Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

Svetislav M. Minčić; Ljubica S. Velimirović

Infinitesimal bending of a subspace of a space with non-symmetric basic tensor

Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 44 (2005), No. 1, 115--130

Persistent URL: http://dml.cz/dmlcz/133373

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **44** (2005) 115–130

Infinitesimal Bending of a Subspace of a Space with Non-Symmetric Basic Tensor

Svetislav M. MINČIĆ, Ljubica S. VELIMIROVIĆ,

Faculty of Science and Mathematics, University of Niš,
Višegradska 33, 18000 Niš, Serbia and Montenegro
e-mail: vljubica@pmf.ni.ac.yu

(Received February 15, 2005)

Abstract

In this work infinitesimal bending of a subspace of a generalized Riemannian space (with non-symmetric basic tensor) are studied. Based on non-symmetry of the connection, it is possible to define four kinds of covariant derivative of a tensor. We have obtained derivation formulas of the infinitesimal bending field and integrability conditions of these formulas (equations).

Key words: Generalized Riemannian space, infinitesimal bending, infinitesimal deformation, subspace.

2000 Mathematics Subject Classification: 53C25, 53A45, 53B05

0 Introduction

0.1. A generalized Riemannian space GR_N is a differentiable manifold, endowed with non-symmetric basic tensor $G_{ij}(x^1,\dots,x^N)$ [2], whose symmetric part is $G_{\underline{ij}}$, and antisymmetric part $G_{\underset{\vee}{ij}}$.

By equations

$$x^i = x^i(u^1,\dots,u^M) \equiv x^i(u^\alpha), \quad \operatorname{rank}(B^i_\alpha) = M, \quad (B^i_\alpha = \partial x^i/\partial u^\alpha), \tag{0.1}$$

in local coordinates is defined a *subspace* $GR_M \subset GR_N$, with metric tensor

$$g_{\alpha\beta} = B^i_\alpha B^j_\beta G_{ij}, \tag{0.2}$$

which is generally also non-symmetric. Remark that in the present work Latin indices $i, j, k, \dots$ take values $1, \dots, N$, while Greek indices $\alpha, \beta, \gamma, \dots$ take values $1, \dots, M$, $(M < N)$ and refer to the subspace.

For the lowering and raising of indices in GR_N one uses the tensor $G_{\underline{ij}}$ respectively $G^{\underline{ij}}$, where $(G^{\underline{ij}}) = (G_{\underline{ij}})^{-1}$.

Christoffel symbols at GR_N are

$$\Gamma_{i.jk} = \frac{1}{2}(G_{ji,k} - G_{jk,i} + G_{ik,j}), \quad \Gamma^i_{jk} = G^{\underline{ip}}\Gamma_{p.jk}, \tag{0.3a, b}$$

where, by the comma a partial derivative is denoted.

The scalar product and the orthogonality one expresses in usual way in the GR_N by $G_{\underline{ij}}$, and in the GR_M by $g_{\underline{\alpha\beta}}$.

On subspaces of generalized Riemannian spaces there exist many works, eg. [7]–[16], [19]–[23]. The present work is continuation and widening of our work [21].

0.2. If in the points of GR_M a vector field $z^i(u^\alpha)$ is defined, the equations

$$\bar{x}^i = x^i(u^\alpha) + \varepsilon z^i(u^\alpha), \tag{0.4}$$

where ε is an infinitesimal, define an *infinitesimal deformation* of the subspace GR_M. Obtained subspace will be denoted $\overline{GR}_M$. The vector field $z^i(u^\alpha)$ is an *infinitesimal deformation field.* In this study of infinitesimal deformations, according to (0.4), magnitudes of a degree higher than the first with respect to ε are omitted.

Among numerous, we refer on papers on infinitesimal deformations of spaces and subspaces, and related topics [4]–[9], [17], [18], [21]–[23].

0.3. A particular case of infinitesimal deformations is *infinitesimal bending* (see e.g. [7], [8], [9], [21]). By virtue of (0.4), for $\overline{g}_{\alpha\beta}$ one obtains [21]:

$$\overline{g}_{\alpha\beta} = g_{\alpha\beta} + \varepsilon(B^i_\alpha B^j_\beta G_{ij,k} z^k + B^i_\alpha z^j_{,\beta} G_{ij} + z^i_{,\alpha} B^j_\beta G_{ij}) \tag{0.5}$$

and, by definition, the subspace $\overline{GR}_M \subset GR_N$ is *infinitesimal bending of the subspace* $GR_M \subset GR_N$ iff (the equation (1.5) in [21]):

$$G_{ij,k} z^k B^i_\alpha B^j_\beta + G_{ij}(B^i_\alpha z^j_{,\beta} + z^i_{,\alpha} B^j_\beta) = 0, \tag{0.6}$$

1 Derivational formulas of the bending field

1.0. Let be $GR_M \subset GR_N$, where GR_M is defined by virtue of (0.1). Consider at points of GR_M $N-M$ mutually orthogonal unit vectors N^i_A, $(A = M+1, \dots, N)$, which are also orthogonal to GR_M, i.e. to the vectors $B^i_\alpha = \partial x^i/\partial u^\alpha$. So, here we are using also the third kind of indices:

$$A, B, C \dots \in \{M+1, \dots, N\}.$$

From the exposed, we have the relations

$$G_{\underline{ip}}G^{\underline{pj}} = \delta_i^j, \quad g_{\underline{\alpha\pi}}g^{\underline{\pi\beta}} = \delta_\alpha^\beta, \tag{1.1a,b}$$

$$G_{\underline{ij}}N_A^iB_\alpha^j = 0, \quad G_{\underline{ij}}N_A^iN_B^j = e_A\delta_{AB}, \quad (e_A = \pm 1), \tag{1.2a,b}$$

where $g^{\underline{\alpha\beta}}$ is obtained analogously to $G^{\underline{ij}}$. Similarly to (0.3), we can define Cristoffel symbols $\widetilde{\Gamma}_{\beta\gamma}^\alpha$ by means of $g_{\alpha\beta}$. These symbols are in general also non-symmetric. Based on that, for a tensor defined in the points of the subspace we have 4 kinds of covariant derivative. For example [13]:

$$B^i_{\underset{\substack{1\\2\\3\\4}}{\alpha|\mu}} = B^i_{\alpha,\mu} + \Gamma^i_{\substack{pm\\mp\\pm\\mp}} B^p_\alpha B^m_\mu - \widetilde{\Gamma}^\pi_{\substack{\alpha\mu\\\mu\alpha\\\mu\alpha\\\alpha\mu}} B^i_\pi \tag{1.3a–d}$$

$$N^i_{\underset{\substack{1\\2}}{A|\mu}} = N^i_{\underset{\substack{3\\4}}{A|\mu}} = N^i_{A,\mu} + \Gamma^i_{\substack{pm\\mp}} N^p_A B^m_\mu. \tag{1.4a,b}$$

From here one obtains 4 kinds of *derivational formulae* of the subspace $GR_M \subset GR_N$ [13,14]:

$$B^i_{\underset{\theta}{\alpha|\mu}} = \underset{\theta}{\Phi}{}^\pi_{\alpha\mu} B^i_\pi + \sum_{A=M+1}^N \Omega_{A\alpha\mu} N^i_A, \tag{1.5a}$$

$$N^i_{\underset{\theta}{B|\mu}} = -e_B g^{\underline{\pi\sigma}} \underset{\theta}{\Omega}_{B\sigma\mu} B^i_\pi + \sum_{A=M+1}^N \underset{\theta}{\Psi}_{AB\mu} N^i_A, \underset{\theta}{\Psi}_{BB\mu} = 0, \tag{1.5b}$$

where $\theta \in \{1,2,3,4\}$ designates the kind of covariant derivative. With respect to (4a,b) is:

$$\underset{\substack{1\\2}}{\Omega}_{A\alpha\beta} = \underset{\substack{3\\4}}{\Omega}_{A\alpha\beta} \tag{1.6a,b}$$

$$\underset{\substack{1\\2}}{\Psi}_{AB\mu} = \underset{\substack{3\\4}}{\Psi}_{AB\mu} \tag{1.7a,b}$$

and by virtue of (48′) in [13]:

$$\underset{2}{\Phi}{}^\alpha_{\beta\gamma} = -\underset{1}{\Phi}{}^\alpha_{\beta\gamma}, \quad \underset{3}{\Phi}{}^\alpha_{\beta\gamma} = \underset{1}{\Phi}{}^\alpha_{\beta\gamma} + 2\tilde{\Gamma}^\alpha_{\underset{\vee}{\beta\gamma}},$$
$$\underset{4}{\Phi}{}^\alpha_{\beta\gamma} = -\underset{1}{\Phi}{}^\alpha_{\beta\gamma} - 2\tilde{\Gamma}^\alpha_{\underset{\vee}{\beta\gamma}} \tag{1.8 a,c}$$

1.1. The infinitesimal bending field z^i can be expressed by tangential and normal component with respect to GR_M:

$$z^i = p^\sigma B^i_\sigma + \sum_A q_A N^i_A. \tag{1.9}$$

Using this value, the condition (0.6) becomes

$$G_{ij,k}B^i_\alpha B^j_\beta(p^\sigma B^k_\sigma + \sum_A q_A N^k_A)$$
$$+ g_{\alpha\sigma}p^\sigma_{,\beta} + G_{ij}B^i_\alpha B^j_{\sigma,\beta}p^\sigma + G_{ij}B^i_\alpha \sum_A (q_{A,\beta}N^j_A + q_A N^j_{A,\beta})$$
$$+ g_{\sigma\beta}p^\sigma_{,\alpha} + G_{ij}B^j_\beta B^i_{\sigma,\alpha}p^\sigma + G_{ij}B^j_\beta \sum_A (q_{A,\alpha}N^i_A + q_A N^i_{A,\alpha}) = 0. \quad (1.10)$$

Taking covariant derivative of the kind θ with respect to u^μ and using (5), we get

$$z^i_{\underset{\theta}{|}\mu} = p^\sigma_{\underset{\theta}{|}\mu}B^i_\sigma + p^\sigma B^i_{\sigma\underset{\theta}{|}\mu} + \sum_A (q_{A\underset{\theta}{|}\mu}N^i_A + q_A N^i_{A\underset{\theta}{|}\mu})$$
$$= p^\sigma_{\underset{\theta}{|}\mu}B^i_\sigma + p^\sigma(\underset{\theta}{\Phi}{}^\pi_{\sigma\mu}B^i_\pi + \sum_A \underset{\theta}{\Omega}{}_{A\sigma\mu}N^i_A) + \sum_A q_{A\underset{\theta}{|}\mu}N^i_A$$
$$+ \sum_A q_A(-e_A \underline{g^{\pi\sigma}}\underset{\theta}{\Omega}{}_{A\sigma\mu}B^i_\pi + \sum_B \underset{\theta}{\Psi}{}_{BA\mu}N^i_B),$$

that is

$$z^i_{\underset{\theta}{|}\mu} = \underset{\theta}{P}{}^\pi_\mu B^i_\pi + \sum_A \underset{\theta}{Q}{}_{A\mu}N^i_A, \quad (1.11)$$

where

$$\underset{\theta}{P}{}^\pi_\mu = p^\pi_{\underset{\theta}{|}\mu} + p^\sigma \underset{\theta}{\Phi}{}^\pi_{\sigma\mu} - \sum_A e_A q_A \underset{\theta}{\Omega}{}_{A\sigma\mu}g^{\pi\sigma}, \quad (1.12)$$

$$\underset{\theta}{Q}{}_{A\mu} = p^\sigma \underset{\theta}{\Omega}{}_{A\sigma\mu} + q_{A\underset{\theta}{|}\mu} + \sum_B q_B \underset{\theta}{\Psi}{}_{AB\mu}. \quad (1.13)$$

The equation (11) is *derivational formula of the infinitesimal bending field* z^i. So, we have

Theorem 1.1 *If the infinitesimal bending field z^i of the subspace $GR_M \subset GR_N$ is expressed by the tangential and the normal component with respect to the GR_M in the form (9), then the derivation formula (11) is valid, where $\underset{\theta}{|}\mu$ is covariant derivative of the kind θ according to u^μ, and $\underset{\theta}{P}$, $\underset{\theta}{Q}$ are given in (12) and (13) respectively.*

2 Integrability conditions of derivational formula of the infinitesimal bending field

2.0. Applying to (1.11) covariant derivative of the kind ω with respect to u^ν, we get

$$z^i_{\underset{\theta}{|}\mu\underset{\omega}{|}\nu} = \underset{\theta}{P}{}^\pi_{\mu\underset{\omega}{|}\nu}B^i_\pi + \underset{\theta}{P}{}^\pi_\mu B^i_{\pi\underset{\omega}{|}\nu} + \sum_A (\underset{\theta}{Q}{}_{A\mu\underset{\omega}{|}\nu}N^i_A + \underset{\theta}{Q}{}_{A\mu}N^i_{A\underset{\omega}{|}\nu}),$$

and substituting $B^i_{\pi\underset{\omega}{|}\nu}$ and $N^i_{A\underset{\omega}{|}\nu}$ with respect to (1.5), after arranging one obtains

$$z^i_{\underset{\theta}{|}\mu\underset{\omega}{|}\nu} = [\underset{\theta}{P}{}^\pi_{\mu\underset{\omega}{|}\nu} + \underset{\theta}{P}{}^\sigma_\mu \underset{\omega}{\Phi}{}^\pi_{\sigma\nu} - \sum_A e_A \underset{\theta}{Q}_{A\mu} g^{\underline{\pi\sigma}} \underset{\omega}{\Omega}_{A\sigma\nu}] B^i_\pi$$
$$+ \sum_A [\underset{\theta}{P}{}^\pi_\mu \underset{\omega}{\Omega}_{A\pi\nu} + \underset{\theta}{Q}_{A\mu\underset{\omega}{|}\nu} + \sum_B \underset{\theta}{Q}_{B\mu} \underset{\omega}{\Psi}_{AB\nu}] N^i_A, \tag{2.1}$$

where the tensors $\underset{\theta}{P}$, $\underset{\theta}{Q}$ are given at (1.12,13). From (1) one gets

$$z^i_{\underset{\theta}{|}\mu\underset{\omega}{|}\nu} - z^i_{\underset{\omega}{|}\nu\underset{\theta}{|}\mu} = [\underset{\theta}{P}{}^\pi_{\mu\underset{\omega}{|}\nu} - \underset{\omega}{P}{}^\pi_{\nu\underset{\theta}{|}\mu} + \underset{\theta}{P}{}^\sigma_\mu \underset{\omega}{\Phi}{}^\pi_{\sigma\nu} - \underset{\omega}{P}{}^\sigma_\nu \underset{\theta}{\Phi}{}^\pi_{\sigma\mu}$$
$$- \sum_A e_A g^{\underline{\pi\sigma}} (\underset{\theta}{Q}_{A\mu} \underset{\omega}{\Omega}_{A\sigma\nu} - \underset{\omega}{Q}_{A\nu} \underset{\theta}{\Omega}_{A\sigma\mu})] B^i_\pi$$
$$+ \sum_A [\underset{\theta}{P}{}^\pi_\mu \underset{\omega}{\Omega}_{A\pi\nu} - \underset{\omega}{P}{}^\pi_\nu \underset{\theta}{\Omega}_{A\pi\mu} + \underset{\theta}{Q}_{A\mu\underset{\omega}{|}\nu} - \underset{\omega}{Q}_{A\mu\underset{\theta}{|}\mu}$$
$$+ \sum_B (\underset{\theta}{Q}_{B\mu} \underset{\omega}{\Psi}_{AB\nu} - \underset{\omega}{Q}_{B\nu} \underset{\theta}{\Psi}_{AB\mu})] N^i_A. \tag{2.2}$$

On the other hand applying the Ricci type identities [11,12], we obtain

$$z^i_{\underset{\frac{1}{2}}{|}\mu\nu} - z^i_{\underset{\frac{1}{2}}{|}\nu\mu} = \underset{\frac{1}{2}}{R}{}^i_{pmn} z^p B^m_\mu B^n_\nu + 2\tilde{\Gamma}^\pi_{\underset{\vee}{\mu\nu}} z^i_{\underset{\frac{1}{2}}{|}\pi}, \tag{2.3a, b}$$

$$z^i_{\underset{1}{|}\mu\underset{2}{|}\nu} - z^i_{\underset{2}{|}\nu\underset{1}{|}\mu} = \underset{3}{R}{}^i_{p\mu\nu} z^p, \tag{2.4}$$

$$z^i_{\underset{\frac{3}{4}}{|}\mu\nu} - z^i_{\underset{\frac{3}{4}}{|}\nu\mu} = \underset{\frac{1}{2}}{R}{}^i_{pmn} z^p B^m_\mu B^n_\nu \pm 2\tilde{\Gamma}^\pi_{\underset{\vee}{\mu\nu}} z^i_{\underset{\frac{1}{2}}{|}\pi}, \tag{2.5a, b}$$

$$z^i_{\underset{3}{|}\mu\underset{4}{|}\nu} - z^i_{\underset{4}{|}\nu\underset{3}{|}\mu} = \underset{4}{R}{}^i_{p\mu\nu} z^p, \tag{2.6}$$

where [11,12]:

$$\underset{1}{R}{}^i_{jmn} = \Gamma^i_{jm,n} - \Gamma^i_{jn,m} + \Gamma^p_{jm}\Gamma^i_{pn} - \Gamma^p_{jn}\Gamma^i_{pm}, \tag{2.7}$$

$$\underset{2}{R}{}^i_{jmn} = \Gamma^i_{mj,n} - \Gamma^i_{nj,m} + \Gamma^p_{mj}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{mp}, \tag{2.8}$$

$$\underset{3}{R}{}^i_{j\mu\nu} = (\Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^p_{jm}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{pm}) B^m_\mu B^n_\nu$$
$$+ 2\Gamma^i_{\underset{\vee}{jm}} (B^m_{\mu,\nu} - \tilde{\Gamma}^\pi_{\nu\mu} B^m_\pi), \tag{2.9}$$

$$\underset{4}{R}{}^i_{j\mu\nu} = (\Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^p_{jm}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{pm}) B^m_\mu B^n_\nu$$
$$+ 2\Gamma^i_{\underset{\vee}{jm}} (B^m_{\mu,\nu} - \tilde{\Gamma}^\pi_{\mu\nu} B^m_\pi). \tag{2.10}$$

The magnitudes $\underset{1}{R}{}^i_{jmn}$, $\underset{2}{R}{}^i_{jmn}$ are curvature tensors of the first and the second kind respectively of the space GR_N, while the magnitudes $\underset{3}{R}{}^i_{j\mu\nu}$, $\underset{4}{R}{}^i_{j\mu\nu}$ are also tensors and we called them in [11,12] curvature tensors of the space GR_N with respect to the subspace GR_M.

2.1. The cases (3.a,b) can be written in the form

$$z^i_{\underset{\theta}{|}\mu\nu} - z^i_{\underset{\theta}{|}\nu\mu} = \underset{\theta}{R}{}^i_{pmn} z^p B^m_\mu B^n_\nu + 2(-1)^\theta \tilde{\Gamma}^\pi_{\underset{\vee}{\mu\nu}} z^i_{\underset{\theta}{|}\pi}, \theta \in \{1,2\}. \tag{2.11}$$

Taking in (2) $\omega = \theta \in \{1,2\}$, we obtain an equation with the same left side as in (11). Substituting $z^i_{\underset{\theta}{|}\pi}$ in (11) by virtue of (1.11) and equaling the right sides of cited equations, we obtain *the first and the second integrability condition of derivational formula* (1.11) of the infinitesimal bending field z^i of the subspace (for $\theta = 1$, $\theta = 2$):

$$\begin{aligned}
&\underset{\theta}{R}{}^i_{pmn} z^p B^m_\mu B^n_\nu + 2(-1)^\theta \tilde{\Gamma}^\pi_{\underset{\vee}{\mu\nu}} (\underset{\theta}{P}{}^\sigma_\pi B^i_\sigma + \sum_A \underset{\theta}{Q}{}_{A\pi} N^i_A) \\
&= [\underset{\theta}{P}{}^\pi_{\mu\underset{\theta}{|}\nu} - \underset{\theta}{P}{}^\pi_{\nu\underset{\theta}{|}\mu} + \underset{\theta}{P}{}^\sigma_\mu \underset{\theta}{\Phi}{}^\pi_{\sigma\nu} - \underset{\theta}{P}{}^\sigma_\nu \underset{\theta}{\Phi}{}^\pi_{\sigma\mu} \\
&- \sum_A e_A g^{\underline{\pi\sigma}} (\underset{\theta}{Q}{}_{A\mu} \underset{\theta}{\Omega}{}_{A\sigma\nu} - \underset{\theta}{Q}{}_{A\nu} \underset{\theta}{\Omega}{}_{A\sigma\mu})] B^i_\pi \\
&+ \sum_A [\underset{\theta}{P}{}^\pi_\mu \underset{\theta}{\Omega}{}_{A\pi\nu} - \underset{\theta}{P}{}^\pi_\nu \underset{\theta}{\Omega}{}_{A\pi\mu} + \underset{\theta}{Q}{}_{A\mu\underset{\theta}{|}\nu} - \underset{\theta}{Q}{}_{A\nu\underset{\theta}{|}\mu} \\
&+ \sum_B (\underset{\theta}{Q}{}_{B\mu} \underset{\theta}{\Psi}{}_{AB\nu} - \underset{\theta}{Q}{}_{B\nu} \underset{\theta}{\Psi}{}_{AB\mu})] N^i_A, \quad \theta = 1, 2.
\end{aligned} \tag{2.12}$$

a) Multiplying this equation with $G_{\underline{il}} B^l_\lambda$ and using (0.2), (1.1,2), we obtain

$$\begin{aligned}
&\underset{\theta}{R}{}_{lpmn} B^l_\lambda z^p B^m_\mu B^n_\nu + 2(-1)^\theta \tilde{\Gamma}^\pi_{\underset{\vee}{\mu\nu}} \underset{\theta}{P}{}^\sigma_\pi g_{\underline{\lambda\sigma}} \\
&= \Big[\underset{\theta}{P}{}^\pi_{\mu\underset{\theta}{|}\nu} - \underset{\theta}{P}{}^\pi_{\nu\underset{\theta}{|}\mu} + \underset{\theta}{P}{}^\sigma_\mu \underset{\theta}{\Phi}{}^\pi_{\sigma\nu} - \underset{\theta}{P}{}^\sigma_\nu \underset{\theta}{\Phi}{}^\pi_{\sigma\mu} - \sum_A e_A g^{\underline{\pi\sigma}} (\underset{\theta}{Q}{}_{A\mu} \underset{\theta}{\Omega}{}_{A\sigma\nu} - \underset{\theta}{Q}{}_{A\nu} \underset{\theta}{\Omega}{}_{A\sigma\mu})\Big] g_{\underline{\lambda\pi}}.
\end{aligned}$$

Taking into consideration (1.1b) and substituting $\underset{\theta}{P}$, $\underset{\theta}{Q}$ according to (1.12,13), the previous equation becomes

$$
\begin{aligned}
&\underset{\theta}{R}_{lpmn}B^{l}_{\lambda}z^{p}B^{m}_{\mu}B^{n}_{\nu}+2(-1)^{\theta}\tilde{\Gamma}^{\pi}_{\underset{\vee}{\mu\nu}}g_{\underline{\lambda\sigma}}(p^{\sigma}_{\underset{\theta}{|}\pi}+p^{\rho}\underset{\theta}{\Phi}{}^{\sigma}_{\rho\pi}-\sum_{A}e_{A}q_{A}\underset{\theta}{\Omega}_{A\rho\pi}g^{\underline{\sigma\rho}})\\
&=[p^{\pi}_{\underset{\theta}{|}\mu\nu}+p^{\sigma}_{\underset{\theta}{|}\nu}\underset{\theta}{\Phi}{}^{\pi}_{\sigma\mu}+p^{\sigma}\underset{\theta}{\Phi}{}^{\pi}_{\sigma\mu\underset{\theta}{|}\nu}-\sum_{A}e_{A}(q_{A\underset{\theta}{|}\nu}\underset{\theta}{\Omega}_{A\sigma\mu}+q_{A}\underset{\theta}{\Omega}_{A\sigma\mu\underset{\theta}{|}\nu})g^{\underline{\pi\sigma}}\\
&\quad-p^{\pi}_{\underset{\theta}{|}\nu\mu}-p^{\sigma}_{\underset{\theta}{|}\mu}\underset{\theta}{\Phi}{}^{\pi}_{\sigma\nu}-p^{\sigma}\underset{\theta}{\Phi}{}^{\pi}_{\sigma\nu\underset{\theta}{|}\mu}+\sum_{A}e_{A}(q_{A\underset{\theta}{|}\mu}\underset{\theta}{\Omega}_{A\sigma\nu}+q_{A}\underset{\theta}{\Omega}_{A\sigma\nu\underset{\theta}{|}\mu})g^{\underline{\pi\sigma}}\\
&\quad+(p^{\sigma}_{\underset{\theta}{|}\mu}+p^{\rho}\underset{\theta}{\Phi}{}^{\sigma}_{\rho\mu}-\sum_{A}e_{A}q_{A}\underset{\theta}{\Omega}_{A\rho\mu}g^{\underline{\sigma\rho}})\underset{\theta}{\Phi}{}^{\pi}_{\sigma\nu}\\
&\quad-(p^{\sigma}_{\underset{\theta}{|}\nu}+p^{\rho}\underset{\theta}{\Phi}{}^{\sigma}_{\rho\nu}-\sum_{A}e_{A}q_{A}\underset{\theta}{\Omega}_{A\rho\nu}g^{\underline{\sigma\rho}})\underset{\theta}{\Phi}{}^{\pi}_{\sigma\mu}]g_{\underline{\lambda\pi}}\\
&\quad-\sum_{A}e_{A}[(p^{\sigma}\underset{\theta}{\Omega}_{A\sigma\mu}+q_{A\underset{\theta}{|}\mu}+\sum_{B}q_{B}\underset{\theta}{\Psi}_{AB\mu})\underset{\theta}{\Omega}_{A\lambda\nu}\\
&\quad-(p^{\sigma}\underset{\theta}{\Omega}_{A\sigma\nu}+q_{A\underset{\theta}{|}\nu}+\sum_{B}q_{B}\underset{\theta}{\Psi}_{AB\nu})\underset{\theta}{\Omega}_{A\lambda\mu}].
\end{aligned}\tag{2.13}
$$

Substituting the dummy indices l,p with i,j respectively and z^{j} according to (1.9), using the Ricci type identity

$$
p^{\pi}_{\underset{\theta}{|}\mu\nu}-p^{\pi}_{\underset{\theta}{|}\nu\mu}=\underset{\theta}{\tilde{R}}{}^{\pi}_{\rho\mu\nu}p^{\rho}+2(-1)^{\theta}\tilde{\Gamma}^{\rho}_{\underset{\vee}{\mu\nu}}p^{\pi}_{\underset{\theta}{|}\rho},\quad \theta=1,2 \tag{2.14}
$$

where $\underset{\theta}{\tilde{R}}{}^{\pi}_{\rho\mu\nu}$ are the corresponding curvature tensors of the subspace (formed by means of $\tilde{\Gamma}$) and denoting

$$
p_{\lambda}=g_{\underline{\lambda\sigma}}p^{\sigma},\quad \underset{\theta}{\Phi}_{\lambda\rho\pi}=g_{\underline{\lambda\sigma}}\underset{\theta}{\Phi}{}^{\sigma}_{\rho\pi},
$$
$$
\underset{\theta}{\Omega}_{A}{}^{\sigma}{}_{\mu}=g^{\underline{\rho\sigma}}\underset{\theta}{\Omega}_{A\rho\mu},
$$

the equation (13) becomes

$$
\begin{aligned}
&\underset{\theta}{R}_{ijmn}B^{i}_{\lambda}(p^{\sigma}B^{j}_{\sigma}+\sum_{A}q_{A}N^{j}_{A})B^{m}_{\mu}B^{n}_{\nu}\\
&+2(-1)^{\theta}\tilde{\Gamma}^{\sigma}_{\underset{\vee}{\mu\nu}}(p^{\rho}\underset{\theta}{\Phi}_{\lambda\rho\sigma}-\sum_{A}e_{A}q_{A}\underset{\theta}{\Omega}_{A\lambda\sigma})\\
&=p^{\sigma}(\underset{\theta}{\tilde{R}}_{\lambda\sigma\mu\nu}+\underset{\theta}{\Phi}_{\lambda\sigma\mu\underset{\theta}{|}\nu}-\underset{\theta}{\Phi}_{\lambda\sigma\nu\underset{\theta}{|}\mu}+\underset{\theta}{\Phi}{}^{\rho}_{\sigma\mu}\underset{\theta}{\Phi}_{\lambda\rho\nu}-\underset{\theta}{\Phi}{}^{\rho}_{\sigma\nu}\underset{\theta}{\Phi}_{\lambda\rho\mu})\\
&+\sum_{A}e_{A}[q_{A}(\underset{\theta}{\Phi}_{\lambda\sigma\mu}\underset{\theta}{\Omega}_{A}{}^{\sigma}{}_{\nu}-\underset{\theta}{\Phi}_{\lambda\sigma\nu}\underset{\theta}{\Omega}_{A}{}^{\sigma}{}_{\mu}-\underset{\theta}{\Omega}_{A\lambda\mu\underset{\theta}{|}\nu}+\underset{\theta}{\Omega}_{A\lambda\nu\underset{\theta}{|}\mu})\\
&+p^{\sigma}(\underset{\theta}{\Omega}_{A\lambda\mu}\underset{\theta}{\Omega}_{A\sigma\nu}-\underset{\theta}{\Omega}_{A\lambda\nu}\underset{\theta}{\Omega}_{A\sigma\mu})\\
&+\sum_{B}q_{B}(\underset{\theta}{\Omega}_{A\lambda\mu}\underset{\theta}{\Psi}_{AB\nu}-\underset{\theta}{\Omega}_{A\lambda\nu}\underset{\theta}{\Psi}_{AB\mu})],\quad \theta=1,2.
\end{aligned}\tag{2.15}
$$

b) By multiplying (12) with $G_{\underline{il}}N^l_c$ and taking into consideration (1.1, 2), one obtains

$$
\begin{aligned}
&\underset{\theta}{R}_{lpmn}N^l_C z^p B^m_\mu B^n_\nu + 2(-1)^\theta \tilde{\Gamma}^\pi_{\underset{\vee}{\mu\nu}} \underset{\theta}{Q}_{C\pi} e_C \\
&= e_C[\underset{\theta}{P}^\pi_{\ \mu} \underset{\theta}{\Omega}_{C\pi\nu} - \underset{\theta}{P}^\pi_{\ \nu} \underset{\theta}{\Omega}_{C\pi\mu} + \underset{\theta}{Q}_{C\mu\underset{\theta}{|}\nu} - \underset{\theta}{Q}_{C\nu\underset{\theta}{|}\mu} \\
&\quad + \sum_B (\underset{\theta}{Q}_{B\mu}\underset{\theta}{\Psi}_{CB\nu} - \underset{\theta}{Q}_{B\nu}\underset{\theta}{\Psi}_{CB\mu})].
\end{aligned}
$$

Substituting $\underset{\theta}{P}, \underset{\theta}{Q}$ as in the previous case, from here we have

$$
\begin{aligned}
&\underset{\theta}{R}_{ijmn}N^i_C z^j B^m_\mu B^N_\nu \\
&+ 2(-1)^\theta \tilde{\Gamma}^\pi_{\underset{\vee}{\mu\nu}} e_C (p^\sigma \underset{\theta}{\Omega}_{C\sigma\pi} + q_{C\underset{\theta}{|}\pi} + \sum_B q_B \underset{\theta}{\Psi}_{CB\pi}) \\
&= e^C\{(p^\pi_{\underset{\theta}{|}\mu} + p^\sigma \underset{\theta}{\Phi}^\pi_{\sigma\mu} - \sum_A e_A q_A \underset{\theta}{\Omega}_{A\sigma\mu} g^{\underline{\pi\sigma}})\underset{\theta}{\Omega}_{C\pi\nu} \\
&\quad - (p^\pi_{\underset{\theta}{|}\nu} + p^\sigma \underset{\theta}{\Phi}^\pi_{\sigma\nu} - \sum_A e_A q_A \underset{\theta}{\Omega}_{A\sigma\nu} g^{\underline{\pi\sigma}})\underset{\theta}{\Omega}_{C\pi\mu} \\
&\quad + p^\sigma_{\underset{\theta}{|}\nu}\underset{\theta}{\Omega}_{C\sigma\mu} + p^\sigma \underset{\theta}{\Omega}_{C\sigma\mu\underset{\theta}{|}\nu} + q_{C\underset{\theta}{|}\mu\underset{\theta}{|}\nu} \\
&\quad + \sum_B (q_{B\underset{\theta}{|}\nu}\underset{\theta}{\Psi}_{CB\mu} + q_B \underset{\theta}{\Psi}_{CB\mu\underset{\theta}{|}\nu}) \\
&\quad - p^\sigma_{\underset{\theta}{|}\mu}\underset{\theta}{\Omega}_{C\sigma\nu} - p^\sigma \underset{\theta}{\Omega}_{C\sigma\nu\underset{\theta}{|}\mu} - q_{C\underset{\theta}{|}\nu\underset{\theta}{|}\mu} \\
&\quad - \sum_B (q_{B\underset{\theta}{|}\mu}\underset{\theta}{\Psi}_{CB\nu} + q_B \underset{\theta}{\Psi}_{CB\nu\underset{\theta}{|}\mu}) \\
&\quad + \sum_B [(p^\sigma \underset{\theta}{\Omega}_{B\sigma\mu} + q_{B\underset{\theta}{|}\mu} + \sum_A q_A \underset{\theta}{\Psi}_{BA\mu})\underset{\theta}{\Psi}_{CB\nu} \\
&\quad - (p^\sigma \underset{\theta}{\Omega}_{B\sigma\nu} + q_{B\underset{\theta}{|}\nu} + \sum_A q_A \underset{\theta}{\Psi}_{BA\nu})\underset{\theta}{\Psi}_{CB\mu}]\}.
\end{aligned}
$$

Multiplying the both sides of this equation with $e_C = \pm 1$, and taking into count that

$$
\begin{aligned}
q_{C\underset{\theta}{|}\mu} &= \partial q_C/\partial u^\mu = q_{C,\mu}, \quad \theta = 1, 2, \\
q_{C\underset{1\atop 2}{|}\mu\underset{1\atop 2}{|}\nu} &= (q_{C\underset{1\atop 2}{|}\mu})_{,\nu} - \tilde{\Gamma}^\pi_{\underset{\nu\mu}{\mu\nu}} q_{C\underset{1\atop 2}{|}\pi} = q_{C,\mu\nu} - \tilde{\Gamma}^\pi_{\underset{\nu\mu}{\mu\nu}} q_{C,\pi}
\end{aligned}
$$

from where $q_{C\underset{\theta}{|}\mu\nu} - q_{C\underset{\theta}{|}\nu\mu} = 2(-1)^\theta \tilde{\Gamma}^\pi_{\underset{\vee}{\nu\mu}} q_{C,\pi}$, the previous equation can be written in the form

$$
\begin{aligned}
&e_C \underset{\theta}{R}_{ijmn} N^i_C (p^\sigma B^j_\sigma + \sum_A q_A N^j_A) B^m_\mu B^n_\nu \\
&+(-1)^\theta \tilde{\Gamma}^\pi_{\underset{\vee}{\mu\nu}} (p^\sigma \underset{\theta}{\Omega}_{C\sigma\pi} + \sum_B q_B \underset{\theta}{\Psi}_{CB\pi}) \\
&= p^\sigma (\underset{\theta}{\Phi}^\pi_{\sigma\mu} \underset{\theta}{\Omega}_{C\pi\nu} - \underset{\theta}{\Phi}^\pi_{\sigma\nu} \underset{\theta}{\Omega}_{C\pi\mu} + \underset{\theta}{\Omega}_{C\sigma\mu \underset{\theta}{|} \nu} - \underset{\theta}{\Omega}_{C\sigma\nu \underset{\theta}{|} \mu}) \\
&+ \sum_A e_A q_A (\underset{\theta}{\Omega}_{C\pi\mu} \underset{\theta}{\Omega}_{A\nu}{}^\pi - \underset{\theta}{\Omega}_{C\pi\nu} \underset{\theta}{\Omega}_{A\mu}{}^\pi) \\
&+ \sum_A [p^\sigma (\underset{\theta}{\Omega}_{A\sigma\mu} \underset{\theta}{\Psi}_{CA\nu} - \underset{\theta}{\Omega}_{A\sigma\nu} \underset{\theta}{\Psi}_{CA\mu}) \\
&+ q_A (\underset{\theta}{\Psi}_{CA\mu \underset{\theta}{|} \nu} - \underset{\theta}{\Psi}_{CA\nu \underset{\theta}{|} \mu}) \\
&+ \sum_B q_B (\underset{\theta}{\Psi}_{AB\mu} \underset{\theta}{\Psi}_{CA\nu} - \underset{\theta}{\Psi}_{AB\nu} \underset{\theta}{\Psi}_{CA\mu})].
\end{aligned} \tag{2.16}
$$

2.2 Substituting $\theta = 1$, $\omega = 2$ into (2) and using (4), we obtain *the third integrability condition* of derivational formula (1.11) of z^i:

$$
\begin{aligned}
\underset{3}{R}^i_{p\mu\nu} z^p &= [\underset{1}{P}^\pi_{\mu \underset{2}{|} \nu} - \underset{2}{P}^\pi_{\nu \underset{1}{|} \mu} + \underset{1}{P}^\sigma_\mu \underset{2}{\Phi}^\pi_{\sigma\nu} - \underset{2}{P}^\sigma_\nu \underset{1}{\Phi}^\pi_{\sigma\mu} \\
&- \sum_A e_A g^{\underline{\pi\sigma}} (\underset{1}{Q}_{A\mu} \underset{2}{\Omega}_{A\sigma\nu} - \underset{2}{Q}_{A\nu} \underset{1}{\Omega}_{A\sigma\mu})] B^i_\pi \\
&+ \sum_A [\underset{1}{P}^\pi_\mu \underset{2}{\Omega}_{A\pi\nu} - \underset{2}{P}^\pi_\nu \underset{1}{\Omega}_{A\pi\mu} + \underset{1}{Q}_{a\mu \underset{2}{|} \nu} - \underset{2}{Q}_{A\nu \underset{1}{|} \mu} \\
&+ \sum_B (\underset{1}{Q}_{B\mu} \underset{2}{\Psi}_{AB\nu} - \underset{2}{Q}_{B\nu} \underset{1}{\Psi}_{AB\mu})] N^i_A.
\end{aligned} \tag{2.17}
$$

a) By multiplying the previous equation with $G_{\underline{il}} B^l_\lambda$ one obtains

$$
\begin{aligned}
\underset{3}{R}_{lp\mu\nu} B^l_\lambda z^p &= [\underset{1}{P}^\pi_{\mu \underset{2}{|} \nu} - \underset{2}{P}^\pi_{\nu \underset{1}{|} \mu} + \underset{1}{P}^\sigma_\mu \underset{2}{\Phi}^\pi_{\sigma\nu} - \underset{2}{P}^\sigma_\nu \underset{1}{\Phi}^\pi_{\sigma\mu} \\
&- \sum_A e_A g^{\underline{\pi\sigma}} (\underset{1}{Q}_{A\mu} \underset{2}{\Omega}_{A\sigma\nu} - \underset{2}{Q}_{A\nu} \underset{1}{\Omega}_{A\sigma\mu})] g_{\underline{\lambda\pi}}.
\end{aligned}
$$

By substitution of $\underset{\theta}{P}, \underset{\theta}{Q}$ with respect to (1.12,13), from here it follows that

$$\begin{aligned}\underset{3}{R}^i_{lp\mu\nu}B^l_\lambda z^p = [&p^\pi_{\underset{1}{|}\mu\underset{2}{|}\nu} + p^\sigma_{\underset{2}{|}\nu}\underset{1}{\Phi}^\pi_{\sigma\mu} + p^\sigma\underset{1}{\Phi}^\pi_{\sigma\mu\underset{2}{|}\nu}\\
&-\sum_A e_A(q_{A\underset{2}{|}\nu}\underset{1}{\Omega}_{A\sigma\mu} + q_A\underset{1}{\Omega}_{A\sigma\mu\underset{2}{|}\nu})g^{\underline{\pi\sigma}}\\
&-p^\pi_{\underset{2}{|}\nu\underset{1}{|}\mu} + p^\sigma_{\underset{1}{|}\mu}\underset{2}{\Phi}^\pi_{\sigma\nu}\\
&+\sum_A e_A(q_{A\underset{1}{|}\mu}\underset{2}{\Omega}_{A\sigma\nu} + q_A\underset{2}{\Omega}_{A\sigma\nu\underset{1}{|}\mu})g^{\underline{\pi\sigma}}\\
&+(p^\sigma_{\underset{1}{|}\mu} + p^\sigma\underset{1}{\Phi}^\sigma_{\rho\mu} - \sum_A e_A q_A\underset{1}{\Omega}_{A\rho\mu}g^{\underline{\sigma\rho}})\underset{2}{\Phi}^\pi_{\sigma\nu}\\
&-(p^\sigma_{\underset{2}{|}\nu} + p^\sigma\underset{2}{\Phi}^\sigma_{\rho\nu} - \sum_A e_A q_A\underset{2}{\Omega}_{A\rho\nu}g^{\underline{\sigma\rho}})\underset{1}{\Phi}^\pi_{\sigma\mu}]g_{\underline{\lambda\pi}}\\
&-\sum_A e_A[(p^\sigma\underset{1}{\Omega}_{A\sigma\mu} + q_{A\underset{1}{|}\mu} + \sum_B q_B\underset{1}{\Psi}_{AB\mu})\underset{2}{\Omega}_{A\lambda\nu}\\
&-(p^\sigma\underset{2}{\Omega}_{A\sigma\nu} + q_{A\underset{2}{|}\nu} + \sum_B q_B\underset{2}{\Psi}_{AB\nu})\underset{1}{\Omega}_{A\lambda\mu}].\end{aligned} \tag{2.18}$$

Substituting the dummy indices l, p with i, j respectively and using the Ricci-type identity [11]:

$$p^\pi_{\underset{1}{|}\mu\underset{2}{|}\nu} - p^\pi_{\underset{2}{|}\nu\underset{1}{|}\mu} = \underset{3}{\tilde{R}}{}^\pi_{\sigma\mu\nu}p^\sigma, \tag{2.19}$$

where

$$\underset{3}{\tilde{R}}{}^\alpha_{\beta\mu\nu} = \tilde{\Gamma}^\alpha_{\beta\mu,\nu} - \tilde{\Gamma}^\alpha_{\nu\beta,\mu} + \tilde{\Gamma}^\sigma_{\beta\mu}\tilde{\Gamma}^\alpha_{\nu\sigma} - \tilde{\Gamma}^\sigma_{\nu\beta}\tilde{\Gamma}^\alpha_{\sigma\mu} + \tilde{\Gamma}^\sigma_{\nu\mu}(\tilde{\Gamma}^\alpha_{\sigma\beta} - \tilde{\Gamma}^\alpha_{\beta\sigma}) \tag{2.20}$$

is the curvature tensor of the 3^{rd} kind of the subspace, the equation (18) becomes

$$\begin{aligned}&\underset{3}{R}_{ij\mu\nu}B^i_\lambda(p^\sigma B^i_\sigma + \sum_A q_A N^i_A)\\
&= p^\sigma(\underset{3}{\tilde{R}}_{\lambda\sigma\mu\nu} + \underset{1}{\Phi}_{\lambda\sigma\mu\underset{2}{|}\nu} - \underset{2}{\Phi}_{\lambda\sigma\nu\underset{1}{|}\mu} + \underset{1}{\Phi}^\rho_{\sigma\mu}\underset{2}{\Phi}_{\lambda\rho\nu} - \underset{2}{\Phi}^\rho_{\sigma\nu}\underset{1}{\Phi}_{\lambda\rho\mu})\\
&+\sum_A e_A[q_A(\underset{1}{\Phi}_{\lambda\sigma\mu}\underset{2}{\Omega}^\sigma_{A\nu} - \underset{2}{\Phi}_{\lambda\sigma\nu}\underset{1}{\Omega}^\sigma_{A\mu} - \underset{1}{\Omega}_{A\lambda\mu\underset{2}{|}\nu} + \underset{2}{\Omega}_{A\lambda\nu\underset{1}{|}\mu})\\
&+p^\sigma(\underset{1}{\Omega}_{A\lambda\mu}\underset{2}{\Omega}_{A\sigma\nu} - \underset{2}{\Omega}_{A\lambda\nu}\underset{1}{\Omega}_{A\sigma\mu})\\
&+\sum_B q_B(\underset{1}{\Omega}_{A\lambda\mu}\underset{2}{\Psi}_{AB\nu} - \underset{2}{\Omega}_{A\lambda\nu}\underset{1}{\Psi}_{AB\mu})].\end{aligned} \tag{2.21}$$

b) Multiplying (17) with $G_{\underline{il}}N^l_c$, one obtains

$$\begin{aligned}\underset{3}{R}_{lp\mu\nu}N^l_c z^p = e_c[&\underset{1}{P}{}^\pi_\mu\underset{2}{\Omega}_{C\pi\nu} - \underset{2}{P}{}^\pi_\nu\underset{1}{\Omega}_{C\pi\mu} + \underset{1}{Q}_{c\mu\underset{2}{|}\nu} - \underset{2}{Q}_{c\nu\underset{1}{|}\mu}\\
&+\sum_B(\underset{1}{Q}_{B\mu}\underset{2}{\Psi}_{CB\nu} - \underset{2}{Q}_{B\nu}\underset{1}{\Psi}_{CB\mu})].\end{aligned}$$

Substituting $\underset{\theta}{P}, \underset{\theta}{Q}$ using that

$$q_{c\underset{1}{|}\mu\underset{2}{|}\nu} - q_{c\underset{2}{|}\nu\underset{1}{|}\mu} = 0,$$

and arranging, we get

$$\begin{aligned} &e_C \underset{3}{R}_{ij\mu\nu} N^i_c (p^\sigma B^j_\sigma + \sum_A q_A N^j_A) \\ &= p^\sigma (\underset{1}{\Phi}^\pi_{\sigma\mu} \underset{2}{\Omega}_{c\pi\nu} - \underset{2}{\Phi}^\pi_{\sigma\nu} \underset{1}{\Omega}_{c\pi\mu} + \underset{1}{\Omega}_{c\sigma\mu\underset{2}{|}\nu} - \underset{2}{\Omega}_{c\sigma\nu\underset{1}{|}\mu}) \\ &\quad + \sum_A e_A q_A (\underset{1}{\Omega}_{c\pi\mu} \underset{2}{\Omega}^\pi_{A\nu} - \underset{2}{\Omega}_{c\pi\nu} \underset{1}{\Omega}^\pi_{A\mu}) \\ &+ \sum_A [p^\sigma (\underset{1}{\Omega}_{A\sigma\mu} \underset{2}{\Psi}_{CA\nu} - \underset{2}{\Omega}_{A\sigma\nu} \underset{1}{\Psi}_{CA\mu}) + q_A (\underset{1}{\Psi}_{CA\mu\underset{2}{|}\nu} - \underset{2}{\Psi}_{CA\nu\underset{1}{|}\mu} \\ &\quad \sum_B q_B (\underset{1}{\Psi}_{AB\mu} \underset{2}{\Psi}_{CA\nu} - \underset{2}{\Psi}_{AB\nu} \underset{1}{\Psi}_{CA\mu})]. \end{aligned} \tag{2.22}$$

2.3. The cases (5a,b) can be given with the equation

$$z^i_{\underset{\theta}{|}\mu\nu} - z^i_{\underset{\theta}{|}\nu\mu} = \underset{\theta-2}{R}{}^{\ i}_{pmn} z^p B^m_\mu B^n_\nu + 2(-1)^{\theta-1} \tilde{\Gamma}^\pi_{\underset{\vee}{\mu\nu}} z^i_{\underset{\theta}{|}\pi}, \quad \theta \in \{3,4\}. \tag{2.23}$$

Substituting $\theta \in \{3,4\}$ in (2), we get the equation with the left side as in (21). According to that we get *the 4th and the 5th integrability condition* of the derivation formula (1.11) (for $\theta \in \{3,4\}$):

$$\begin{aligned} &\underset{\theta-2}{R}{}^{\ i}_{pmn} z^p B^m_\mu B^n_\nu + 2(-1)^{\theta-1} \tilde{\Gamma}^\pi_{\underset{\vee}{\mu\nu}} (\underset{\theta}{P}{}^\sigma_\pi B^i_\sigma + \sum_A \underset{\theta}{Q}_{A\pi} N^i_A) \\ &= [\underset{\theta}{P}{}^\pi_{\mu\underset{\theta}{|}\nu} - \underset{\theta}{P}{}^\pi_{\nu\underset{\theta}{|}\mu} + \underset{\theta}{P}{}^\sigma_\mu \underset{\theta}{\Phi}{}^\pi_{\sigma\nu} - \underset{\theta}{P}{}^\sigma_\nu \underset{\theta}{\Phi}{}^\pi_{\sigma\mu} \\ &\quad - \sum_A e_A g^{\underline{\pi\sigma}} (\underset{\theta}{Q}_{A\mu} \underset{\theta}{\Omega}_{A\sigma\nu} - \underset{\theta}{Q}_{A\nu} \underset{\theta}{\Omega}_{A\sigma\mu})] B^i_\pi \\ &\quad + \sum_A [\underset{\theta}{P}{}^\pi_\mu \underset{\theta}{\Omega}_{A\pi\nu} - \underset{\theta}{P}{}^\pi_\nu \underset{\theta}{\Omega}_{A\pi\mu} + \underset{\theta}{Q}_{A\mu\underset{\theta}{|}\nu} - \underset{\theta}{Q}_{A\nu\underset{\theta}{|}\mu} \\ &\quad + \sum_B (\underset{\theta}{Q}_{B\mu} \underset{\theta}{\Psi}_{AB\nu} - \underset{\theta}{Q}_{B\nu} \underset{\theta}{\Psi}_{AB\mu})] N^i_A, \quad \theta \in \{3,4\}. \end{aligned} \tag{2.24}$$

a) Multiplying this equation with $G_{\underline{il}} B^l_\lambda$, we get

$$\begin{aligned} &\underset{\theta-2}{R}_{lpmn} B^l_\lambda z^p B^m_\mu B^n_\nu + 2(-1)^{\theta-1} \tilde{\Gamma}^\pi_{\underset{\vee}{\mu\nu}} \underset{\theta}{P}{}^\sigma_\pi g_{\underline{\lambda\sigma}} \\ &= [\underset{\theta}{P}{}^\pi_{\mu\underset{\theta}{|}\nu} - \underset{\theta}{P}{}^\pi_{\nu\underset{\theta}{|}\mu} + \underset{\theta}{P}{}^\sigma_\mu \underset{\theta}{\Phi}{}^\pi_{\sigma\nu} - \underset{\theta}{P}{}^\sigma_\nu \underset{\theta}{\Phi}{}^\pi_{\sigma\mu} \\ &\quad - \sum_A e_A g^{\underline{\pi\sigma}} (\underset{\theta}{Q}_{A\mu} \underset{\theta}{\Omega}_{A\sigma\nu} - \underset{\theta}{Q}_{A\nu} \underset{\theta}{\Omega}_{A\sigma\mu})] g_{\underline{\lambda\pi}}. \end{aligned}$$

from where, as in previous cases,

$$\underset{\theta-2}{R}_{lpmn}B^l_\lambda z^p B^m_\mu B^n_\nu + 2(-1)^{\theta-1}\tilde{\Gamma}^\pi_{\underset{\vee}{\mu\nu}} g_{\underline{\lambda\sigma}}(p^\sigma_{\underset{\theta}{|}\pi} + p^\rho \underset{\theta}{\Phi}{}^\sigma_{\rho\pi}$$

$$-\sum_A e_A q_A \underset{\theta}{\Omega}_{A\rho\pi} g^{\underline{\sigma\rho}}) = [p^\pi_{\underset{\theta}{|}\mu\nu} + p^\sigma_{\underset{\theta}{|}\nu}\underset{\theta}{\Phi}{}^\pi_{\sigma\mu} + p^\sigma \underset{\theta}{\Phi}{}^\pi_{\sigma\mu\underset{\theta}{|}\nu}$$

$$-\sum_A e_A(q_{A\underset{\theta}{|}\nu}\underset{\theta}{\Omega}_{A\sigma\mu} + q_A \underset{\theta}{\Omega}_{A\sigma\mu\underset{\theta}{|}\nu}) g^{\underline{\pi\sigma}}$$

$$-p^\pi_{\underset{\theta}{|}\nu\mu} - p^\sigma_{\underset{\theta}{|}\mu}\underset{\theta}{\Phi}{}^\pi_{\sigma\nu} - p^\sigma \underset{\theta}{\Phi}{}^\pi_{\sigma\nu\underset{\theta}{|}\mu}$$

$$+\sum_A e_A(q_{A\underset{\theta}{|}\mu}\underset{\theta}{\Omega}_{A\sigma\nu} + q_A \underset{\theta}{\Omega}_{A\sigma\nu\underset{\theta}{|}\mu}) g^{\underline{\pi\sigma}}$$

$$+(p^\sigma_{\underset{\theta}{|}\mu} + p^\rho \underset{\theta}{\Phi}{}^\sigma_{\rho\mu} - \sum_A e_A q_A \underset{\theta}{\Omega}_{A\rho\mu} g^{\underline{\sigma\rho}})\underset{\theta}{\Phi}{}^\pi_{\sigma\nu}$$

$$-(p^\sigma_{\underset{\theta}{|}\nu} + p^\rho \underset{\theta}{\Phi}{}^\sigma_{\rho\nu} - \sum_A e_A q_A \underset{\theta}{\Omega}_{A\rho\nu} g^{\underline{\sigma\rho}})\underset{\theta}{\Phi}{}^\pi_{\sigma\mu}] g_{\underline{\lambda\pi}}$$

$$-\sum_A e_A[(p^\sigma \underset{\theta}{\Omega}_{A\sigma\mu} + q_{A\underset{\theta}{|}\mu} + \sum_B q_B \underset{\theta}{\Psi}_{AB\mu})\underset{\theta}{\Omega}_{A\lambda\nu}]$$

$$-(p^\sigma \underset{\theta}{\Omega}_{A\sigma\nu} + q_{A\underset{\theta}{|}\nu} + \sum_B q_B \underset{\theta}{\Psi}_{AB\nu})\underset{\theta}{\Omega}_{A\lambda\mu}].$$

According to [12]:

$$p^\pi_{\underset{\theta}{|}\mu\nu} - p^\pi_{\underset{\theta}{|}\nu\mu} = \underset{\theta-2}{\tilde{R}}{}^{\ \pi}_{\sigma\mu\nu} p^\sigma + 2(-1)^{\theta-1}\tilde{\Gamma}^\sigma_{\underset{\vee}{\mu\nu}} p^\pi_{\underset{\theta}{|}\sigma}, \theta \in \{3,4\}, \tag{2.25}$$

the previous equation becomes

$$\underset{\theta-2}{R}_{ijmn}B^i_\lambda(p^\sigma B^j_\sigma + \sum_A q_A N^j_A)B^m_\mu B^n_\nu$$

$$+ 2(-1)^{\theta-1}\tilde{\Gamma}^\pi_{\underset{\vee}{\mu\nu}}{}_{\theta}(p^\sigma \underset{\theta}{\Phi}_{\lambda\sigma\pi} - \sum_A e_A q_A \underset{\theta}{\Omega}_{A\lambda\pi})$$

$$= p^\sigma(\underset{\theta-2}{R}_{\lambda\sigma\mu\nu} + \underset{\theta}{\Phi}_{\lambda\sigma\mu\underset{\theta}{|}\nu} - \underset{\theta}{\Phi}_{\lambda\sigma\nu\underset{\theta}{|}\mu} + \underset{\theta}{\Phi}{}^\rho_{\sigma\mu}\underset{\theta}{\Phi}_{\lambda\rho\nu} - \underset{\theta}{\Phi}{}^\rho_{\sigma\nu}\underset{\theta}{\Phi}_{\lambda\rho\mu})$$

$$+\sum_A e_A[q_A(\underset{\theta}{\Phi}_{\lambda\sigma\mu}\underset{\theta}{\Omega}{}^\sigma_{A\nu} - \underset{\theta}{\Phi}_{\lambda\sigma\nu}\underset{\theta}{\Omega}{}^\sigma_{A\mu}\underset{\theta}{\Omega}_{A\lambda\mu\underset{\theta}{|}\nu} - \underset{\theta}{\Omega}_{A\lambda\nu\underset{\theta}{|}\mu})$$

$$+p^\sigma(\underset{\theta}{\Omega}_{A\lambda\mu}\underset{\theta}{\Omega}_{A\sigma\nu} - \underset{\theta}{\Omega}_{A\lambda\nu}\underset{\theta}{\Omega}_{A\sigma\mu})$$

$$+\sum_B q_B(\underset{\theta}{\Omega}_{A\lambda\mu}\underset{\theta}{\Psi}_{AB\nu} - \underset{\theta}{\Omega}_{A\lambda\nu}\underset{\theta}{\Psi}_{AB\mu})], \quad \theta \in \{3,4\} \tag{2.26}$$

b) Multiplying (23) with $G_{\underline{il}}N^l_c$, we have

$$\underset{\theta-2}{R}_{lpmn}N^l_c z^p B^m_\mu B^n_\nu + 2(-1)^{\theta-1}\tilde{\Gamma}^\pi_{\underset{\vee}{\mu\nu}}\underset{\theta}{Q}_{c\pi}e_c$$

$$= e_c[\underset{\theta}{P}{}^\pi_\mu \underset{\theta}{\Omega}_{C\pi\nu} - \underset{\theta}{P}{}^\pi_\nu \underset{\theta}{\Omega}_{C\pi\mu} + \underset{\theta}{Q}_{c\mu\underset{\theta}{|}\nu} - Q_{c\nu\underset{\theta}{|}\mu}]$$

$$+\sum_B(\underset{\theta}{Q}_{B\mu}\underset{\theta}{\Psi}_{CB\nu} - \underset{\theta}{Q}_{B\nu}\underset{\theta}{\Psi}_{CB\mu})], \quad \theta \in \{3,4\}.$$

Substituting $\underset{\theta}{P}$, $\underset{\theta}{Q}$, one obtains

$$
\begin{aligned}
& e_c \underset{\theta-2}{R}_{ijmn} N^i_c z^j B^m_\mu B^n_\nu \\
& + 2(-1)^{\theta-1} \underset{\vee}{\tilde{\Gamma}}{}^\pi_{\mu\nu} \Big(p^\sigma \underset{\theta}{\Omega}_{C\sigma\pi} + q_{c\underset{\theta}{|}\pi} + \sum_B q_B \underset{\theta}{\Psi}_{CB\pi}\Big) \\
& = \Big(p^\pi_{\underset{\theta}{|}\mu} + p^\sigma \underset{\theta}{\Phi}{}^\pi_{\sigma\mu} - \sum_A e_A q_A \underset{\theta}{\Omega}_{A\sigma\mu} g^{\underline{\pi\sigma}}\Big) \underset{\theta}{\Omega}_{C\pi\nu} \\
& -\Big(p^\pi_{\underset{\theta}{|}\nu} + p^\sigma \underset{\theta}{\Phi}{}^\pi_{\sigma\nu} - \sum_A e_A q_A \underset{\theta}{\Omega}_{A\sigma\nu} g^{\underline{\pi\sigma}}\Big) \underset{\theta}{\Omega}_{C\pi\mu} \\
& + p^\sigma_{\underset{\theta}{|}\nu} \underset{\theta}{\Omega}_{C\sigma\mu} + p^\sigma \underset{\theta}{\Omega}_{C\sigma\mu\underset{\theta}{|}\nu} + q_{C\underset{\theta}{|}\mu\nu} + \sum_B \Big(q_{B\underset{\theta}{|}\nu} \underset{\theta}{\Psi}_{CB\mu} + q_B \underset{\theta}{\Psi}_{CB\underset{\theta}{|}\nu}\Big) \\
& - p^\sigma_{\underset{\theta}{|}\mu} \underset{\theta}{\Omega}_{C\sigma\nu} - p^\sigma \underset{\theta}{\Omega}_{C\sigma\nu\underset{\theta}{|}\mu} - q_{C\underset{\theta}{|}\mu\nu} - \sum_B \Big(q_{B\underset{\theta}{|}\nu} \underset{\theta}{\Psi}_{CB\mu} + q_B \underset{\theta}{\Psi}_{CB\underset{\theta}{|}\nu}\Big) \\
& + \sum_B \Big[\Big(p^\sigma \underset{\theta}{\Omega}_{B\sigma\mu} + q_{B\underset{\theta}{|}\mu} + \sum_A q_A \underset{\theta}{\Psi}_{CB\mu}\Big) \underset{\theta}{\Psi}_{CB\nu} \\
& -\Big(p^\sigma \underset{\theta}{\Omega}_{B\sigma\nu} + q_{B\underset{\theta}{|}\nu} + \sum_A q_A \underset{\theta}{\Psi}_{CB\nu}\Big) \underset{\theta}{\Psi}_{CB\mu}\Big].
\end{aligned}
$$

Having in mind that for $\theta \in \{3,4\}$:

$$
q_{C\underset{\theta}{|}\mu\nu} - q_{C\underset{\theta}{|}\mu\nu} = 2(-1)^{\theta-1} \tilde{\Gamma}^\pi_{\mu\nu} q_{C,\pi}, \tag{2.27}
$$

the previous equation, after putting in order, becomes

$$
\begin{aligned}
& e_c \underset{\theta-2}{R}_{ijmn} N^i_c \Big(P^\sigma B^j_\sigma + \sum_A q_A N^j_A\Big) B^m_\mu B^n_\nu \\
& + 2(-1)^{\theta-1} \underset{\vee}{\tilde{\Gamma}}{}^\pi_{\mu\nu} \Big(p^\sigma \underset{\theta}{\Omega}_{C\sigma\pi} + \sum_B q_B \underset{\theta}{\Psi}{}_{CB\pi}\Big) \\
& = p^\sigma \Big(\underset{\theta}{\Phi}{}^\pi_{\sigma\mu} \underset{\theta}{\Omega}_{C\pi\nu} - \underset{\theta}{\Phi}{}^\pi_{\sigma\nu} \underset{\theta}{\Omega}_{C\pi\mu} + \underset{\theta}{\Omega}_{C\sigma\mu\underset{\theta}{|}\nu} - \underset{\theta}{\Omega}_{C\sigma\nu\underset{\theta}{|}\mu}\Big) \\
& + \sum_A e_A q_A \Big(\underset{\theta}{\Omega}{}^\pi_{C\pi\mu} \underset{\theta}{\Omega}{}^\pi_{A\nu} - \underset{\theta}{\Omega}{}^\pi_{C\pi\nu} \underset{\theta}{\Omega}{}^\pi_{A\mu}\Big) \\
& + \sum_A \Big[p^\sigma \Big(\underset{\theta}{\Omega}_{A\sigma\mu} \underset{\theta}{\Psi}_{CA\nu} - \underset{\theta}{\Omega}_{A\sigma\nu} \underset{\theta}{\Psi}_{CA\mu}\Big) + q_A \Big(\underset{\theta}{\Psi}_{CA\mu\underset{\theta}{|}\nu} - \underset{\theta}{\Psi}_{CA\nu\underset{\theta}{|}\mu}\Big) \\
& + \sum_B q_B \Big(\underset{\theta}{\Psi}_{AB\mu} \underset{\theta}{\Psi}_{CA\nu} - \underset{\theta}{\Psi}_{AB\nu} \underset{\theta}{\Psi}_{CA\mu}\Big)\Big], \quad \theta \in \{3,4\}.
\end{aligned} \tag{2.28}
$$

2.4. For $\theta = 3$, $\omega = 4$ according to (2) and (6) we get

$$\begin{aligned}\underset{4}{R}^{i}_{p\mu\nu}z^{p} = [&\underset{3}{P}^{\pi}_{\mu\underset{4}{|}\nu} - \underset{4}{P}^{\pi}_{\nu\underset{3}{|}\mu} + \underset{3}{P}^{\sigma}_{\mu}\underset{4}{\Phi}^{\pi}_{\sigma\nu} - \underset{4}{P}^{\sigma}_{\nu}\underset{3}{\Phi}^{\pi}_{\sigma\mu}\\ &\sum_{A} e_{A} g^{\underline{\pi\sigma}}(\underset{3}{Q}_{A\mu}\underset{4}{\Omega}_{A\sigma\nu} - \underset{4}{Q}_{A\nu}\underset{4}{\Omega}_{A\sigma\mu})]B^{i}_{\pi}\\ &+\sum_{A}[\underset{3}{P}^{\pi}_{\mu}\underset{4}{\Omega}_{A\pi\nu} - \underset{4}{P}^{\pi}_{\nu}\underset{3}{\Omega}_{A\pi\mu} + \underset{3}{Q}_{A\mu\underset{4}{|}\nu} - \underset{4}{Q}_{A\nu\underset{3}{|}\mu}\\ &\quad+\sum_{B}(\underset{3}{Q}_{B\mu}\underset{4}{\Psi}_{AB\nu} - \underset{4}{Q}_{B\nu}\underset{3}{\Psi}_{AB\mu})]N^{i}_{A}.\end{aligned}\tag{2.29}$$

This is the *6th integrability condition* of the derivational formula (1.11) of the deformation field z^i.

a) Multiplying the previous equation with $G_{\underline{il}}B^{l}_{\lambda}$, we get

$$\begin{aligned}\underset{4}{R}_{lp\mu\nu}B^{l}_{\lambda}z^{p} = [&\underset{3}{P}^{\pi}_{\mu\underset{4}{|}\nu} - \underset{4}{P}^{\pi}_{\nu\underset{3}{|}\mu} + \underset{3}{P}^{\sigma}_{\mu}\underset{4}{\Phi}^{\pi}_{\sigma\nu} - \underset{4}{P}^{\sigma}_{\nu}\underset{3}{\Phi}^{\pi}_{\sigma\mu}\\ &\sum_{A} e_{A} g^{\underline{\pi\sigma}}(\underset{3}{Q}_{A\mu}\underset{4}{\Omega}_{A\sigma\nu} - \underset{4}{Q}_{A\nu}\underset{3}{\Omega}_{A\sigma\mu})]g_{\underline{\lambda\pi}}\end{aligned}\tag{2.30}$$

From here, analogously to the previous cases, using the Ricci type identity [12]

$$p^{\pi}\underset{3}{|}\mu\underset{4}{|}\nu - p^{\pi}\underset{4}{|}\nu\underset{3}{|}\mu = \underset{4}{\tilde{R}}^{\pi}_{\sigma\mu\nu}p^{\sigma},\tag{2.31}$$

where

$$\underset{4}{R}^{\alpha}_{\beta\mu\nu} = \tilde{\Gamma}^{\alpha}_{\beta\mu,\nu} - \tilde{\Gamma}^{\alpha}_{\nu\beta,\mu} + \tilde{\Gamma}^{\sigma}_{\beta\mu}\tilde{\Gamma}^{\alpha}_{\nu\sigma} - \tilde{\Gamma}^{\sigma}_{\nu\beta}\tilde{\Gamma}^{\alpha}_{\sigma\mu} + \tilde{\Gamma}^{\sigma}_{\mu\nu}(\tilde{\Gamma}^{\alpha}_{\sigma\beta} - \tilde{\Gamma}^{\alpha}_{\beta\sigma}),\tag{2.32}$$

is the 4th kind curvature tensor of a subspace, and from (29) we finally get

$$\begin{aligned}&\underset{4}{R}_{ij\mu\nu}B^{i}_{\lambda}(p^{\sigma}B^{j}_{\sigma} + \sum_{A} q_{A}N^{j}_{A})\\ &= p^{\sigma}(\underset{4}{\tilde{R}}_{\lambda\sigma\mu\nu} + \underset{3}{\Phi}_{\lambda\sigma\mu\underset{4}{|}\nu} + \underset{4}{\Phi}_{\lambda\sigma\nu\underset{3}{|}\mu} + \underset{3}{\Phi}^{\rho}_{\sigma\mu}\underset{4}{\Phi}_{\lambda\rho\nu} - \underset{4}{\Phi}^{\rho}_{\sigma\nu}\underset{3}{\Phi}_{\lambda\rho\mu})\\ &+\sum_{A} e_{A}[q_{A}(\underset{3}{\Phi}_{\lambda\rho\mu}\underset{4}{\Omega}^{\sigma}_{A\nu} - \underset{4}{\Phi}_{\lambda\sigma\nu}\underset{3}{\Omega}^{\sigma}_{A\nu} - \underset{3}{\Omega}_{A\lambda\sigma\mu\underset{4}{|}\nu} + \underset{4}{\Omega}_{A\lambda\sigma\nu\underset{3}{|}\mu})\\ &\qquad+p^{\sigma}(\underset{3}{\Omega}_{A\lambda\mu}\underset{4}{\Omega}_{A\sigma\nu} - \underset{4}{\Omega}_{A\lambda\nu}\underset{3}{\Omega}_{A\sigma\mu})\\ &\qquad+\sum_{B} q_{B}(\underset{3}{\Omega}_{A\lambda\mu}\underset{4}{\Psi}_{AB\nu} - \underset{4}{\Omega}_{A\lambda\nu}\underset{3}{\Psi}_{AB\mu})].\end{aligned}\tag{2.33}$$

b) Multiplying (29) with $G_{\underline{il}}N^l_C$ and arranging, we get finally

$$\begin{aligned} &e_C \underset{4}{R}_{ij\mu\nu} N^i_C (p^\sigma B^j_\sigma + \sum_A q_A N^j_A) \\ &= p^\sigma (\underset{3}{\Phi}^\pi_{\sigma\mu} \underset{4}{\Omega}_{C\pi\nu} - \underset{4}{\Phi}^\pi_{\sigma\nu} \underset{3}{\Omega}_{C\pi\mu} + \underset{3}{\Omega}_{C\sigma\mu \underset{4}{|}\nu} - \underset{4}{\Omega}_{C\sigma\nu \underset{3}{|}\mu}) \\ &\quad + \sum_A e_A q_A (\underset{3}{\Omega}_{C\pi\mu} \underset{4}{\Omega}^\pi{}_{A\nu} - \underset{4}{\Omega}_{C\pi\nu} \underset{3}{\Omega}^\pi_{A\mu}) \\ &\quad \sum_A [p^\sigma (\underset{3}{\Omega}_{A\sigma\mu} \underset{4}{\Psi}_{CA\nu} - \underset{4}{\Omega}_{A\sigma\nu} \underset{3}{\Omega}_{CA\mu}) \\ &\quad\quad + q_A (\underset{3}{\Psi}_{CA\mu \underset{4}{|}\nu} - \underset{4}{\Psi}_{CA\nu \underset{3}{|}\mu}) \\ &\quad + \sum_B q_B (\underset{3}{\Psi}_{AB\mu} \underset{4}{\Psi}_{CA\nu} - \underset{4}{\Psi}_{AB\nu} \underset{3}{\Psi}_{CA\mu})] \end{aligned} \tag{2.34}$$

From the above exposed, the next theorem is valid:

Theorem 2.1 *If the infinitesimal bending field z^i of the subspace $GR_M \subset GR_N$ is expressed by virtue of tangent and normal component in the form (1.9), then the coefficients p^σ, q_A satisfy the conditions (15), (16), (21), (22), (26), (28), (33), (34).*

References

[1] Efimov, N. V.: *Kachestvennye voprosy teorii deformacii poverhnostei.* UMN **3.2** (1948), 47–158.

[2] Eisenhart, L. P.: *Generalized Riemann spaces.* Proc. Nat. Acad. Sci. USA **37** (1951), 311–315.

[3] Kon-Fossen, S. E.: Nekotorye voprosy differ. geometrii v celom. *Fizmatgiz, Moskva*, 1959.

[4] Hineva, S. T.: *On infinitesimal deformations of submanifolds of a Riemannian manifold.* Differ. Geom., Banach center publications **12**, *PWN, Warshaw*, 1984, 75–81.

[5] Ivanova-Karatopraklieva, I., Sabitov, I. Kh.: *Surface deformation.* J. Math. Sci., New York, **70**, 2 (1994), 1685–1716.

[6] Ivanova-Karatopraklieva, I., Sabitov, I. Kh.: *Bending of surfaces II.* J. Math. Sci., New York, **74**, 3 (1995), 997–1043.

[7] Lizunova, L. Yu.: *O beskonechno malyh izgibaniyah giperpoverhnostei v rimanovom prostranstve.* Izvestiya VUZ, Matematika **94**, 3 (1970), 36–42.

[8] Markov, P. E.: *Beskonechno malye izgibanya nekotoryh mnogomernyh poverhnostei.* Matemat. zametki **T. 27**, 3 (1980), 469–479.

[9] Mikeš, J.: *Holomorphically projective mappings and their generalizations.* J. Math. Sci., New York, **89**, 3 (1998), 1334–1353.

[10] Mikeš, J., Laitochová, J., Pokorná, O.: *On some relations between curvature and metric tensors in Riemannian spaces.* Suppl. ai Rediconti del Circolo Mathematico di Palermo **2**, 63 (2000), 173–176.

[11] Minčić S. M.: *Ricci type identities in a subspace of a space of non-symmetric affine connexion.* Publ. Inst. Math., NS, t.18(32) (1975), 137–148.

[12] Minčić S. M.: *Novye tozhdestva tipa Ricci v podprostranstve prostranstva nesimmetrichnoi affinoi svyaznosty.* Izvestiya VUZ, Matematika **203**, 4 (1979), 17–27.

[13] Minčić S. M.: *Derivational formulas of a subspace of a generalized Riemannian space.* Publ. Inst. Math., NS, t.34(48) (1983), 125–135.

[14] Minčić S. M.: *Integrability conditions of derivational formulas of a subspace of generalized Riemannian space.* Publ. Inst. Math., NS, t.31(45) (1980), 141–157.

[15] Minčić S. M., Velimirović, L. S.: *O podprostranstvah obob. rimanova prostranstva.* Siberian Mathematical Journal, Dep. v VINITI No.3472-V 98 (1998).

[16] Minčić, S. M., Velimirović, L. S.: *Riemannian Subspaces of Generalized Riemannian Spaces.* Universitatea Din Bacau Studii Si Cercetari Stiintifice, Seria: Matematica, 9 (1999), 111–128.

[17] Minčić, S. M., Velimirović, L. S., Stanković, M. S.: *Infinitesimal Deformations of a Non-Symmetric Affine Connection Space.* Filomat (2001), 175–182.

[18] Velimirović, L. S., Minčić, S. M., Stanković, M. S.: *Infinitesimal deformations and Lie Derivative of a Non-symmetric Affine Connection Space.* Acta Univ. Palacki. Olomuc., Fac. rer. nat., Math. **42** (2003), 111–121.

[19] Mishra, R. S.: *Subspaces of generalized Riemannian space.* Bull. Acad. Roy. Belgique, Cl. sci., (1954), 1058–1071.

[20] Prvanovich, M.: *Équations de Gauss d'un sous-espace plongé dans l'espace Riemannien généralisé.* Bull. Acad. Roy. Belgique, Cl. sci., (1955), 615–621.

[21] Velimirović, L. S., Minčić, S. M.: *On infinitesimal bendings of subspaces of generalized Riemannian spaces.* Tensor (2004), 212–224.

[22] Yano, K.: *Sur la theorie des deformations infinitesimales.* Journal of Fac. of Sci. Univ. of Tokyo **6** (1949), 1–75.

[23] Yano, K.: *Infinitesimal variations of submanifolds.* Kodai Math. J., **1** (1978), 30–44.