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FINITE-VALUED DUALLY RESIDUATED LATTICE-ORDERED MONOIDS

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ABSTRACT. Lattice-ordered groups, as well as GMV-algebras (called also pseudo MV-algebras), are both particular cases of dually residuated lattice-ordered monoids ($DR\ell$-monoids). In the paper we study values in $DR\ell$-monoids, especially if the ideal lattice is a member of the class $\mathcal{IRN}$ of algebraic, distributive lattices whose compact elements form a relatively normal sublattice, and we characterize finite-valued $DR\ell$-monoids whose ideal lattices belong to $\mathcal{IRN}$.

1. Introduction

K. L. N. Swamy [19] introduced commutative dually residuated lattice-ordered monoids ($DR\ell$-semigroups) as a common abstraction of Abelian lattice-ordered groups and Brouwerian algebras (by a Brouwerian algebra is meant a dually relatively pseudo-complemented lattice). J. Rachůnek [13], [14] proved that well-known MV-algebras ([2]), an algebraic counterpart of Łukasiewicz's logic, and BL-algebras ([9]), structures for Hájek's basic logic, that captures the three most significant fuzzy logics (Łukasiewicz logic, Gödel logic and product logic), can be viewed as particular kinds of bounded commutative $DR\ell$-monoids.

In the paper we deal with (non-commutative) $DR\ell$-monoids, which include lattice-ordered groups, and likewise non-commutative generalizations of mentioned MV-algebras and BL-algebras, i.e. GMV-algebras ([15]) called also pseudo MV-algebras ([7]), and pseudo BL-algebras ([4], [5]), respectively. In [17], [18] and [6], the class $\mathcal{IRN}$ of algebraic, distributive lattices whose compact elements form a relatively normal sublattice was examined; it turns out that lattices in $\mathcal{IRN}$ have similar properties as e.g. the lattice of all convex ℓ-subgroups of an ℓ-group. We define and study the notion of a value of a non-zero element of a $DR\ell$-monoid. Further, we show that given a $DR\ell$-monoid

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satisfying an additional identity, its ideal lattice is a member of $\mathcal{IRN}$ and this enables us to describe finite-valued $DR\ell$-monoids that satisfy this identity.

The present concept of a (non-commutative) dually residuated lattice-ordered monoid is due to T. Kovář [10]:

An algebra $(A; +, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ of type $\langle 2,0,2,2,2,2\rangle$ is said to be a *dually residuated lattice-ordered monoid* (*$DR\ell$-monoid*) if

(1) $(A; +, 0, \vee, \wedge)$ is an ℓ-monoid, i.e., $(A; +, 0)$ is a monoid, $(A; \vee, \wedge)$ is a lattice and the monoid operation distributes over the lattice operations;
(2) for any $a, b \in A$, $a \rightharpoonup b$ is the least $x \in A$ such that $x + b \geq a$, and $a \leftharpoondown b$ is the least $y \in A$ such that $b + y \geq a$;
(3) A fulfils the identities

$$\big((x \rightharpoonup y) \vee 0\big) + y \leq x \vee y\,, \qquad y + \big((x \leftharpoondown y) \vee 0\big) \leq x \vee y\,,$$
$$x \rightharpoonup x \geq 0\,, \qquad x \leftharpoondown x \geq 0\,.$$

In the definition, the condition (2) can be equivalently replaced by the following identities ([10], [15]):

$$(x \rightharpoonup y) + y \geq x\,, \qquad y + (x \leftharpoondown y) \geq x\,,$$
$$x \rightharpoonup y \leq (x \vee z) \rightharpoonup y\,, \qquad x \leftharpoondown y \leq (x \vee z) \leftharpoondown y\,,$$
$$(x + y) \rightharpoonup y \leq x\,, \qquad (y + x) \leftharpoondown y \leq x\,.$$

The following lemma catalogues a few basic properties of dually residuated ℓ-monoids:

Lemma 1.1. ([10]) *In any $DR\ell$-monoid we have:*

(1) $x \rightharpoonup x = 0 = x \leftharpoondown x$;
(2) $\big((x \rightharpoonup y) \vee 0\big) + y = x \vee y = y + \big((x \leftharpoondown y) \vee 0\big)$;
(3) $x \rightharpoonup (y + z) = (x \rightharpoonup z) \rightharpoonup y$, $x \leftharpoondown (y + z) = (x \leftharpoondown y) \leftharpoondown z$;
(4) *if* $x \leq y$, *then* $x \rightharpoonup z \leq y \rightharpoonup z$, $x \leftharpoondown z \leq y \leftharpoondown z$, $z \rightharpoonup x \geq z \rightharpoonup y$ *and* $z \leftharpoondown x \geq z \leftharpoondown y$;
(5) $x \leq y$ *iff* $x \rightharpoonup y \leq 0$ *iff* $x \leftharpoondown y \leq 0$;
(6) $x \rightharpoonup (y \wedge z) = (x \rightharpoonup y) \vee (x \rightharpoonup z)$, $x \leftharpoondown (y \wedge z) = (x \leftharpoondown y) \vee (x \leftharpoondown z)$;
(7) $(x \vee y) \rightharpoonup z = (x \rightharpoonup z) \vee (y \rightharpoonup z)$, $(x \vee y) \leftharpoondown z = (x \leftharpoondown z) \vee (y \leftharpoondown z)$.

2. Values in $DR\ell$-monoids

First of all, let us recall necessary facts concerning ideals in $DR\ell$-monoids.

For $x \in A$, let $|x| = x \vee (0 \rightharpoonup x)$, or equivalently, $|x| = x \vee (0 \leftharpoondown x)$, be the *absolute value* of x, and for $X \subseteq A$, let $X^+ = \{x \in X : 0 \leq x\}$.

An *ideal* in A is a subset H such that

(i) $0 \in H$,
(ii) if $x, y \in H$, then $x + y \in H$,
(iii) if $x \in H$, $y \in A$ and $|y| \leq |x|$, then $y \in H$.

One readily sees that the ideals of any $DR\ell$-monoid form a complete lattice, $\mathrm{Id}(A)$, and therefore, for every $\emptyset \neq X \subseteq A$, the set

$$I(X) = \{a \in A : \ |a| \leq |x_1| + \cdots + |x_n| \ \text{for some} \ x_1, \dots, x_n \in X\}$$

is the smallest ideal in A including X. In particular, for any $x \in A$,

$$I(x) = \{a \in A : \ |a| \leq n|x| \ \text{for some} \ n \in \mathbb{N}\}.$$

For any $DR\ell$-monoid A, $\mathrm{Id}(A)$ is an algebraic, distributive lattice whose compact elements are obviously finitely generated ideals. However, by [11; Proposition 12],

$$I(x) \cap I(y) = I(|x| \wedge |y|) \qquad \text{and} \qquad I(x) \vee I(y) = I(|x| \vee |y|),$$

for all $x, y \in A$, and consequently, every finitely generated ideal is principal. Hence the compact elements of $\mathrm{Id}(A)$ are just the principal ideals that obviously form a sublattice of $\mathrm{Id}(A)$.

An ideal H is said to be *normal* if $x + H^+ = H^+ + x$ for all $x \in A$. The normal ideals are precisely the kernels of homomorphisms; if H is a normal ideal, then the corresponding congruence relation Θ_H is given by

$$x \equiv y \quad (\Theta_H) \qquad \text{iff} \qquad (x \rightharpoonup y) \vee (y \rightharpoonup x) \in H,$$

so the quotient $DR\ell$-monoid A/H over H comprises the elements in the form $x/H = \{a \in A : \ (x \rightharpoonup a) \vee (a \rightharpoonup x) \in H\}$. In general, if H is an arbitrary ideal, then $\mathcal{R}_A(H) = \{x/H : \ x \in A\}$ is a distributive lattice in which

$$x/H \leq y/H \qquad \text{iff} \qquad (x \rightharpoonup y) \vee 0 \in H.$$

Since the ideal lattice $\mathrm{Id}(A)$ is algebraic and distributive by [11; Theorem 14], we can use several concepts and results from [17], [18] or [6].

Let L be an algebraic, distributive lattice with the greatest element 1, and let $\mathrm{Com}(L)$ be the join-subsemilattice of all compact elements in L. It is well known that L fulfils the join-infinite distributive law

$$x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I} x \wedge y_i, \tag{JID}$$

and consequently, L is a *Brouwerian lattice*, i.e., for any $a, b \in L$, there exists the greatest $x \in L$ such that $x \wedge a \leq b$.

An element $a \in L \setminus \{1\}$ is said to be

(i) *meet-prime* if $a \geq x \wedge y$ implies $a \geq x$ or $a \geq y$,
(ii) *meet-irreducible* if $a = x$ or $a = y$ whenever $a = x \wedge y$, for all $x, y \in L$.

Observe that the primeness coincides with the irreducibility because of the distributivity of L. The concept of a *completely meet-prime element* and a *completely meet-irreducible element*, respectively, is obtained when allowing arbitrary meets in the above definitions. We should remind that every element of L is the infimum of a set of completely meet-irreducible elements. In addition, one readily sees that each completely meet-prime element is completely meet-irreducible, but the converse holds if L satisfies the meet-infinite distributive law

$$x \vee \bigwedge_{i \in I} y_i = \bigwedge_{i \in I} x \vee y_i\,, \tag{MID}$$

i.e., L is a dually Brouwerian lattice.

If $c \in \mathrm{Com}(L) \setminus \{0\}$, then there is a maximal element $x \in L$, a *value* of c in L, such that $c \nleq x$. The set of all values of c in L is denoted by $\Gamma_L(c)$. By [17; p. 312], and [18; p. 43], an element $a \in L$ is a value of some $c \in \mathrm{Com}(L)$ if and only if a is completely meet-irreducible. Furthermore, completely meet-prime elements are determined by values: an element is completely meet-prime if and only if it is the unique value of some compact element.

Let us return to $DR\ell$-monoids. We define an ideal $H \in \mathrm{Id}(A)$ to be *prime* if it is a meet-prime element of $\mathrm{Id}(A)$, i.e. for all $J, K \in \mathrm{Id}(A)$, if $J \cap K \subseteq H$, then $J \subseteq H$ or $K \subseteq H$. By [12; Theorem 2.2], for every proper ideal H and $a \notin H$, there is a prime ideal P such that $H \subseteq P$ and $a \notin P$, and consequently, any ideal is equal to the intersection of all prime ideals exceeding it. If a $DR\ell$-monoid satisfies the identities

$$\begin{aligned}(x \rightharpoonup y) \wedge (y \rightharpoonup x) &\leq 0\,,\\ (x \leftharpoondown y) \wedge (y \leftharpoondown x) &\leq 0\,,\end{aligned} \tag{$*$}$$

then we have several criteria for primeness of ideals (see [12; Theorem 2.12]):

PROPOSITION 2.1. *Let A be a $DR\ell$-monoid satisfying $(*)$. Then for any $H \in \mathrm{Id}(A)$, the following conditions are equivalent:*

(1) *H is prime.*
(2) *If $x \wedge y \in H$, then $x \in H$ or $y \in H$.*
(3) *If $x \wedge y = 0$, then $x \in H$ or $y \in H$.*
(4) *For any $x, y \in A$, $(x \rightharpoonup y) \vee 0 \in H$ or $(y \rightharpoonup x) \vee 0 \in H$.*
(5) *$\mathcal{R}_A(H)$ is linearly ordered.*
(6) *The set of all ideals containing H is a chain.*
(7) *H is the intersection of a chain of completely meet-irreducible ideals.*

In analogy with ℓ-groups or GMV-algebras we define the notion of a value of a non-zero element in a $DR\ell$-monoid. Let $a \in A \setminus \{0\}$. By Zorn's lemma, the set of all ideals that do not contain a has a maximal element; such an ideal is called a *value* of a. We use $\Gamma_A(a)$ to denote the set of all values of a in A, and

$\Gamma(A)$ denotes the set comprising all values of all $a \in A \setminus \{0\}$. It is easy to see that $\Gamma_A(a) = \Gamma_{\mathrm{Id}(A)}\big(I(a)\big)$ for any $a \in A \setminus \{0\}$.

An element $a \in A$ is said to be *special* if it has the unique value V in A; in this case, V is called a *special value*.

PROPOSITION 2.2. *Let A be a $DR\ell$-monoid. Then the following conditions are equivalent for every ideal $H \in \mathrm{Id}(A)$:*

(1) $H \in \Gamma(A)$.
(2) *H is completely meet-irreducible.*
(3) *H has the unique cover H^* in the ideal lattice* $\mathrm{Id}(A)$.

Moreover, if A fulfils $(*)$ and H is normal, then each of the above is equivalent to

(4) *A/H is linearly ordered and the ideal lattice $\mathrm{Id}(A/H)$ contains the unique atom.*

P r o o f . As pointed out before, since H is a value of $a \in A \setminus \{0\}$ iff it is a value of $I(a)$ in $\mathrm{Id}(A)$, the conditions (1) and (2) are equivalent by [17; p. 312]. The equivalence of (2) and (3) is obvious.

CLAIM. *If H is a normal ideal of A, then* $\mathrm{Id}(A/H) \cong [H) \subseteq \mathrm{Id}(A)$.

One readily verifies that if $J \in \mathrm{Id}(A)$, then $J/H = \{a/H : a \in J\}$ is an ideal in A/H, and conversely, $\overline{K} = \{a \in A : a/H \in K\}$ is an ideal in A with $H \subseteq \overline{K}$ provided $K \in \mathrm{Id}(A/H)$. In addition, it can be easily proved that $J \mapsto J/H$ and $K \mapsto \overline{K}$ are mutually inverse order preserving bijections between $\mathrm{Id}(A/H)$ and $[H) = \big\{J \in \mathrm{Id}(A) : H \subseteq J\big\}$.

We are now ready to verify the latter statement.

(3) $\Longrightarrow$ (4): Since H is a prime ideal, it follows by Proposition 2.1(5) that A/H is linearly ordered and it should be evident by the claim that H^*/H is the only atom in $\mathrm{Id}(A/H)$.

(4) $\Longrightarrow$ (3): By the claim. □

By [11; Theorem 13], every ideal H in a $DR\ell$-monoid A is a convex subalgebra of A, and hence our next aim is to describe the connections between the values of $a \in H$ in A and the values of a in H.

PROPOSITION 2.3. *Let A be a $DR\ell$-monoid, $H \in \mathrm{Id}(A)$ and $a \in H \setminus \{0\}$. Then the mapping*

$$V \mapsto H \cap V, \qquad V \in \Gamma_A(a),$$

is a bijection of $\Gamma_A(a)$ onto $\Gamma_H(a)$.

P r o o f . Let $\mathrm{Spec}(H)$ be the set of all proper prime ideals in H and $\mathcal{S}(H)$ the set of all prime ideals in A that do not include H. By [12; Proposition 2.6], the mappings

$$\varphi\colon P \mapsto H \cap P\,, \qquad P \in \mathcal{S}(H)\,,$$

and

$$\psi\colon Q \mapsto H * Q\,, \qquad Q \in \operatorname{Spec}(H)\,,$$

where

$$H * Q = \big\{x \in A:\ |x| \wedge |y| \in Q \ \text{ for all } \ y \in H\big\}$$

is the relative pseudo-complement of H with respect to Q in the ideal lattice $\operatorname{Id}(A)$, are mutually inverse, order preserving bijections between $\mathcal{S}(H)$ and $\operatorname{Spec}(H)$. It is easily seen that $a \in P$ iff $a \in H \cap P$ and $a \in Q$ iff $a \in H * Q$. Further, $\Gamma_A(a) \subseteq \mathcal{S}(H)$ and $\Gamma_H(a) \subseteq \operatorname{Spec}(H)$. If $V \in \Gamma_A(a)$, then there exists $W \in \Gamma_H(a)$ such that $H \cap V \subseteq W$, whence $V \subseteq H * W$. It is clear that $H \cap V = W$ since otherwise $V \subset H * W$ and so $V \notin \Gamma_A(a)$. Similarly, if $W \in \Gamma_H(a)$, then $H * W \in \Gamma_A(a)$. Therefore, $V \in \Gamma_A(a)$ iff $H \cap V \in \Gamma_H(a)$ and $W \in \Gamma_H(a)$ iff $H * W \in \Gamma_A(a)$, so that $\varphi\restriction_{\Gamma_A(a)}$ and $\psi\restriction_{\Gamma_H(a)}$ are mutually inverse bijections. □

Now, let us recall some facts from [3]. Again, L is an algebraic, distributive lattice. We say that L is *generated* by its set of all meet-irreducible elements Γ if each element of L is the meet of some filter in Γ. If, moreover, $\bigwedge F_1 = \bigwedge F_2$ entails $F_1 = F_2$, then L is *freely generated* by Γ. Thus Γ freely generates L if there is a natural one-to-one correspondence between the elements of L and the filters in Γ.

A lattice L is called *completely distributive* if

$$\bigwedge_{i\in I}\bigvee_{j\in J} a_{ij} = \bigvee_{\varphi\colon I\to J}\ \bigwedge_{i\in I} a_{i\varphi(i)} \tag{CD}$$

whenever the indicated suprema and infima exist in L. By [1; p. 232, Theorem 17], (CD) and its dual are in complete lattices equivalent.

A *root-system* P is a poset in which for all a, the principal filter $[a) = \{x \in P:\ x \geq a\}$ is a chain. A maximal chain in a root-system is called a *root*.

THEOREM 2.4. *Let A be a DRℓ-monoid satisfying* $(*)$. *Then $\Gamma(A)$ is a root-system that generates* $\operatorname{Id}(A)$ *and the following statements are equivalent:*

(1) $\Gamma(A)$ *freely generates* $\operatorname{Id}(A)$.
(2) $\operatorname{Id}(A)$ *is completely distributive.*
(3) $\operatorname{Id}(A)$ *is dually Brouwerian, i.e.,* $\operatorname{Id}(A)$ *fulfils* (MID).
(4) *Every value is special.*
(5) $\operatorname{Id}(A)$ *is bialgebraic.*

Proof. In view of Proposition 2.1(6), the set of all prime ideals in A is a root-system, and hence so is $\Gamma(A)$. Since the values are precisely the completely meet-irreducible ideals, it follows that the ideal lattice $\mathrm{Id}(A)$ is generated by $\Gamma(A)$. The conditions (1)–(4) are equivalent by [3; Theorem 2.1, Corollary to Proposition 2.4] since a lattice L is freely generated by Γ iff L is completely distributive iff Γ satisfies (MID) iff every $a \in \Gamma$ is completely meet-prime, i.e., a is the unique value of a compact element. Finally, e.g. by [17], Lemma 1.1, an algebraic, distributive lattice is bialgebraic (algebraic and dually algebraic) iff every completely meet-irreducible element is even completely meet-prime. □

THEOREM 2.5. *If a $DR\ell$-monoid A satisfies $(*)$ and $\Gamma(A)$ contains only a finite number of roots, then $\mathrm{Id}(A)$ is freely generated by $\Gamma(A)$.*

Proof. By [3; Theorem 2.3], if Γ is a root-system that generates L and contains only finitely many roots and if $D\Big(\bigwedge_{i\in I} a_i\Big) = \bigcup_{i\in I} D(a_i)$ for each chain $\{a_i\}_{i\in I} \subseteq \Gamma$, where $D(a) = \{x \in \Gamma : \ x \geq a\}$, then L is freely generated by Γ. Therefore it suffices to show that $D\Big(\bigcap_{i\in I} V_i\Big) = \bigcup_{i\in I} D(V_i)$ for every chain of values $\{V_i\}_{i\in I}$ in A.

Obviously, $\bigcup_{i\in I} D(V_i) \subseteq D\Big(\bigcap_{i\in I} V_i\Big)$. Conversely, let $W \in D\Big(\bigcap_{i\in I} V_i\Big)$, that is, $V = \bigcap_{i\in I} V_i \subseteq W$. Since V is a prime ideal in A, by Proposition 2.1(7), it follows from (6) of Proposition 2.1 that W is comparable with every V_i. If $W \subset V_i$ for all $i \in I$, then $W \subseteq V$, and so $W = V$, which yields $W = V_{i_0}$ for some $i_0 \in I$ since $W \in \Gamma(A)$, which is a contradiction. Thus there exists $i_0 \in I$ such that $V_{i_0} \subseteq W$, so $W \in D(V_{i_0}) \subseteq \bigcup_{i\in I} D(V_i)$ proving $D\Big(\bigcap_{i\in I} V_i\Big) \subseteq \bigcup_{i\in I} D(V_i)$. □

3. Finite-valued $DR\ell$-monoids satisfying $(*)$

A lower-bounded, distributive lattice L is said to be *relatively normal* if its prime ideals form a root-system. This term is suggested by topological considerations: a topological space is hereditarily normal (not necessarily a T_2-space) if and only if the lattice of its open sets is relatively normal.

The class of the ideal lattices of relatively normal lattices is denoted by $\mathcal{IRN}$. If L is algebraic and distributive and $\mathrm{Com}(L)$ is a sublattice in L, then L is obviously isomorphic with the ideal lattice of $\mathrm{Com}(L)$ and the poset of the meet-prime elements of L is order-isomorphic to the poset of the prime ideals in $\mathrm{Com}(L)$. Therefore, $\mathrm{Com}(L)$ is a relatively normal lattice if and only if the meet-prime elements of L form a root-system, and so L belongs to $\mathcal{IRN}$ iff

L is an algebraic, distributive lattice such that $\mathrm{Com}(L)$ is a sublattice and the meet-prime elements of L form a root-system.

THEOREM 3.1. *If A satisfies $(*)$, then its ideal lattice $\mathrm{Id}(A)$ is a member of the class $\mathcal{IRN}$.*

Proof. We already know that $\mathrm{Id}(A)$ is an algebraic and distributive lattice and $\mathrm{Com}(\mathrm{Id}(A))$ is a sublattice of $\mathrm{Id}(A)$ as the compact elements in $\mathrm{Id}(A)$ are the principal ideals. In addition, due to Proposition 2.1, the meet-prime elements of $\mathrm{Id}(A)$, i.e. the prime ideals in A, form a root-system. □

It can be easily seen that an ideal H is a value of $a \in A \setminus \{0\}$ if and only if H is a value of the principal ideal $I(a)$ in $\mathrm{Id}(A)$. This allows to apply some results from [6] and [17], [18], particularly if A fulfils $(*)$.

In an algebraic, distributive lattice L, $a \in L$ is called *completely join-prime* if $a \leq \bigvee_{i \in I} x_i$ implies $a \leq x_{i_0}$ for some $i_0 \in I$; clearly, a is completely join-prime iff it is completely join-irreducible since L fulfils (JID). Similarly as completely meet-prime elements, by [17; p. 312] or [18; p. 43], likewise completely join-primes can be characterized in terms of values in L: an element is completely join-prime iff it is compact and has a unique value.

We say that $a, b \in L$ are *orthogonal* if $a \wedge b = 0$.

THEOREM 3.2. *Let A be a $DR\ell$-monoid satisfying $(*)$ and let $a \in A^+$. Then the following statements are equivalent:*

(1) *$\Gamma_A(a)$ is finite.*
(2) *Every value of a is special.*
(3) *$I(a)$ is the unique join of finitely many pairwise orthogonal completely join-prime ideals.*

Proof. Since $\mathrm{Id}(A) \in \mathcal{IRN}$ and $\Gamma_A(a) = \Gamma_{\mathrm{Id}(A)}(I(a))$, this is an immediate consequence of [17; Lemma 2.3] or [18; Lemma 3.5], stating that if $L \in \mathcal{IRN}$, then the following are equivalent, for $c \in \mathrm{Com}(L)$:

(1) c has only a finite number of values;
(2) every value of c is completely meet-prime, i.e. the only value of some compact element;
(3) c can be written uniquely as a finite join of pairwise orthogonal completely join-prime elements.

□

We define a $DR\ell$-monoid A to be *finite-valued* if $\Gamma_A(a)$ is finite for all $a \in A$. It is known that an ℓ-group G is finite-valued if and only if every value in G is special. The same holds for GMV-algebras by [16; Theorem 6].

COROLLARY 3.3. *A $DR\ell$-monoid A satisfying $(*)$ is finite-valued if and only if every $V \in \Gamma(A)$ is special. If $\Gamma(A)$ contains only finitely many roots, then A is finite-valued.*

P r o o f. The former statement is just another formulation of the previous theorem, the latter one follows from the simple observation that for $a \in A \setminus \{0\}$, $\Gamma_A(a)$ is an antichain in $\Gamma(A)$, so it is necessarily finite provided $\Gamma(A)$ has only a finite number of roots. □

In what follows, we use two technical lemmata to turn the condition (3) of Theorem 3.2 in the form that generalizes another description of finite-valued ℓ-groups (see [3; Theorem 3.9], and [17; Theorem 2.5]): an ℓ-group G is finite-valued if and only if each positive element of G is a finite sum of pairwise orthogonal special elements.

We shall call $a, b \in A$ *orthogonal* if $|a| \wedge |b| = 0$.

LEMMA 3.4. *Let A be any $DR\ell$-monoid. If $0 \le b \le a_1 + \dots + a_n$ for some $a_1, \dots, a_n \in A^+$, then $b = b_1 + \dots + b_n$ for some $b_i \in A^+$ with $b_i \le a_i$ $(1 \le i \le n)$.*

P r o o f. The proof proceeds by induction on n. For $n = 1$, the result is clear. Assume that $b \le a_1 + \dots + a_n$ and let $b_n = b \wedge a_n$. Then

$$c = b \rightharpoonup b_n = b \rightharpoonup (b \wedge a_n) = (b \rightharpoonup b) \vee (b \rightharpoonup a_n) = 0 \vee (b \rightharpoonup a_n).$$

Further, $b \le a_1 + \dots + a_n$ implies

$$b \rightharpoonup a_n \le (a_1 + \dots + a_{n-1} + a_n) \rightharpoonup a_n \le a_1 + \dots + a_{n-1}.$$

Hence $0 \le c = 0 \vee (b \rightharpoonup a_n) \le 0 \vee (a_1 + \dots + a_{n-1}) = a_1 + \dots + a_{n-1}$, and so by induction, $c = b_1 + \dots + b_{n-1}$ for some $0 \le b_i \le a_i$, where $1 \le i \le n-1$. Since $b_n \le b$, we have $b = (b \rightharpoonup b_n) + b_n$. Therefore $c + b_n = (b \rightharpoonup b_n) + b_n = b$ and consequently $b = b_1 + \dots + b_{n-1} + b_n$ for $b_i \le a_i$, $1 \le i \le n$. □

LEMMA 3.5. *Let A be any $DR\ell$-monoid. If $a_1, \dots, a_k \in A^+$ are pairwise orthogonal elements, then*

$$a_1 + \dots + a_k = a_1 \vee \dots \vee a_k,$$
$$n(a_1 \vee \dots \vee a_k) = na_1 \vee \dots \vee na_k$$

for every $n \in \mathbb{N}$.

P r o o f. If $a \wedge b = 0$, then

$$(a \rightharpoonup b) \vee 0 = (a \rightharpoonup b) \vee (a \rightharpoonup a) = a \rightharpoonup (a \wedge b) = a \rightharpoonup 0 = a.$$

Therefore $a \vee b = \big((a \rightharpoonup b) \vee 0\big) + b = a + b$. The rest is an easy induction. □

PROPOSITION 3.6. *Let A be a $DR\ell$-monoid with $(*)$ and let $a \in A^+$. Then $I(a)$ fulfils the condition (3) of Theorem 3.2 if and only if a can be uniquely expressed as a finite sum of positive, pairwise orthogonal special elements.*

Proof. The completely join-prime ideals are precisely principal ideals generated by special elements. Therefore, if a positive element a is the unique finite sum of positive, pairwise orthogonal special elements, then $I(a)$ satisfies (3) in Theorem 3.2.

Conversely, let us suppose that $I(a)$ has the unique representation

$$I(a) = I(b_1) \vee \dots \vee I(b_k) = I(b_1 \vee \dots \vee b_n), \tag{3.1}$$

where $I(b_i)$ are pairwise orthogonal completely join-prime ideals, i.e., every b_i is a special element. Since $I(b_i) \cap I(b_j) = I(b_i \wedge b_j) = \{0\}$ for all $i \neq j$, it follows that $b_i \wedge b_j = 0$ for all $i \neq j$. Clearly, $a \in I(b_1 \vee \dots \vee b_k)$, and so $a \leq n(b_1 \vee \dots \vee b_k) = nb_1 \vee \dots \vee nb_k = nb_1 + \dots + nb_k$ by Lemma 3.5. In view of Lemma 3.4 this implies $a = c_1 + \dots + c_k = c_1 \vee \dots \vee c_k$ for some $0 \leq c_i \leq nb_i$ $(1 \leq i \leq k)$. Therefore, $I(a) = I(c_1) \vee \dots \vee I(c_k)$ and thus $I(c_i) = I(b_i)$ as the expression (3.1) is unique. Altogether, a is the sum of pairwise orthogonal special elements $c_1, \dots, c_k$. □

COROLLARY 3.7. *A $DR\ell$-monoid satisfying $(*)$ is finite-valued if and only if every positive element has the unique expression as the sum (= the join) of a finite number of positive, pairwise orthogonal special elements.*

We are now going to show that the part "if" of Corollary 3.3 is true even for an arbitrary $DR\ell$-monoid A, i.e., if every value is special, then A is finite-valued.

LEMMA 3.8. *Let H be an ideal of an arbitrary $DR\ell$-monoid A, $a \in A^+ \setminus \{0\}$ and let*

$$\gamma(H) = \bigcap \{V \in \Gamma(A) : V \nsubseteq H\}.$$

Then $a \in \gamma(H)$ if and only if $W \subseteq H$ for all $W \in \Gamma_A(a)$. In addition, $H \in \Gamma(A)$ is special if and only if $\gamma(H) \nsubseteq H$.

Proof. Let $a \notin \gamma(H)$. Then $a \notin V$ for some $V \nsubseteq H$, and so there exists a value W of a such that $V \subseteq W$. Therefore $W \nsubseteq H$ as $W \subseteq H$ would imply $V \subseteq H$. Conversely, let $a \in \gamma(H)$ and $W \in \Gamma_A(a)$. If $W \nsubseteq H$, then $a \in W$, which is impossible. Thus $W \subseteq H$.

For the last claim, note that $H \in \Gamma(A)$ is a special value of some $a \in A$ if and only if $a \in \gamma(H) \setminus H$. Indeed, if $a \in \gamma(H) \setminus H$, then every value of a is a subset of H and $a \notin H$, so H is the only value of a. Conversely, if a is special with the unique value H, then $a \notin H$ and $a \in \gamma(H)$ as $H \subseteq H$. □

THEOREM 3.9. *Let A be any $DR\ell$-monoid. If every value of $a \in A^+ \setminus \{0\}$ is special, then a has finitely many values.*

Proof. Let K be the ideal generated by $\bigcup\{\gamma(V) : V \in \Gamma_A(a)\}$. If $a \notin K$, then there is $W \in \Gamma_A(a)$ with $K \subseteq W$, whence $\gamma(W) \subseteq K \subseteq W$. However, W is special and so $\gamma(W) \nsubseteq W$ by the last lemma, which is a contradiction. Thus $a \in K$. Lemma 3.4 now yields that $a = a_1 + \cdots + a_n$, where a_i is a positive element in $\gamma(V_i)$ for some $V_i \in \Gamma_A(a)$ $(1 \leq i \leq n)$. Moreover, any value of a_i is a subset of V_i since $a_i \in \gamma(V_i)$. If now V is a value of a, then V is a value of some a_i since $a \in H$ iff $a_1, \ldots, a_n \in H$ for any ideal H, and consequently, $V \subseteq V_i$. However, $V, V_i \in \Gamma_A(a)$, which entails $V = V_i$, and so $\Gamma_A(a)$ is finite. □

4. Main theorem

Combining Theorem 2.4 and Corollaries 3.3 and 3.7 we have obtained the following characterization of finite-valued $DR\ell$-monoid verifying $(*)$:

THEOREM 4.1. *For any dually residuated lattice-ordered monoid A satisfying $(*)$, the following statements are equivalent:*

(1) $\mathrm{Id}(A)$ *is freely generated by* $\Gamma(A)$.
(2) $\mathrm{Id}(A)$ *is completely distributive.*
(3) $\mathrm{Id}(A)$ *is dually Brouwerian.*
(4) $\mathrm{Id}(A)$ *is bialgebraic.*
(5) *Every value is completely meet-prime.*
(6) *Every value is special.*
(7) A *is finite-valued.*
(8) *Every positive element of A has the unique expression as the sum (= the join) of a finite number of positive pairwise orthogonal special elements.*

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