász). The inequality (6) gives also possibility to extend and refine in various ways the theorem of Pólya according which an integral-function $f(z) = \sum_{\nu=1}^{\infty} a_\nu z^{\lambda_\nu}$ with the Fabry-condition (5) increases in an arbitrary small angle essentially as fast as on the whole plane.

Another application refers to the theory of quasi-analytic functions. The main-problem here — somewhat generally expressed as usual — to characterise classes of functions defined in $[a, b]$ such that if any two functions $f_1(x)$ and $f_2(x)$ behave similarly in a certain sense *in a neighbourhood* of an inner point c of $[a, b]$, then $f_1(x) = f_2(x)$ almost everywhere *in the whole interval* $[a, b]$; The classical instances are the class of functions, analytic in $[a, b]$ and the class of functions, differentiable infinitely often and satisfying limitations

$$|f^{(n)}(x)| \leqq M_n$$
$$n = 0, 1, 2, \ldots, \quad a \leqq x \leqq b$$

with

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{M_n}} = \infty.$$

S. BERNSTEIN and S. MANDELBROJT showed the existence quasi-analytic classes — interpreting properly the phrase „similar behaviour in a point c“ — containing functions which are not even continuous. Mandelbrojt defines the „similar behaviour“ of the integrable $f_1(x)$ and $f_2(x)$ in c by

$$\varliminf_{h \to 0} e^{\frac{1}{|h|^\varrho}} \int_c^{c+h} |f_1(t) - f_2(t)| \mathrm{d}t < \infty \tag{7}$$

and determines his classes by expanding the functions in Fourier-series (supposed $[a, b] = [0, 2\pi]$)

$$f(x) \sim \sum_{\nu=0}^{\infty} (a_\nu \cos\nu x + b_\nu \sin\nu x) \tag{8}$$

and restricting the density of ν Fourier-*exponents* for whose

$$a_\nu^2 + b_\nu^2 \neq 0.$$

Now one can obtain quasi-analytic classes using theorem I in a suitable way based on the Fourier-*coefficients* of the expansion (8). Instead of (7) we require a little more restrictively

$$\overline{\lim_{h \to 0}}\; c^{\frac{1}{|h|^\varrho}} \max_{c \leqq x \leqq c+h} |f_1(x) - f_2(x)| < \infty$$

in the definition of the quasi-analyticity. These results can be extended to H. Bohr's almost-periodical functions too.

The next application shows clearly why in some situations theorem I. gives sharper results, than Dirichlet's theorem. In connection with the so-called Riemann-Siegel asymptotical representation of $\zeta(\frac{1}{2} + it)$ the question arises how many real zeros the polynomial

$$G(x) = \sum_{\nu=0}^{n} a_\nu \cos\lambda_\nu x \tag{9}$$

$$0 = \lambda_0 < \lambda_1 < \ldots < \lambda_n$$

can have in the interval $(0 <)\ a - d \leqq x \leqq a$. We restrict ourselves to the case $a_\nu \geqq 0$ $(\nu = 0, 1, \ldots, n)$. It is rather straightforward to overlap the interval $a - 2d \leqq x \leqq a + d$ by a suitable circle and apply Jensen's inequality. Difficulty arises at the lower estimation of $G(z)$ in the centre of the circle. Dirichlet's theorem assures the existence of a real x_1 with

$$G(x_1) > \frac{1}{100}(a_0 + a_1 + \ldots + a_n) \tag{10}$$

but this x_1 can be restricted only to be less than 100^n and hence nothing essentially better can be said about the radius of the covering circle than it is $\leqq 100^n + a$. But this implies that for the absolute maximum of $G(z)$ in this circle no essentially better estimation can be stated than

$$\max |G(z)| < (a_0 + \ldots + a_n)\, e^{\lambda_n(100^n + a)} \tag{11}$$

i. e. Jensen's inequality furnishes only as upper estimation a quantity about $\lambda_n(100^n + a)$. Using theorem I instead of Dirichlet's theorem we can assure the existence of an x_2 in $a - d \leqq x \leqq a$ such that

$$|G(x_2)| > \left(\frac{11d}{a}\right)^n (a_0 + \ldots + a_n)$$

which is much worse an estimation than (10) but, since x_2 is much better localised, we can choose an overlapping circle around x_2 with the radius $2d$ and in this circle we have the estimation

$$|G(z)| < (a_0 + \dots + a_n)e^{\lambda n 2d}$$

which is generally much better than the estimation (11). This gives — if $N(G, a, d)$ denotes the number of real zeros of $G(z)$ in $a - d \leqq x \leqq a$ — the estimation

$$N(G, a, d) < 6n \log \frac{24(|a| + d)}{d} + 6\,d\lambda_n.$$

Now I turn to applications in the theory of zeta-function of Riemann. This is defined — denoting the complex variable by $s = \sigma + ti$ — for $\sigma > 1$ by

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \dots + \frac{1}{n^s} + \dots$$

It is classical that in this domain

$$\zeta(s) = \prod_p \frac{1}{1 - \frac{1}{p^s}}$$

where p runs over the primes and that $\zeta(s) - \frac{1}{s-1}$ can be continued over the whole plane. The famous unsolved hypothesis of Riemann states that $\zeta(s)$ does not vanish in the half-plane $\sigma > \frac{1}{2}$. This and the similar questions referring to other „zeta-functions" constitute the central problems in the analytical theory of numbers. There is a tendency — expressed in the papers of Davenport-Hasse and A. Weil mainly — to transform the problem into estimations of certain exponential-sums. Theorem II. gave possibility to do this for the zeta-function of Riemann itself. The critical exponential-sums are of the form

$$S(t) = \sum_{N' \leqq p \leqq N''} e^{it \log p} \tag{12}$$

where the summation refers again to the primes in $[N', N'']$, further

$$\frac{N}{2} \leqq N' < N'' \leqq N. \tag{13}$$

If α is an arbitrary prescribed number with

$$\alpha \geqq 2 \tag{14}$$

then N and t are restricted by

$$N \geqq |t|^\alpha. \tag{15}$$

Now it turned out that the existence of numerical β and A with

$$0 < \beta \leqq 1, \quad A \geqq 1 \tag{16}$$

for which the inequality

$$|S(t)| \leqq A \frac{Ne^{20(\log\log N)^2}}{|t|^\beta} \tag{17}$$

holds (with the provisions (13), (14), (15) of course) implies the existence of a half-plane

$$\sigma > \Theta_0 = \Theta_0(\alpha, \beta)(< 1) \tag{18}$$

containing at most a finite number of zeros of $\zeta(s)$. The converse statement i. e. that the existence of a half-plane $\sigma > \Theta_0(< 1)$ containing only a finite number of zeros of $\zeta(s)$ implies (17) for *all* $\alpha \geqq 2$, with some β and A satisfying (16), is almost trivial. Hence the inequality (17) with its provisions (13), (14), (15) and (16) constitutes the *necessary and sufficient* condition for the truth of the quasiriemannian hypothesis i. e. for the existence of a non-trivial half-plane $\sigma > \Theta_0(< 1)$ containing at most a finite number of zeros of $\zeta(s)$.

This result shows the interesting fact that the position of this Θ_0 depends only upon α and β. This suggests the problem to determine this dependence *exactly* and even to introduce for functions $B(s)$ defined by general Dirichlet-series the abscissa σ_ν as the lower limit of such Θ's for which the half plane $\sigma > \Theta$ contains at most a finite number of zeros of $B(s)$. According to an example of H. Bohr this abscissa does not exist for all $B(s)$; but I determined σ_ν for a class of the functions $B(s)$ *exactly*. This class contains $\zeta(s)$ only if the so-called Lindelöf-hypothesis is true, i. e. if in the half-plane $\sigma \geqq \frac{1}{2} + \varepsilon$

$$\frac{|\zeta(s)|}{(1 + |t|)^\varepsilon}$$

is bounded by a quantity depending only upon ε (ε arbitrary small positive number). In this case the σ_ν abscissa for $\zeta(s)$ is

$$1 - \overline{\lim} \frac{\beta}{\alpha}. \tag{19}$$

where the $\overline{\lim}$ is taken for all (α, β) pairs, for whose the inequality (17) with the previsions (13), (14), (15), (16) holds. The main tool of the proof is again theorem II. This shows that the expression $S(t)$ is naturally connected to the problem.

Omitting other equivalence-theorems in whose the role of the expression (12) is played by

$$H_\gamma(t) = \sum_{N' \leqq p \leqq N''} e^{it\log^\gamma p} \tag{20}$$

resp.

$$V(t) = \sum_{N' \leqq p \leqq N''} e^{ipt} \tag{21}$$

we turn to another application of theorem II. Denoting by $N(\sigma_0, T)$, as usual, the number of zeros of $\zeta(s)$ in the parallelogram

$$\sigma > \sigma_0, \quad |t| \leqq T.$$

Carlson proved the estimation (which I mention in a slightly generalised form)

$$N(\sigma_0, T) < cT^{4\sigma_0(1-\sigma_0)} \log^6 T \tag{22}$$

uniformly for $\frac{1}{2} \leqq \sigma_0 \leqq 1$, where c denotes a numerical constant. This became extremely important recently in the analytical theory of numbers; I mention only the papers of Hoheisel, Ingham and Linnik. More exactly an estimation of the form

$$N(\sigma_0, T) < cT^{\lambda(1-\sigma_0)} \log^6 T \tag{23}$$

valid uniformly in $\frac{1}{2} \leqq \sigma_0 \leqq 1$, is what is essential for the upper estimation of the difference of the consecutive primes, according to the discovery of Hoheisel. The smaller λ can be chosen, the better an estimation we obtain for $p_{n+1} - p_n$. (22) gave $\lambda = 4$; it is obvious that this can be improved in any part of $\frac{1}{2} \leqq \sigma_0 \leqq 1$ which does not contain the neighbourhood of $\sigma_0 = 1$. Ingham replaced in (22) the factor $4\sigma_0$ by

$$\min(\tfrac{8}{3}, 1 + 2\sigma_0)$$

i. e. he proved (23) with $\lambda = \frac{8}{3}$; also his result shows that the most difficult is to improve λ for such σ_0 — values which are near to 1. On the other hand it is easy to show that $\lambda \geqq 2$ and essentially Hoheisel's analysis shows that an estimation

$$N(\sigma_0, T) < cT^{2(1-\sigma_0)} \log^6 T \tag{24}$$

valid with a numerical constant c uniformly for $\frac{1}{2} \leqq \sigma_0 \leqq 1$ would imply for arbitrary small positive ε

$$p_{n+1} - p_n < c_0(\varepsilon) p_n^{\frac{1}{2}+\varepsilon}. \tag{25}$$

Now using theorem II. I proved the existence of a sufficiently small numerical b and sufficiently large numerical c_1, and c_2 such that for

$$1 - b \leqq \sigma_0 \leqq 1, \quad T > c_1$$

we have

$$N(\sigma_0, T) < c_2 T^{2(1-\sigma_0)+320(1-\sigma_0)^{\frac{15}{14}}} \log^6 T. \tag{26}$$

This gives essentially the required estimation (24) near to $\sigma_0 = 1$; it is not quite impossible that even (26) implies (25).

A further field of applications refers to the remainder-term of the prime-number formula. Denoting the zeros of $\zeta(s)$ by $\varrho = \sigma_\varrho + it_\varrho$ and

$$\sup_{\varrho} \sigma_\varrho = \Theta \tag{27}$$

it is well-known that

$$\Delta(x) = \sum_{n \leqq x} \Lambda(n) - x = O(x^\Theta \log^2 x) \tag{28}$$

or if $\Theta > \frac{1}{2}$, according to an unpublished remark of Ingham even

$$\Delta(x) = \mathrm{O}(x^{\Theta})$$

and to an arbitrary positive ε there is a sequence

$$x_1 < x_2 < \ldots < x_n > \ldots \to \infty$$

such that

$$|\Delta(x_\nu)| > x_\nu^{\Theta - \varepsilon}. \tag{29}$$

Hence there is a natural connection between the exponent of x in the remainder-term of (28) and the abscissa of non-vanishing (27). But in fact we know only domains of the form

$$\sigma > 1 - \frac{1}{\omega(t)} \tag{30}$$

with $\omega(t)$ even, positive, increasing monotonically for $t > t_0$ and tending to ∞ if $t \to \infty$, where we can assure the non-vanishing of $\zeta(s)$ and correspondingly we can prove actually for $\Delta(x)$ only estimations of the form

$$\Delta(x) = O\left(\frac{x}{\nu(x)}\right) \tag{31}$$

with

$$\lim_{x \to \infty} \frac{\log \nu(x)}{\log x} = 0. \tag{32}$$

It is a natural question to ask whether or not an inference from (31), (32) to (30) is possible at all. Now theorem II. gives way to such inference whose possibility was indicated to me by Prof. HEILBRONN. in a letter.

Concerning (29) LITTLEWOOD wrote in a paper from 1937 in which he proved a special case of our theorem I. ,,... But this proof is curiously indirect; if $\Theta > \frac{1}{2}$ and we are given a particular $\varrho = \varrho_0$ for which $\beta = \beta_0 > \frac{1}{2}$ they provide no explicit X depending only on β_0, γ_0 and ε such that $|\Delta(x)| > X^{\beta_0 - \varepsilon}$ for some x in (O, X). There is no known way of showing (for any explicit X) that the single term $\dfrac{x^{\beta_0 + i\gamma_0}}{\beta_0 + i\gamma_0}$ of $\Delta(x)$ is not interfered with by the other terms of the series over the range (O, X) ... Now theorem II. leads to the theorem that if $c_3, c_4, \ldots$ are explicitely given numerical constants and $M_0 = M_0(\gamma_0)$ is such that for $t > M_0$

$$\sqrt[4]{t}\, e^{-c_3 \frac{\log t \log_3 t}{\log_2 t}} > \gamma_0^{10 \frac{\log t}{\log_2 t}}$$

than for $X > \max(c_4, M_0)$ we have

$$\max_{0 < x < X} |\Delta(x)| > \frac{X^{\beta_0}}{(\gamma_0)^{10 \frac{\log X}{\log_2 X}}} e^{-c_5 \frac{\log X \log_3 X}{\log_2 X}}$$

Finally I mention another field where applications will be possible after an improvement of our theorems I. and II. Linnik succeeded in 1943 in proving that the smallest prime $P(k, l)$ of the arithmetical progression $\equiv l(\mathrm{mod} k)$, $(k, l) = 1$ is $< k^{c_6}$ with a numerical c_6 what I could deduce previously only using an unproved hypothesis concerning the zeros of L-functions of Dirichlet. His proof for this important theorem is extremely complicated; an improvement of theorem II. with respect to the dependence upon the b_ν-coefficients would give a much shorter proof.

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Výtah. — Summary.

O nové methodě v analyse s aplikacemi.

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Autor rozřešil dva problémy (theorém 1. a 2. v textu), které lze považovati za modifikace Kroneckerova theorému o diofantických aproximacích. Podává řadu aplikací. Tyto se týkají theorie mocninných a Dirichletových řad s mezerami, theorie kvasianalytických funkcí, rozdělení kořenů skoro periodických polynomů; v analytické theorii čísel vedou k větám o ekvivalenci, k zjemnění důležité věty Carlsonovy a k různým výsledkům o zbytkovém členu v prvočíselné formuli. Další možné aplikace jsou uvedeny.