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Two Different However Equivalent Methods for Derivation of Estimators of Parameters in Deformation Measurements

Lucie EXNEROVÁ

Department of Mathematical Analysis and Applications of Mathematics,
Faculty of Science, Palacký University,
Tomkova 40, 779 00 Olomouc, Czech Republic
e-mail: exnerova.lucie@seznam.cz

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Abstract

The aim of this paper is to develop two different methods for an executing of the deformation measurement and to prove that these two methods are equivalent which is a advantage for a conclusive verification of the results of the experiment in a practice.

Key words: Multiepoch linear model, multivariate regression model with constraints.

2000 Mathematics Subject Classification: 62J05, 62H15

1 Introduction

The aim is to develop two different methods for an executing of four epochs experiment in which the movements of the reference points on a dam during the gradual filling of the dam have been measured. According to the instructions of a structural designer these points should move along the specific trajectories. The aim of this experiment is to compare these theoretical trajectories with empirical ones. In the first method coordinates of the reference points and the parameters that describe trajectories of these points are estimated at the same time. In the second method the coordinates of the points are estimated first of all and these estimates are used for a calculation of the trajectories. The corrected coordinates from the second method must be equal to the estimated coordinates from the first method. The first procedure can be realized after realization of the 4th epoch measurement only. Since it is necessary to know preliminary results (shifts of the reference points) during the single epoch, we must estimate coordinates after each epoch separately. At the end of the 4th epoch estimations of the coordinates of the all reference points are at our disposal and the parameters of the trajectories can be estimated by means of second method. However at the same time both coordinates and trajectories parameters can be estimated simultaneously in another, however equivalent model (first method). Both methods should give the same result for the parameters of the trajectories.

2 Notation and auxiliary statements

Let $\underline{\mathbf{Y}}$ be $n \times m$ random matrix (observation matrix), $\underline{\mathbf{Y}} = (\mathbf{Y}_1, \ldots, \mathbf{Y}_m)$, with the mean value $\mathrm{E}(\underline{\mathbf{Y}}) = \mathbf{XB}$. $\mathbf{X}$ is an $n \times k$ given design matrix, $\mathbf{B}$ is an $k \times m$ matrix of unknown parameters (coordinates of the reference points) and $\mathbf{C}$ is an $k \times q$ matrix of unknown parameters (parameters of the trajectories). $\mathbf{I} \otimes \mathbf{\Sigma}$ is the covariance matrix of the observation vector $\mathrm{vec}(\underline{\mathbf{Y}}) = (\mathbf{Y}'_1, \ldots, \mathbf{Y}'_{\mathrm{m}})'$ and the constraints are given in a form $\mathbf{BH} + \mathbf{CZ} + \mathbf{G} = \mathbf{0}$. Here the matrix $\mathbf{H}, \mathbf{Z}, \mathbf{G}$ are known.

The model

$$\underline{\mathbf{Y}} \sim (\mathbf{XB}, \mathbf{I} \otimes \mathbf{\Sigma}),$$

is regular if $r(\mathbf{X}_{n,k}) = k < n$ and $\mathbf{\Sigma}$ is positive definite. The constraints $\mathbf{BH} + \mathbf{CZ} + \mathbf{G} = \mathbf{0}$ are regular if $r(\mathbf{H}'_{m,r}, \mathbf{Z}'_{q,r}) = r < m + q$ and $\mathrm{r}(\mathbf{Z}_{q,r}) = q < r$. In the following text it is also assumed $r(\mathbf{H}_{m,r}) = r < m$.

In the following $\mathbf{A}^+$ denotes the Moore–Penrose generalized inverse of the matrix $\mathbf{A}$ (i.e. $\mathbf{AA}^+\mathbf{A} = \mathbf{A}$, $\mathbf{A}^+\mathbf{AA}^+ = \mathbf{A}^+$, $\mathbf{AA}^+ = (\mathbf{AA}^+)'$, $\mathbf{A}^+\mathbf{A} = (\mathbf{A}^+\mathbf{A})'$ cf. [3]).

The symbol $\mathbf{M}_X$ means the projection matrix $\mathbf{I} - \mathbf{P}_X$, where $\mathbf{I}$ is the identity matrix and $\mathbf{P}_X$ is the projection matrix (in the Euclidean norm) on the subspace $\mathcal{M}(\mathbf{X}) = \{\mathbf{Xu} : \mathbf{u} \in \mathrm{R}^k\}$. Here R^k means the k dimensional real vector space.

Lemma 1 *Let the model and the constraints*

$$\underline{\mathbf{Y}} \sim_{nm} (\mathbf{XB}, \mathbf{I} \otimes \mathbf{\Sigma}), \quad \mathbf{B}_{k,m}\mathbf{H}_{m,r} + \mathbf{C}_{k,q}\mathbf{Z}_{q,r} + \mathbf{G} = \mathbf{0}_{k,r} \tag{1}$$

be regular. Then the best linear unbiased estimators (BLUE) of the matrices $\mathbf{B}$ *a* $\mathbf{C}$ *are*

$$\begin{aligned}\hat{\hat{\mathbf{B}}} = & -\mathbf{G}\Big[\mathbf{I} - (\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}'[\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}']^{-1}\mathbf{Z}\Big](\mathbf{H}'\mathbf{H})^{-1}\mathbf{H}' + \\ & + \hat{\mathbf{B}}\Big[\mathbf{M}_H + \mathbf{H}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}'[\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}']^{-1}\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{H}'\Big],\end{aligned} \tag{2}$$

$$\hat{\hat{\mathbf{C}}} = -\mathbf{G}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}'[\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}']^{-1} - \hat{\mathbf{B}}\mathbf{H}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}'[\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}']^{-1} \tag{3}$$

and

$$\mathrm{Var}[\mathrm{vec}(\widehat{\widehat{\mathbf{B}}})] =$$
$$= \Big[\mathbf{M}_H + \mathbf{H}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}'[\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}']^{-1}\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{H}'\Big] \otimes (\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1},$$
$$\mathrm{Var}[\mathrm{vec}(\widehat{\widehat{\mathbf{C}}})] = [\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}']^{-1} \otimes (\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}.$$

Here $\widehat{\mathbf{B}} = (\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1}\underline{\mathbf{Y}}$.

Proof In the univariate regular model

$$\mathbf{Y} \sim (\mathbf{X}\boldsymbol{\beta}_1, \mathbf{\Sigma}), \quad \mathbf{B}_1\boldsymbol{\beta}_1 + \mathbf{B}_2\boldsymbol{\beta}_2 + \mathbf{b} = \mathbf{0},$$

the BLUE of $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ are

$$\begin{pmatrix} \hat{\hat{\boldsymbol{\beta}}}_1 \\ \hat{\hat{\boldsymbol{\beta}}}_2 \end{pmatrix} = -\begin{pmatrix} (\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{B}_1'\mathbf{Q}_{1,1} \\ \mathbf{Q}_{2,1} \end{pmatrix}\mathbf{b} + \begin{pmatrix} \mathbf{I} - (\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{B}_1'\mathbf{Q}_{1,1}\mathbf{B}_1 \\ -\mathbf{Q}_{2,1}\mathbf{B}_1 \end{pmatrix}\hat{\boldsymbol{\beta}}_1$$

and

$$\begin{aligned} \mathrm{var}(\hat{\hat{\boldsymbol{\beta}}}_1) &= (\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1} - (\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{B}_1'\mathbf{Q}_{1,1}\mathbf{B}_1(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}, \\ \mathrm{var}(\hat{\hat{\boldsymbol{\beta}}}_2) &= -\mathbf{Q}_{2,2}, \end{aligned}$$

where $\hat{\boldsymbol{\beta}}_1 = (\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{Y}$ and

$$\begin{pmatrix} \mathbf{Q}_{1,1}, & \mathbf{Q}_{1,2} \\ \mathbf{Q}_{2,1}, & \mathbf{Q}_{2,2} \end{pmatrix} = \begin{pmatrix} \mathbf{B}_1(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{B}_1', & \mathbf{B}_2 \\ \mathbf{B}_2', & \mathbf{0} \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} \{\mathbf{M}_{B_2}\mathbf{A}\mathbf{M}_{B_2}\}^+, & (\mathbf{B}_2')^-_{m(A)} \\ \Big[(\mathbf{B}_2')^-_{m(A)}\Big]', & -\Big[(\mathbf{B}_2')^-_{m(A)}\Big]'\mathbf{A}(\mathbf{B}_2')^-_{m(A)} \end{pmatrix}.$$

Here $\mathbf{A} = \mathbf{B}_1(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{B}_1'$ and $(\mathbf{B}_2')^-_{m(A)}$ denotes minimum $\mathbf{A}$-seminorm generalized inverse of the matrix $\mathbf{B}_2'$. (cf. theory of the Pandora-Box matrix in [3])

Now it suffices to write the multivariate model in the form

$$\mathrm{vec}(\underline{\mathbf{Y}}) \sim [(\mathbf{I} \otimes \mathbf{X})\mathrm{vec}(\mathbf{B}), \mathbf{I} \otimes \mathbf{\Sigma}],$$
$$(\mathbf{H}' \otimes \mathbf{I})\mathrm{vec}(\mathbf{B}) + (\mathbf{Z}' \otimes \mathbf{I})\mathrm{vec}(\mathbf{C}) + \mathrm{vec}(\mathbf{G}) = \mathbf{0}$$

and use the equalities

$$\begin{aligned}
\mathbf{Q}_{1,1} &= \left\{\mathbf{M}_{(Z'\otimes I)}[(\mathbf{H}'\mathbf{H})\otimes(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}]\mathbf{M}_{(Z'\otimes I)}\right\}^{+}\\
&= [(\mathbf{H}'\mathbf{H})^{-1}\otimes(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})] - [(\mathbf{H}'\mathbf{H})^{-1}\otimes(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})](\mathbf{Z}'\otimes\mathbf{I})\\
&\quad\times\left[(\mathbf{Z}\otimes\mathbf{I})[(\mathbf{H}'\mathbf{H})^{-1}\otimes(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})](\mathbf{Z}'\otimes\mathbf{I})\right]^{-1}\\
&\quad\times(\mathbf{Z}\otimes\mathbf{I})[(\mathbf{H}'\mathbf{H})^{-1}\otimes(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})]\\
&= [(\mathbf{H}'\mathbf{H})^{-1}\otimes(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})]\left\{(\mathbf{I}\otimes\mathbf{I}) - \left[\mathbf{Z}'[\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}']^{-1}\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\right]\otimes\mathbf{I}\right\},\\
\mathbf{Q}_{2,1} &= \left[(\mathbf{Z}\otimes\mathbf{I})^{-}_{m[(H'H)\otimes(X'\Sigma^{-1}X)^{-1}]}\right]'\\
&= \left[(\mathbf{Z}\otimes\mathbf{I})[(\mathbf{H}'\mathbf{H})^{-1}\otimes(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})](\mathbf{Z}'\otimes\mathbf{I})\right]^{-1}\\
&\quad\times(\mathbf{Z}\otimes\mathbf{I})[(\mathbf{H}'\mathbf{H})^{-1}\otimes(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})] = \left\{[\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}']^{-1}\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\right\}\otimes\mathbf{I},\\
\mathbf{Q}_{2,2} &= -\left[(\mathbf{Z}\otimes\mathbf{I})^{-}_{m[(H'H)\otimes(X'\Sigma^{-1}X)^{-1}]}\right]'[(\mathbf{H}'\mathbf{H})\otimes(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}]\\
&\quad\times(\mathbf{Z}\otimes\mathbf{I})^{-}_{m[(H'H)\otimes(X'\Sigma^{-1}X)^{-1}]}\\
&= -\left[(\mathbf{Z}\otimes\mathbf{I})[(\mathbf{H}'\mathbf{H})^{-1}\otimes(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})](\mathbf{Z}'\otimes\mathbf{I})\right]^{-1}\\
&= -[\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}']^{-1}\otimes(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}
\end{aligned}$$

and $\mathrm{vec}(\mathbf{AXB}) = (\mathbf{B}'\otimes\mathbf{A})\mathrm{vec}(\mathbf{X})$. □

Lemma 2 *The BLUEs of the matrices* $\mathbf{B}$ *and* $\mathbf{C}$ *are the same in the model (1) and in the model*

$$\underline{\widehat{\mathbf{B}}} \sim_{km} [\mathbf{B}, \mathbf{I}\otimes(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}], \quad \mathbf{B}_{k,m}\mathbf{H}_{m,r} + \mathbf{C}_{k,q}\mathbf{Z}_{q,r} + \mathbf{G} = \mathbf{0}_{k,r} \tag{4}$$

respectively.

Proof We write the model (4) in the form

$$\begin{gathered}
\mathrm{vec}(\underline{\widehat{\mathbf{B}}}) \sim [(\mathbf{I}\otimes\mathbf{I})\mathrm{vec}(\mathbf{B}), \mathbf{I}\otimes(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}],\\
(\mathbf{H}'\otimes\mathbf{I})\mathrm{vec}(\mathbf{B}) + (\mathbf{Z}'\otimes\mathbf{I})\mathrm{vec}(\mathbf{C}) + \mathrm{vec}(\mathbf{G}) = \mathbf{0}
\end{gathered}$$

and use the relations from the proof of Lemma 1

$$\begin{pmatrix}\mathbf{Q}_{1,1}, & \mathbf{Q}_{1,2}\\ \mathbf{Q}_{2,1}, & \mathbf{Q}_{2,2}\end{pmatrix} = \begin{pmatrix}(\mathbf{H}'\otimes\mathbf{I})\{(\mathbf{I}\otimes\mathbf{I})[\mathbf{I}\otimes(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})](\mathbf{I}\otimes\mathbf{I})\}^{-1}(\mathbf{H}\otimes\mathbf{I}), & \mathbf{Z}'\otimes\mathbf{I}\\ \mathbf{Z}\otimes\mathbf{I}, & \mathbf{0}\end{pmatrix}^{-1},$$

$$\begin{aligned}
\mathbf{Q}_{1,1} &= \left\{\mathbf{M}_{(Z'\otimes I)}[(\mathbf{H}'\mathbf{H})\otimes(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}]\mathbf{M}_{(Z'\otimes I)}\right\}^{+}\\
&= [(\mathbf{H}'\mathbf{H})^{-1}\otimes(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})]\left\{(\mathbf{I}\otimes\mathbf{I}) - \left[\mathbf{Z}'[\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}']^{-1}\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\right]\otimes\mathbf{I}\right\},\\
\mathbf{Q}_{2,1} &= \left[(\mathbf{Z}\otimes\mathbf{I})^{-}_{m[(H'H)\otimes(X'\Sigma^{-1}X)^{-1}]}\right]' = \left\{[\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}']^{-1}\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\right\}\otimes\mathbf{I}
\end{aligned}$$

and

$$\mathbf{Q}_{2,2} = -\left[(\mathbf{Z}\otimes\mathbf{I})^{-}_{m[(H'H)\otimes(X'\Sigma^{-1}X)^{-1}]}\right]'[(\mathbf{H}'\mathbf{H})\otimes(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}] \times (\mathbf{Z}\otimes\mathbf{I})^{-}_{m[(H'H)\otimes(X'\Sigma^{-1}X)^{-1}]} = -[\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}']^{-1}\otimes(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}.$$

Thus the corrected estimate $\hat{\hat{\mathbf{B}}}$ of the preliminary estimate $\hat{\mathbf{B}}$ is given by the relation

$$\begin{pmatrix}\mathrm{vec}(\hat{\hat{\mathbf{B}}})\\ \mathrm{vec}(\hat{\hat{\mathbf{C}}})\end{pmatrix} = -\begin{pmatrix}\{(\mathbf{I}\otimes\mathbf{I})[\mathbf{I}\otimes(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})](\mathbf{I}\otimes\mathbf{I})\}^{-1}(\mathbf{H}\otimes\mathbf{I})\mathbf{Q}_{1,1}\\ \mathbf{Q}_{2,1}\end{pmatrix}\mathrm{vec}(\mathbf{G})$$
$$+\begin{pmatrix}(\mathbf{I}\otimes\mathbf{I})-\{(\mathbf{I}\otimes\mathbf{I})[\mathbf{I}\otimes(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})](\mathbf{I}\otimes\mathbf{I})\}^{-1}(\mathbf{H}\otimes\mathbf{I})\mathbf{Q}_{1,1}(\mathbf{H}'\otimes\mathbf{I})\\ -\mathbf{Q}_{2,1}(\mathbf{H}'\otimes\mathbf{I})\end{pmatrix}$$
$$\times\{\mathbf{I}\otimes[(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1}]\}\mathrm{vec}(\underline{\mathbf{Y}}),$$

i.e.

$$\hat{\hat{\mathbf{B}}} = -\mathbf{G}\left[\mathbf{I}-(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}'[\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}']^{-1}\mathbf{Z}\right](\mathbf{H}'\mathbf{H})^{-1}\mathbf{H}' + \hat{\mathbf{B}}\left[\mathbf{M}_H+\mathbf{H}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}'[\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}']^{-1}\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{H}'\right],$$
$$\hat{\hat{\mathbf{C}}} = -\mathbf{G}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}'[\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}']^{-1} - \hat{\mathbf{B}}\mathbf{H}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}'[\mathbf{Z}(\mathbf{H}'\mathbf{H})^{-1}\mathbf{Z}']^{-1}.$$

(cf. the relations (2), (3)). □

Remark 1 The analogous lemma for univariate model without constraints cf. [2], p. 398, Theorem 9.2.12.

3 Statistical model of experiment

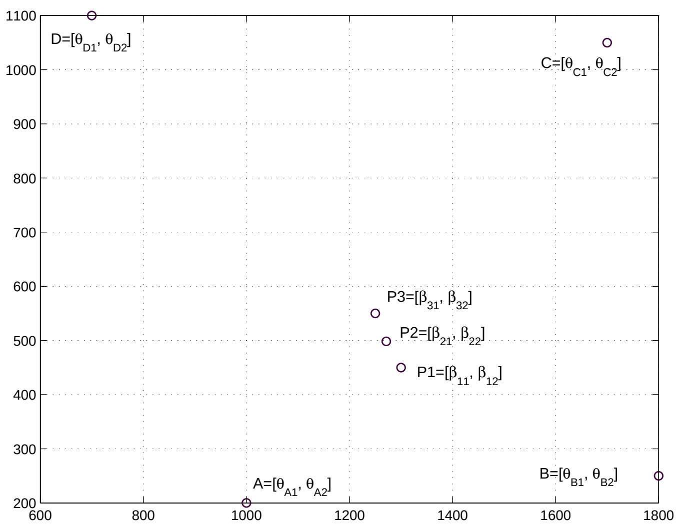

Fig. 1: Position of the points A, B, C, D and the reference points P_1, P_2, P_3.

A deformation measurement is realized according the scheme given by Fig. 1. Here A, B, C, D are points with given coordinates and the reference points are described as P_1, P_2, P_3. The distances are measured in meters with the standard deviation $\sigma_s = 0.01$m and the angles are measured with standard deviation $\sigma_\omega = \frac{3}{206265}$rad. A model of four epochs experiment is considered in the form (1) and (4), where the ith column of $\underline{\mathbf{Y}}$ is the observation vector of the ith epoch, $i = 1, \ldots 4$ minus values calculated from approximate coordinates,

$$\mathbf{Y}_i = \begin{pmatrix} \sqrt{(\beta_{11}^{(i)} - \theta_{A1})^2 + (\beta_{12}^{(i)} - \theta_{A2})^2} \\ \vdots \\ \arctan \frac{\beta_{12}^{(i)} - \theta_{A2}}{\beta_{11}^{(i)} - \theta_{A1}} - \arctan \frac{\theta_{B2} - \theta_{A2}}{\theta_{B1} - \theta_{A1}} \\ \vdots \end{pmatrix}, \quad \mathbf{f}_0 = \mathbf{f}(\boldsymbol{\beta}^0).$$

A choice of the approximate coordinates $\boldsymbol{\beta}^0$ is the same for each epoch. Thus the design matrix

$$\mathbf{X} = \frac{\partial \mathbf{E}(\mathbf{Y}_i)}{\partial (\boldsymbol{\beta}^{(i)})'} \Big|_{\boldsymbol{\beta}^{(i)} = \boldsymbol{\beta}^0}$$

is common for all epochs.

Estimation of parameter $\boldsymbol{\beta}$ in each epoch is a base for calculation of parameter $\boldsymbol{\gamma}$ in the relations $y = \gamma_1 + \gamma_2 x + \gamma_3 x^2$ that describe trajectories of the reference points, e.g. in the case of the reference point P_1

$$\beta_{12}^{(i)} = \gamma_1 + \gamma_2 \beta_{11}^{(i)} + \gamma_3 (\beta_{11}^{(i)})^2, \quad i = 1, 2, 3, 4.$$

Estimation of parameters γ_1, γ_2 and γ_3 is executed by linearized regression model with constraint of type II because estimated coordinates $\hat{\boldsymbol{\beta}}^{(i)}$ are result of the measurement. Therefore the constraint is

$$\mathbf{BH} + \mathbf{CZ} + \mathbf{G} = \mathbf{0},$$

where $\mathbf{C}$ is a matrix of the parameter $\boldsymbol{\gamma}$ and

$$\mathbf{H} = \begin{pmatrix} \gamma_{12}^0 + 2\gamma_{11}^0 \beta_{11}^0 & -1 & 0 & 0 & \ldots \\ 0 & 0 & \gamma_{22}^0 + 2\gamma_{21}^0 \beta_{21}^0 & -1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\mathbf{Z} = \begin{pmatrix} 1 & \beta_{11}^0 & (\beta_{11}^0)^2 \\ 1 & \beta_{21}^0 & (\beta_{21}^0)^2 \\ \vdots & \vdots & \vdots \end{pmatrix},$$

$$\mathbf{G} = \begin{pmatrix} \gamma_{11}^0 + \gamma_{12}^0 \beta_{11}^0 + \gamma_{13}^0 (\beta_{11}^0)^2 - \beta_{12}^0 \\ \gamma_{21}^0 + \gamma_{22}^0 \beta_{21}^0 + \gamma_{23}^0 (\beta_{21}^0)^2 - \beta_{22}^0 \\ \vdots \end{pmatrix}.$$

4 Numerical example

In the experiment the distances of the reference points from the points

$$A = [\theta_{A1}, \theta_{A2}],\ B = [\theta_{B1}, \theta_{B2}],\ C = [\theta_{C1}, \theta_{C2}],\ D = [\theta_{D1}, \theta_{D2}]$$

and the angles between these points are measured. Approximate coordinates are

$$P_1 = [1300.0\,\mathrm{m}, 450.0\,\mathrm{m}],\ P_2 = [1271.4\,\mathrm{m}, 498.2\,\mathrm{m}],\ P_3 = [1250.0\,\mathrm{m}, 550.0\,\mathrm{m}].$$

$$\mathbf{\Sigma} = \sigma_s^2 \begin{pmatrix} \mathbf{I}_{4\times4} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{8\times8} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{4\times4} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}_{8\times8} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{4\times4} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}_{8\times8} \end{pmatrix} + \sigma_\omega^2 \begin{pmatrix} \mathbf{0}_{4\times4} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{8\times8} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_{4\times4} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{8\times8} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}_{4\times4} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{8\times8} \end{pmatrix},$$

$\sigma_s^2 = (0.01\,\mathrm{m})^2$ and $\sigma_\omega^2 = (\frac{3}{206265})^2$, where ω is an angle measured in radians.

The origin of the system of the coordinates is moved to the coordinates $[1200\,\mathrm{m}, 400\,\mathrm{m}]$.

The structural designer gives these trajectories:

$$\begin{array}{rcrcrcrcl} -222172.44 & + & 4444.4444\boldsymbol{\beta}_{11} & - & 22.2222\boldsymbol{\beta}_{11}^2 & - & \boldsymbol{\beta}_{12} & = & 0, \\ 80.35555 & + & 0.25\boldsymbol{\beta}_{21} & & & - & \boldsymbol{\beta}_{22} & = & 0, \\ 55705.61 & - & 2222.2222\boldsymbol{\beta}_{31} & + & 22.2222\boldsymbol{\beta}_{31}^2 & - & \boldsymbol{\beta}_{32} & = & 0. \end{array}$$

The corrected coordinates $\widehat{\widehat{\mathbf{B}}}$ given in meters from the model (4) are given in the following Table 1:

	1st epoch	2nd epoch	3th epoch	4th epoch
P1	[99.991,50.005]	[100.008,50.003]	[100.022,49.989]	[100.035,49.979]
P2	[71.413,98.206]	[71.431,98.214]	[71.444,98.223]	[71.464,98.233]
P3	[50.000,150.004]	[50.008,150.004]	[50.033,150.012]	[50.035,150.023]

Table 1

and

$$\begin{array}{rcrcrcrcl} -383388.15 & + & 7668.548048\boldsymbol{\beta}_{11} & - & 38.341665\boldsymbol{\beta}_{11}^2 & - & \boldsymbol{\beta}_{12} & = & 0, \\ 60.29 & + & 0.530932\boldsymbol{\beta}_{21} & & & - & \boldsymbol{\beta}_{22} & = & 0, \\ 94104.42 & - & 3757.486631\boldsymbol{\beta}_{31} & + & 37.567967\boldsymbol{\beta}_{31}^2 & - & \boldsymbol{\beta}_{32} & = & 0. \end{array}$$

Although in the model (4) the estimations of the parameter $\boldsymbol{\gamma}$ are different from the parameter given by the structural designer, the estimated trajectories are practically the same in the sector in which the movements of the reference point have been measured.

The figures 2–4 show the specific inconsistence between the theory and the numerical results. The corrected coordinates should lie exactly on the each trajectories. The inconsistence evident from the figures seems to be made by the linearization of the nonlinear model. An influence of the nonlinearity will be characterized by the bias

$$\mathbf{E}(\widehat{\widehat{\delta\boldsymbol{\beta}}}) - \delta\boldsymbol{\beta},$$

where $\mathbf{E}(\widehat{\widehat{\delta\boldsymbol{\beta}}})$ is calculated under the assumption that the nonlinear model is quadratized. The expression

$$\mathbf{E}(\widehat{\widehat{\delta\boldsymbol{\beta}}}) - \delta\boldsymbol{\beta}$$

can be obtained in our case from the formula in [1], p. 248, Corollary VI. 2.2.3.5. For the numerical demonstration we use the relations

$$\mathbf{E}(\widehat{\widehat{\delta\boldsymbol{\beta}}}) - \delta\boldsymbol{\beta} = \mathbf{P}^{\mathbf{C}}_{\mathbf{C}^{-1}\mathbf{H}'\mathbf{M_Z}}[\mathbf{C}^{-1}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}' + \mathbf{Z}\mathbf{Z}')^{-1}]\delta\boldsymbol{\mu}\delta\gamma_2$$

for the point P_2 and

$$\begin{aligned}\mathbf{E}(\widehat{\widehat{\delta\boldsymbol{\beta}}}) - \delta\boldsymbol{\beta} &= \mathbf{P}^{\mathbf{C}}_{\mathbf{C}^{-1}\mathbf{H}'\mathbf{M_Z}}[\mathbf{C}^{-1}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}' + \mathbf{Z}\mathbf{Z}')^{-1}]\\ &\times \left(\delta\gamma_2\delta\beta_{11}^{(i)} + \gamma_3^0(\delta\beta_{11}^{(i)})^2 + 2\delta\gamma_3\delta\beta_{11}^{(i)}\beta_{11}^{0(i)}\right)_{i=1}^{4}\end{aligned}$$

for the points P_1, P_3, where $\mathbf{C} = \mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X}$, $\delta\gamma_i = \sqrt{\operatorname{var}\gamma_i}$, $\delta\boldsymbol{\mu} = 4 \times 1$ matrix containing arbitrary combination of numbers $0, 1, -1$.

Numerical results verify the influence of the nonlinearity.

$$\mathbf{E}(\widehat{\widehat{\delta\boldsymbol{\beta}}}) - \delta\boldsymbol{\beta} = \begin{array}{r}-0.002\\ 0.004\\ -0.002\\ 0.008\\ 0.002\\ -0.006\\ 0.004\\ -0.020\end{array}$$

for the point P_2, where $\delta\boldsymbol{\mu} = [0; 1; 0; 1]$ and $\delta\gamma_2 = 0.053$.

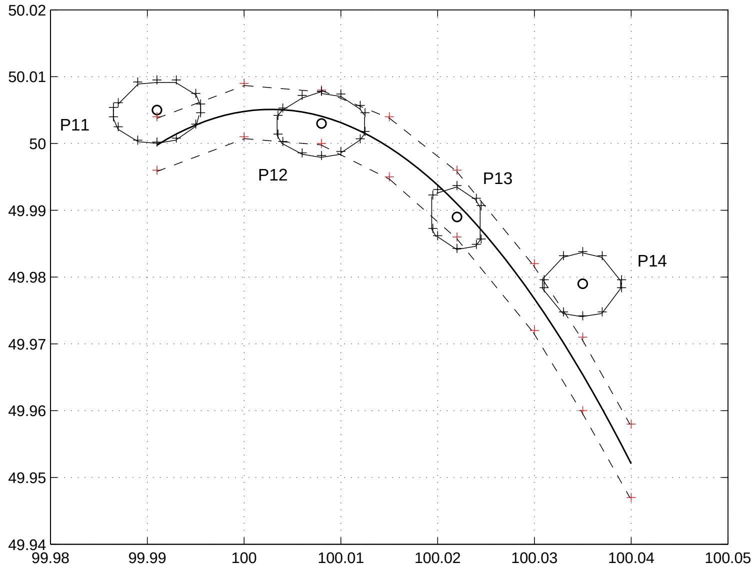

Fig. 2: P_1: Estimation of the trajectory + confidence region in the model (4).

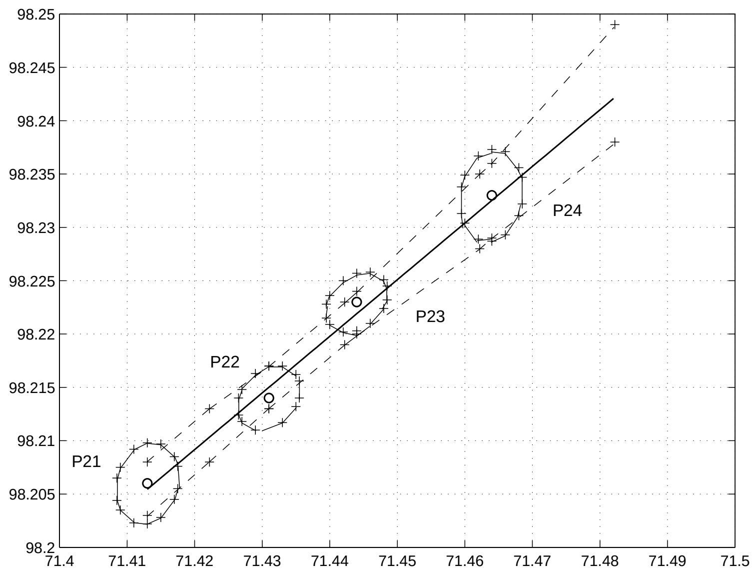

Fig. 3: P_2: Estimation of the trajectory + confidence region in the model (4).

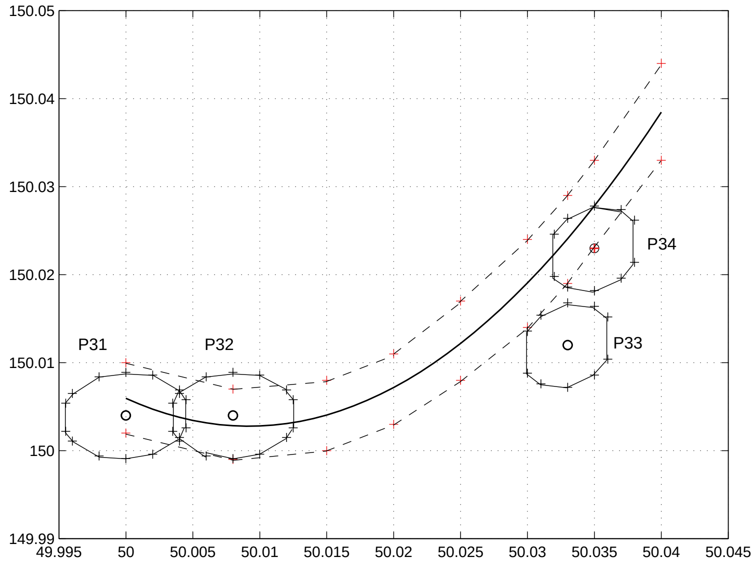

Fig. 4: P_3: Estimation of the trajectory + confidence region in the model (4).

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