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POPRODUCT OF LATTICES

ZUZANA LADZIANSKA

The poproduct was introduced for the class of distributive lattices by R. Balbes and A. Horn [2] under the name order sum. The notion of the $\mathcal{K}$-poproduct for an arbitrary equational class $\mathcal{K}$ of lattices was defined in [11]. The $\mathcal{K}$-poproduct is a generalization of the $\mathcal{K}$-free product and the ordinal sum of lattices. M. Höft [8] defined the order sum for the class of partially ordered algebras and showed that the order sum exists in each quasi-equational class. If $\mathcal{K}$ is an equational class of lattices, then the $\mathcal{K}$-poproduct always exists and coincides with the order sum. The $\mathcal{K}$-poproduct was considered by T. G. Kucera and B. Sands [10] under the name $F_{\mathcal{K}}\left(\sum_{p\in P} L_p\right)$.

The present paper consists of five parts, in which various problems concerning a poproduct are considered.

1. The word problem for the $\mathcal{L}$-poproduct of lattices
2. Minimal representation of the elements of poproduct
3. Free-lattice-like sublattices of the poproduct of lattices
4. The poproduct decomposition of a lattice
5. Poproduct and direct (inverse) limits of lattices

1. The word problem for the $\mathcal{L}$-poproduct of lattices

In [11] we have investigated the word problem for the $\mathcal{L}$-poproduct. But the solution stated there is not correct, as we worked with an inadequate definition of the cover of an element. In the present paper we shall improve this result.

Let R be a poset and let L_r, $r\in R$ be pairwise disjoint lattices. The lattice operations in each L_r will be denoted by $\vee$, $\wedge$. Let $Q=\cup(L_r; r\in R)$ be partially ordered in the following way: for $a, b\in Q$ we put $a\leqq b$ if and only if one of the conditions (1) and (2) holds:

(1) there is an $r\in R$ such that $a, b\in L_r$ and the relation $a\leqq b$ holds in L_r;
(2) *there are* $p, r\in R$ such that $a\in L_p$, $b\in L_r$ and the relation $p<r$ holds in the poset R.

If f is a mapping from Q into a lattice M, then f_r denotes its restriction on L_r.

Definition 1.1. *Let $\mathcal{K}$ be an equational class of lattices. Let L, $L_r \in \mathcal{K}$ for $r \in R$ and let R be a poset. The lattice L is said to be the $\mathcal{K}$-poproduct of the lattices L_r if:*

(1) *there is an isotone injection $i:Q \to L$ such that for each $r \in R$, i_r is a lattice homomorphism;*
(2) *if $M \in \mathcal{K}$, then for every isotone mapping $f:Q \to M$ such that for each $r \in R$, f_r is a lattice homomorphism, there exists uniquely a lattice homomorphism $g:L \to M$ such that $g \circ i = f$.*

From the definition it follows that L is generated by the set $\iota(Q)$. If it does not cause ambiguity we say simply that $i:Q \to L$ is a canonical embedding. We shall mostly identify the sets Q and $i(Q)$. We also say that Q is a *skeleton* of L.

The $\mathcal{K}$-poproduct of the lattices L_r, $r \in R$ will be denoted by $P_{\mathcal{K}}(L_r; r \in R)$. It is easy to see that a poproduct exists in each equational class of lattices. From the definition it follows that the $\mathcal{K}$-poproduct form the $\mathcal{K}$-free product if and only if R is an antichain and that the $\mathcal{K}$-poproduct forms the ordinal sum if and only if R is a chain.

We shall consider the word problem for an $\mathcal{L}$-poproduct of lattices, where $\mathcal{L}$ is the class of all lattices. $\mathcal{L}$-poproduct of the lattices L_r, $r \in R$ will be briefly called poproduct and denoted by $P(L_r; r \in R)$. The poproduct is a special case of the $FL(Q, A, B)$, the free lattice generated by the partially ordered set Q and preserving finite joins and meets of elements of Q, defined by R. A. Dean in [3] In our case, the set $A = B$ consists of all comparable pairs of the set Q and of all finite subsets of every lattice L_r, $r \in R$.

Throughout the paper, Q will denote a skeleton of a poproduct Let us denote by $W(Q)$ the set of lattice polynomials (words, terms) over Q. These polynomials are formed from symbols denoting elements of Q and from the symbols $\vee$, $\wedge$. For $a, b \in W(Q)$ the symbol $a \equiv b$ means that a equals b as the elements of the absolutely free algebra. In [3], the relation $\leqq$ between the elements of $W(Q)$ was defined, from that relation we get the equivalence $\cong$ and $FL(Q, A, B) = W(Q)/\cong$. For simplicity, we shall identify classes $[a]$ in the equivalence $\simeq$ with their representatives a, thus the lattices L_r will be considered as sublattices of the poproduct. Instead of $a \cong b$ we shall usually write only $a = b$. Similarly as in [3], let us denote $J(a) = \{p: p \in Q, p \leqq a\}$, $M(a) = \{p: p \in Q, p \geqq a\}$. For each $a \in W(Q)$ define a natural number $l(a)$ — the *length* of a — as follows: if $a \in Q$, then $l(a) = 1$, if $a, b \in W(Q)$, then $l(a \vee b) = 1(a \wedge b) = l(a) + l(b)$. For $r \in R$, denote by $I(L_r)$ the lattice of all nonempty ideals of L_r and by $D(L_r)$ the lattice of all nonempty dual ideals (filters) of L_r. Denote $I_0(L_r) = I(L_r) \cup \{\emptyset\}$ and $D_0(L_r) = D(L_r) \cup \{\emptyset\}$. The operations in lattices $I_0(L_r)$ and $D_0(L_r)$ will be denoted by ∇ (join) and Δ (meet). Similarly as in [9], for each $r \in R$ we shall define a homomorphism $T_r: W(Q) \to I_0(L_r)$ and a dual homomorphism $T^r: W(Q) \to D_0(L_r)$ as follows:

$T_r(a)=\{x\in L_r\colon x\leqq a\}=(a]$, $T^r(a)=\{x\in L_r\colon x\geqq a\}=[a)$ if $a\in L_r$,
$T_r(a)=L_r$, $T^r(a)=\emptyset$ if $a\in L_p$, $p>r$,
$T_r(a)=\emptyset$, $T^r(a)=L_r$ if $a\in L_p$, $p<r$,
$T_r(a)=\emptyset$, $T^r(a)=\emptyset$ if $a\in L_p$, $p\parallel r$ (p, r incomparable).

Since $W(Q)$ is an absolutely free algebra, there exist uniquely the extensions of the given mappings onto homomorphism and dual homomorphism, respectively, hence the following holds:

$T_r(a\vee b)=T_r(a)\nabla T_r(b)$,
$T^r(a\vee b)=T^r(a)\Delta T^r(b)=T^r(a)\cap T^r(b)$,
$T^r(a\wedge b)=T^r(a)\nabla T^r(b)$,
$T_r(a\wedge b)=T_r(a)\Delta T_r(b)=T_r(a)\cap T_r(b)$.

Lemma 1.1. For $a\in W(Q)$ both $T_r(a)=J(a)\cap L_r$, and $T^r(a)=M(a)\cap L_r$ hold.

Proof. By induction with respect to the length of a.

Denote by 0,1 two new elements, which do not belong to the skeleton Q and extend the partial ordering from the set Q to the set $Q\dot{\cup}\{0,1\}$ ($\dot{\cup}$ denotes the disjuint union of sets) in the following way: for each $q\in Q$ the relation $0<q<1$ holds.

Similarly as in [6], for each $a\in W(Q)$ and each $r\in R$ the *upper r-cover* $a^{(r)}$ and the *lower r-cover* $a_{(r)}$ are defined as follows:

Definition 1.2.

1. *Let $a\in L_p$.*
   If $p=r$, then $a_{(r)}=a^{(r)}=a$.
   If $p\parallel r$, i.e. p and r and incomparable, then $a_{(r)}=0$, $a^{(r)}=1$.
   If $p<r$, then $a_{(r)}=0$, $a^{(r)}=0$.
   If $p>r$, then $a_{(r)}=1$, $a^{(r)}=1$.
2. *Let $a=p(a_1,\dots,a_n)$.*
   Then $a_{(r)}=p((a_1)_{(r)},\dots,(a_n)_{(r)})$ and
   $a^{(r)}=p((a_1)^{(r)},\dots,(a_n)^{(r)})$.
   If $a_{(r)}$ or $a^{(r)}\in L_r$, it is called a proper cover.

Proceeding by induction on the length of $a=p(a_1,\dots,a_n)$ one can easily prove the following six propositions.

Proposition 1.1. *Let $a=p(a_1,\dots,a_n)$. If $a_{(r)}$ or $a^{(r)}$ is proper, then there is at least one i, $1\leqq i\leqq n$ such that $a_i\in L_r$. Hence, for a given $a\in W(Q)$ there exists only a finite number of proper covers.*

Proposition 1.2. *If $a=p(a_1,\dots,a_n)$ and $a_{(r)}$ is proper, then there exists a polynomial $p'(b_1,\dots,b_m)$ such that $\{b_1,\dots,b_m\}\subseteqq\{a_1,\dots,a_n\}\cap L_r$ and $a_{(r)}=p'(b_1,\dots,b_m)$. And dually for $a^{(r)}$.*

Proposition 1.3. *If $a_{(r)}$ is proper, then $a_{(r)} \leqq a$. Dually, if $a^{(r)}$ is proper, then $a^{(r)} \geqq a$*

If $T_r(a)$ is a principal ideal, denote its generator by a_r, i.e. $T_r(a) = (a_r]$. If $T^r(a)$ is a principal filter, denote its generator by a^r, i.e. $T^r(a) = [a^r)$. If L_r has the smallest element, denote such an element by o_r. If L_r has the greatest element, denote such an element by i_r.

Proposition 1.4. *If $a_{(r)}$ is proper, then a_r exists and $a_{(r)} = a_r$. If $a^{(r)}$ is proper, then a^r exists and $a^{(r)} = a^r$.*

Conversely, if a_r exists, it need not imply that $a_{(r)}$ is proper and $a_{(r)} = a_r$, for it can happen that $a_r = i_r$ and $a_{(r)}$ is not proper.

Proposition 1.5. *If a_r exists and $a_r \neq i_r$, then $a_{(r)}$ is proper and $a_r = a_{(r)}$. If a^r exists and $a^r \neq o_r$, then $a^{(r)}$ is proper and $a^r = a^{(r)}$.*

Proposition 1.6. *If i_r does not exist, then $a_{(r)}$ is proper if and only if a_r exists. In such a case $a_{(r)} = a_r$. If o_r does not exist, then $a^{(r)}$ is proper if and only if a^r exists. In such a case, $a^{(r)} = a^r$.*

The following theorem gives a solution of the word problem for the poproduct

Theorem 1.1. *Let $L = P(L_r;\, r \in R)$, let Q be a skeleton of L, let $a, b \in W(Q)$. Then $a \leqq b$ if and only if one of the following holds:*

(1) $a \equiv a_1 \vee a_2$, *where* $a_1 \leqq b$ *and* $a_2 \leqq b$,
(2) $a \equiv a_1 \wedge a_2$, *where* $a_1 \leqq b$ *or* $a_2 \leqq b$,
(3) $b \equiv b_1 \wedge b_2$, *where* $a \leqq b_1$ *and* $a \leqq b_2$,
(4) $b \equiv b_1 \vee b_2$, *where* $a \leqq b_1$ *or* $a \leqq b_2$,
(5) *there are $p, r \in R$ $(p \leqq r)$ such that $a^{(p)}$, $b_{(r)}$ are proper and $a^{(p)} \leqq b_{(r)}$ holds.*

Proof. Throughout the proof, the following two lemmas will be used, which easily follow from the definition of an r-cover.

Lemma 1.2. *Let $a \equiv a_1 \vee a_2$, $T^r(a) \neq \emptyset$, $a^{(r)}$ be not proper. Then $T^r(a_1) = L_r$, $T^r(a_2) = L_r$. Dually, let $a \equiv a_1 \wedge a_2$, $T_r(a) \neq \emptyset$, $a_{(r)}$ be not proper. Then $T_r(a_1) = L_r$, $T_r(a_2) = L_r$.*

Lemma 1.3. *Let $a \equiv a_1 \wedge a_2$, $T^r(a) \neq \emptyset$, $a^{(r)}$ be not proper. Then either $T^r(a_1) = L_r$ or $T^r(a_2) = L_r$. Dually, let $a \equiv a_1 \vee a_2$, $T_r(a) \neq \emptyset$, $a_{(r)}$ be not proper. Then either $T_r(a_1) = L_r$ or $T_r(a_2) = L_r$.*

Now we shall prove Theorem 1.1. Theorem 7 of [3] characterizes $a \leqq b$ in the free lattice $FL(Q, A, B)$, using conditions (1)—(4) of Theorem 1.1 and the condition

$$M(a) \cap J(b) \neq \emptyset. \tag{5'}$$

Since $P(L_r;\, r \in R) = FL(Q, A, B)$ for suitable A and B, the conditions (1)—(5) are sufficient for $a \leqq b$. Conversely, let $a, b \in W(Q)$ and let $a \leqq b$. Then by

Theorem 7 of [3], either one of (1)—(4) holds or (5') is true, i.e. there is an $x \in Q$ such that $a \leqq x \leqq b$. Suppose that there exists an $x \in Q$ such that $a \leqq x \leqq b$ and let $x \in L_r$, $r \in R$. Therefore $x \in M(a) \cap L_r = T^r(a)$, $x \in J(b) \cap L_r = T_r(b)$. If both $a^{(r)}$, $b_{(r)}$ are proper, then $a^{(r)} \leqq x \leqq b_{(r)}$ and (5) holds. Let at least one of $a^{(r)}$, $b_{(r)}$ not be proper. If $l(a)=1$, $l(b)=1$, then $a \in L_p$, $b \in L_q$ and $a = a^{(p)} \leqq x \leqq b_{(q)} = b$ and (5) holds. Let at least one of $l(a)$, $l(b)$ greater than 1.

If $a^{(r)}$ is not proper and $l(a)>1$, then if $a \equiv a_1 \vee a_2$, using Lemma 1.2 we get (1), and if $a \equiv a_1 \wedge a_2$, using Lemma 1.3, we get (2).
If $b_{(r)}$ is not proper and $l(b)>1$, then if $b \equiv b_1 \wedge b_2$, using Lemma 1.2 we get (3) and if $b \equiv b_1 \vee b_2$, using Lemma 1.3, we get (4).
If $a^{(r)}$ is proper and $l(a)>1$, $l(b)=1$, we get (5).
If $b_{(r)}$ is proper and $l(b)>1$, $l(a)=1$, we get (5).

It is now easy to see that every case leads to one of the above mentioned possibilities and the theorem is proved.

Since for a given $a \in W(Q)$ there exists only a finite number of proper covers $a^{(r)}$, $a_{(r)}$, $r \in R$, the word problem is recursively solvable.

The word problem for a free product of lattices [6] is a corollary of Theorem 1.1.

The following result is a generalization of Theorem 4.1 from [9].

Let L_r, $r \in R$ be pairwise disjoint sublattices of the lattice L such that $Q = \cup(L_r; r \in R)$ (with the usual ordering) generates L, $[Q]=L$. Denote by T: $W(Q) \to L$ the function assigning to every word its value in L. Then T is an epimorphism and $T(x)=x$ for $x \in Q$. Homomorphisms T_r, T^r are defined similarly as for the poproduct.

Theorem 1.2. *L is isomorphic to $P(L_r; r \in R)$ if and only if for every p, $q \in R$, $x \in L_p$, $y \in L_q$, a, b, c, $d \in W(Q)$ the following conditions hold:*

(1) *$x \leqq y$ implies $p \leqq q$,*
(2) *$x \leqq T(a)$ if and only if $x \in T_p(a)$,*
(3) *$x \geqq T(a)$ if and only if $x \in T^p(a)$,*
(4) *$T(a \wedge b) \leqq T(c \vee d)$ if and only if at least one of the following five conditions holds: $T(a \wedge b) \leqq T(c)$, $T(a \wedge b) \leqq T(d)$, $T(a) \leqq T(c \vee d)$, $T(b) \leqq T(c \vee d)$, $T^r(a \wedge b) \cap T_r(c \vee d) \neq \emptyset$ for some $r \in R$.*

Proof. The proof is similar to that of the corresponding theorem from [9].

A poset satisfies the m — *chain condition* if it contains no chain of cardinality m (where m denotes an infinite cardinal). In [1] it was shown that the $\mathcal{V}$-free product (where $\mathcal{V} = \mathcal{L}$ or $\mathcal{D}$) preserves the m-chain condition for uncountable regular m.

First it was shown that the completely free lattice $CFL(P)$ preserves the m-chain condition for uncountable regular m. Then it was proved that for $\mathcal{V} = \mathcal{L}$ or $\mathcal{D}$, the $\mathcal{V}$-free product of a family $(L_i; i \in I)$ of lattices can be embedded into the completely free lattice generated by $\cup(L_i\,;\, i\in I)$. Especially Sorkin's theorem for the free product is an immediate consequence of this embedding.

Analogically, it can be shown that for $\mathcal{V}=\mathcal{L}$ or $\mathcal{D}$ the $\mathcal{V}$-poproduct $P_{\mathcal{V}}(L_r\,;\, r\in R)$ can be embedded into the completely free lattice generated by Q, where Q is the skeleton of the poproduct.

Hence $P_{\mathcal{V}}(L_r\,;\, r\in R)$, where $\mathcal{V}=\mathcal{L}$ or $\mathcal{D}$ preserves the m-chain condition for an uncountable regular m.

As an immediate consequence of this embedding we have the following

Theorem 1.3 *(the generalised Sorkin's theorem). Let $L=P(L_r\,;\, r\in R)$, let M be a lattice, let Q be a skeleton of L and let $i\colon Q\to L$ be the canonical embedding. Let $f\colon Q\to M$ be an isotone mapping. Then there exists (not necessarily uniquely) an isotone mapping $g\colon L\to M$ such that $g\circ i=f$.*

2. Minimal representation of the elements of poproduct

In this paragraph generalizations of the results of [13] are given.

Each element $a\in W(Q)$ represents an element $[a]\in L = P(L_r\,;\, r\in R) = W(Q)/\cong$. A polynomial $a\in W(Q)$ is said to be *minimal* if no shorter polynomial in $W(Q)$ represents $[a]$, we also say that a is a minimal representation of $[a]$ ([13]). A polynomial $a\in W(Q)$ is said to be a $\vee$*-polynomial* if $a\equiv b\vee c$ where $a\neq b$, $a\neq c$. The dual concept is a $\wedge$*-polynomial*. Thus each $a\in W(Q)$ is either a $\vee$-polynomial or a $\wedge$-polynomial ([13]).

Theorem 2.1. *Let Q be a skeleton of an L-poproduct $P(L_r\,;\, r\in R)$. Let $a\in W(Q)$.*

(a) *If $l(a)=1$, then a is minimal.*

(b) *If a is a $\vee$-polynomial and if $a\equiv a_0\vee\ldots\vee a_{k-1}$, $k>1$, with no a_i a $\vee$-polynomial, then a is minimal if and only if the following five conditions hold:*

(1) *each a_i, $i<k$ is minimal,*

(2) *for each $i<k$ $a_i\not\leqq a_0\vee\ldots\vee a_{i-1}\vee a_{i+1}\vee\ldots\vee a_{k-1}$,*

(3) *if $i<k$, $l(a_i)>$, then $M(a_i)\cap J(a)=\emptyset$,*

(4) *if $l(a_i)>1$, $a_i\equiv c\wedge d$, then $c\not\leqq a$ and $d\not\leqq a$,*

(5) *if i, $j<k$, p, $q\in R$, $p\leqq q$, $a_i\in L_p$, $a_j\in L_q$, then $p=q$ and $i=j$.*

Proof. The proof is similar to the proof of the corresponding theorem of [13]. Part (a) is clear. The necessity of the five conditions in part (b) can be established in a similar way to that [13]. We now establish their sufficiency. Let a satisfy these conditions and let $b\in W(Q)$ be a minimal polynomial such that $a=b$. We shall show that $l(a)=l(b)$.

We first show that $l(b)>1$. If $l(b)=1$, then there is a $q\in R$ such that $b\in L_q$. From $a\leqq b$ it follows that $b\in M(a)$. In the same way $a_i\leqq a$ infers $M(a)\subseteq M(a_i)$

and consequently $b \in M(a_i)$. From $b \leqq a$ it follows that $b \in J(a)$. Therefore $b \in M(a_i) \cap J(a)$ and $M(a_i) \cap J(a) \neq \emptyset$ for each $i < k$. By condition (3) $l(a_i) = 1$ holds for each $i < k$. Then by condition (5) the set $S = \{p: p \in R$ and there is an i, $i < k$, such that $a_i \in L_p\}$ is an antichain. For each $i < k$ there holds $a_i \leqq a \leqq b$, hence for $p \in S$ there is $p \leqq q$. Since $b \leqq a$, by the definition of $\leqq$ there exists $i < k$ such that $a_i \in L_p$, $p \in S$, $p \geqq q$, $b \leqq a_i$ in Q. Therefore $p = q$ and $b \leqq a_i$ in L. Since S is an antichain and for each $p \in S$ $p \leqq q$, the cardinality of S is 1, therefore $l(a) = 1$, a contradiction. Hence $l(b) > 1$.

We show next that b cannot be a $\wedge$-polynomial. If $b = b_0 \wedge b_1$, then, since $a \leqq b$, for each $i < k$ there holds $a_i \leqq b$. We have also $b \leqq a$. If $b \leqq a$ arises by condition (5) of theorem 1.1, then $M(b) \cap J(a) \neq \emptyset$, hence there exists an $u \in Q$ such that $b \leqq u \leqq a \leqq b$, therefore $b \equiv u$, which contradicts the minimality of b. Thus $b \leqq a$ arises either by (2) or by (4) of theorem 1.1, that means that either $b_j \leqq a$ for some $j < 2$, or $b \leqq a_{k-1}$, or $b \leqq a_0 \vee \ldots \vee a_{k-2}$. The inequality $b \leqq a_0 \vee \ldots \vee a_{k-2}$ cannot be derived by condition (5) of theorem 1.1, because $M(b) \cap J(a_0 \vee \ldots \vee a_{k-2}) \neq \emptyset$ and $J(a) \supseteqq J(a_0 \vee \ldots \vee a_{k-2})$ imply $M(b) \cap J(a) \neq \emptyset$, therefore the existence of a $u \in Q$ such that $b \leqq u \leqq a \leqq b$, hence $b \equiv u$, which contradicts the minimality of b. Continuing in this vein we find that either $b_j \leqq a$ for some $j < 2$ or $b \leqq a_i$ for some $i < k$. If $b_j \leqq a$, then $b \leqq b_j \leqq a \leqq b$, hence $b = b_j$, which contradicts the minimality of b. If $b \leqq a_i$, then $a_j \leqq a \leqq b \leqq a_i$ for $i \neq j$, which contradicts condition (2) of the present theorem.

Consequently, b is a $\vee$-polynomial, $b \equiv b_0 \vee \ldots \vee b_{n-1}$, $n \geqq 1$, where no b_j is a $\vee$-polynomial. We observe that conditions (1)—(5) of the present theorem hold for b, because it is a minimal polynomial. Suppose $i < k$ and $l(a_i) > 1$. There holds $a_i \leqq b$. Since a_i is a $\wedge$-polynomial and b is a $\vee$-polynomial, $a_i \equiv b$ cannot hold. If $a_i \leqq b$ arises by condition (5) of theorem 1.1, then $M(a_i) \cap J(b) \neq \emptyset$ and from $b \leqq a$ it follows that $J(b) \subseteqq J(a)$, therefore $M(a_i) \cap J(a) \neq \emptyset$, which is a contradiction with (3). If $a_i \leqq b$ arises by condition (2) of theorem 1.1, it contradicts condition (4). Therefore $a_i \leqq b$ is derived by condition (4) of theorem 1.1. Continuing the argument in this way we conclude that there is an $f(i) < n$ such that $a_i \leqq b_{f(i)}$. If $b_{f(i)} \in L_p$, $p \in R$, then from $a_i \leqq b_{f(i)}$ it follows that $M(a_i) \cap J(b_{f(i)}) \neq \emptyset$. Now $J(b_{f(i)}) \subseteqq J(b)$ and from $b \leqq a$ it follows that $J(b) \subseteqq J(a)$, hence $J(b_{f(i)}) \subseteqq J(a)$ and $M(a_i) \cap J(a) \neq \emptyset$, contradicting condition (3). Hence $l(b_{f(i)}) > 1$. Since b also satisfies conditions (1)—(5), we get that for each $j < n$ such that $l(b_j) > 1$ there is a $g(j) < k$ such that $b_j \leqq a_{g(j)}$, hence $g(f(i))$ exists and $a_i \leqq b_{f(i)} \leqq a_{g(f(i))}$. By condition (2) $g(f(i)) = i$ holds, and thus $a_i = b_{f(i)}$, by condition (1) $l(a_i) = l(b_{f(i)})$. Thus we have established the following statement:

($*$) *For each $i < k$ such that $l(a_i) > 1$ there is an $f(i) < n$ such that $b_{f(i)} = a_i$ and $l(b_{f(i)}) = l(a_i)$, and, similarly, for each $j < n$ such that $l(b_j) > 1$ there is a $g(j) < k$ such that $a_{g(j)} = b_j$ and $l(a_{g(j)}) = l(b_j)$; furthermore, $g(f(i)) = i$ and $f(g(j)) = j$.*

Now suppose $i < k$ and $l(a_i) = 1$, hence $a_i \in L_p$ for some $p \in R$. Since $a_i \leqq b$, there holds $T_p(b)\neq\emptyset$ and $T_p(a_t)\subseteqq T_p(b)$. By the definition of T_p we can suppose that $0<t\leqq n$ and $T_p(b_j)\neq\emptyset$ if and only if $j<t$. Hence $T_p(b) = T_p(b_0)\nabla\ldots\nabla T_p(b_{t-1})$, where ∇ denotes the lattice join in the lattice of all ideals of the lattice L_p. If $l(b_j)>1$ for all $j<t$, then by (*) there is $l(a_{g(j)})>1$ and $T_p(a_{g(j)}) = T_p(b_j)$ holds for all $j<t$. Hence $g(j)=i$ for all $j<t$. Now we have $T_p(b_0)\nabla \ldots \nabla T_p(b_{t-1}) = T_p(a_{g(0)})\nabla \ldots \nabla T_p(a_{g(t-1)}) \subseteqq T_p(a_0\vee \ldots \vee a_{i-1}\vee a_{i+1}\vee \ldots \vee a_{k-1})$. Thus $a_i \in T_p(a_i) \subseteqq T_p(b) \subseteqq T_p(a_0\vee \ldots \vee a_{i-1}\vee a_{i+1}\vee \ldots \vee a_{k-1})$, hence $a_i\leqq a_0\vee \ldots \vee a_{i-1}\vee a_{i+1}\vee \ldots \vee a_{k-1})$, contradicting condition (2). Consequently, there is an $f(i)<n$ such that $T_p(b_{f(i)})\neq\emptyset$ and $l(b_{f(i)})=1$, that is, $b_{f(i)} \in L_{F(p)}$, $F(p)\geqq p$. Similarly, if $j<n$ and $b_j\in L_q$ for some $q\in R$, then there exists $g(j)<k$ such that $a_{g(j)} \in L_{G(q)}$, $G(q)\geqq q$. By condition (5), $G(F(p))=p$, hence $F(p)=p$, $G(q)=q$ and $f(g(j))=j$ for each $j<n$ such that $l(b_j)=1$. Thus we have established the following statement:

(**) *There are mappings*

$$f\colon \{0, \ldots, k-1\}\to\{0, \ldots, n-1\},\quad g\colon \{0, \ldots, n-1\}\to\{0, \ldots, k-1\}$$

satisfying the conditions

(i) *if $i<k$, then $g(f(i))=i$ and if $j<n$, then $f(g(j))=j$,*
(ii) *if $i<k$ and $l(a_i)>1$, then $b_{f(i)}=a_i$ and $l(b_{f(i)}) = l(a_i)$ and similarly for any $j<n$ such that $l(b_j)>1$,*
(iii) *if $i<k$ and $a_i\in L_p$, $p\in R$, then $b_{f(i)}\in L_p$, and similarly for any $j<n$ such that $b_j\in L_q$, $q\in R$.*

Consequently, $k=n$ and f, g are permutations of the set $\{0, \ldots, k-1\}$. Since $l(a_i) = l(b_{f(i)})$ for all $i<k$, it follows that $l(a)=l(b)$. Since b is minimal, a is also minimal. Theorem 2.1 is proved.

If the set $\{u\colon u\in Q,\ a_i\leqq u\leqq a\}$ is finite for each $i<k$, then the proof of the necessity in theorem 2.1 provides an algorithm for reducing any polynomial to an equivalent minimal polynomial.

We now present an algorithm determining the case of the two minimal polynomials representing the same element of L.

Theorem 2.2. *Let Q be a skeleton of an L-poproduct $L=P(L_r;\, r\in R)$. Let $a, b\in W(Q)$ be minimal polynomials. If $l(a)=1$, then $a=b$ if and only if $a\equiv b$. If $a\equiv a_0\vee\ldots\vee a_{k-1}$, $k>1$, where no a_i is a $\vee$-polynomial, then $a=b$ if and only if b can be written in the form $b\equiv b_0\vee\ldots\vee b_{k-1}$ such that the following conditions hold:*

(1) *no b_i is a $\vee$-polynomial,*
(2) *for each $i<k$ and $p\in R$, $a_i\in L_p$ if and only if $b_i\in L_p$,*
(3) *for each $i<k$, $l(a_i)>1$ if and only if $l(b_i)>1$ and in this event $a_i=b_i$,*
(4) *for each $i<k$ and $p\in R$ from $a_i\in L_p$ it follows that $a_i\in T_p(b)$ and from $b_i\in L_p$ it follows that $b_i\in T_p(a)$.*

The proof is similar to the one of the corresponding theorem of [13].

In general, an element of L has several different minimal representations ([13]). In a special case we can choose one, well-defined up to commutativity and associativity, which we call the *normal representation.*

Suppose that for each $a \in L$, the ideal $T_p(a)$ and the dual ideal $T^p(a)$, if non-empty, are principal.

Definition 2.1. *If $a \in W(Q)$ and $l(a)=1$, then a is a normal polynomial. If $l(a)>1$, then a is normal if and only if the following two conditions hold:*

(1) *a is a minimal polynomial,*
(2) *if a is a $\vee$-polynomial, i.e. $a=a_0 \vee \ldots \vee a_{k-1}$, $k>1$ and no a_i is a $\vee$-polynomial, then each a_i is normal and if for some $i<k$ there is $a_i \in L_r$, then $T_r(a_i)=T_r(a)$. Dually for a $\wedge$-polynomial a.*

Theorem 2.3. *Let for each $a \in L$ be $T_r(a)$ and $T^r(a)$, if non-empty, principal for every $r \in R$. Then there holds:*

(1) *Each $x \in L$ has a normal representation,*
(2) *for each $x \in L$, its normal representation is unique up to commutativity and associativity.*

The proof is similar to the one of the corresponding theorem of [13].

The poproduct is said to admit *canonical representations* if a minimal representation of every element is unique up to commutativity and associativity ([13]). Under the assumptions of theorem 2.3, the poproduct admits canonical representations if and only if every minimal polynomial is normal.

3. Free-lattice-like sublattices of the poproduct of lattices

In this paragraph we generalize the results of [5]. We shall show that certain sublattices of the poproduct of lattices satisfy the same conditions as the sublattices of a free lattice.

A free lattice is known to satisfy the following conditions ([5]):

(*F*) *if $x \wedge y \leqq u \vee v$, then one of the following four possibilities occurs: $x \leqq u \vee v$, $y \leqq u \vee v$, $x \wedge y \leqq u$, $x \wedge y \leqq v$;*
(*F2*) *if $u=x \vee y=x \vee z$, then $u=x \vee (y \wedge z)$;*
(*F3*) *if $u=x \wedge y=x \wedge z$, then $u=x \wedge (y \vee z)$.*

Suppose that for every $r \in R$ the lattice L_r contains the greatest element i_r and the smallest element o_r. Hence for every nonempty ideal $T_r(a)$ there exists its generator a_r and for every nonempty filter $T^r(a)$ there exists its generator a^r.

Lemma 3.1. *Let Q be a skeleton of a poproduct $P(L_r\,;\, r \in R)$ of the bounded* *lattices* L_r, $r \in R$. *Let* $a, b, c, d \in W(Q)$. *Then* $a \wedge b \leqq c \vee d$ *if and only if one of the following conditions holds:*

(1) *there are* $p, q \in R$ *such that there exist* $a^p, b^p \in L_p$, $c_q, d_q \in L_q$ *and* $a^p \wedge b^p \leqq c_q \vee d_q$;
(2) $a \wedge b \leqq c$ or $a \wedge b \leqq d$;
(3) $a \leqq c \vee d$ or $b \leqq c \vee d$.

Proof. The necessity of the conditions. By the solution of the word problem we can restrict ourselves to the case that neither (2) nor (3) holds and there are p, $q \in R$ $(p \leqq q)$ such that $(a \wedge b)^{(p)}$, $(c \vee d)_{(q)}$ exist and $(a \wedge b)^{(p)} \leqq (c \vee d)_{(q)}$. By the definition of covers there are the following possibilities for the covers $(a \wedge b)^{(p)}$ and $(c \vee d)_{(q)}$, respectively: The $(a \wedge b)^{(p)}$ equals one of the following: $a^{(p)} \wedge b^{(p)}$, $a^{(p)}$, $b^{(p)}$. The $(c \vee d)_{(q)}$ equals one of the following: $c_{(q)} \vee d_{(q)}$, $c_{(q)}$, $d_{(q)}$. Now if $(a \wedge b)^{(p)} = a^{(p)}$ (or $b^{(p)}$), then $a \leqq a^{(p)} = (a \wedge b)^{(p)} \leqq (c \vee d)_{(q)} = c \vee d$ (or $b \leqq c \vee d$), hence (3) holds, which contradicts the assumption. If $(c \vee d)_{(q)} = c_{(q)}$ (or $d_{(q)}$), then $c \geqq c_{(q)} = (c \vee d)_{(q)} \geqq (a \wedge b)^{(p)} \geqq a \wedge b$ (or $d \geqq a \wedge b$), hence (2) holds, a contradiction. Therefore, $(a \wedge b)^{(p)} = a^{(p)} \wedge b^{(p)}$ and $(c \vee d)_{(q)} = c_{(q)} \vee d_{(q)}$. Consequently $a^p \wedge b^p = a^{(p)} \wedge b^{(p)} < c_{(q)} \vee d_{(q)} = c_q \vee d_q$.

The sufficiency of the conditions: If $a^p \wedge b^p \leqq c_q \vee d_q$, then from $a \leqq a^p$, $b \leqq b^p$, $c_q \leqq c$, $d_q \leqq d$ there follows $a \wedge b \leqq a^p \wedge b^p \leqq c_q \vee d_q \leqq c \vee d$. The lemma is proved.

Lemma 3.2. *Let* Q *be a skeleton of a poproduct* $P(L_r; r \in R)$. *Let* $u \in W(Q)$. *Then* u *can be written as* $u = u_0 \vee \ldots \vee u_{n-1}$, $n \geqq 1$, *where* u_j, $j < n$ *satisfy the following conditions:*

(1) *if* $u_j \notin \cup(L_r; r \in R)$, *then* $u_j = a_j \wedge b_j$ for some $a_j, b_j \in W(Q)$ *with* $u_j < a_j$, $u_j < b_j$;
(2) *for each* $j < n$, $n > 1$ *there holds* $u_j \neq u_0 \vee \ldots \vee u_{j-1} \vee u_{j+1} \vee \ldots \vee u_{n-1}$;
(3) *if* $j < n$ *and* $u_j \notin \cup(L_r; r \in R)$, *then* $M(u_j) \cap J(u) = \emptyset$;
(4) *if for* $j < n$ *we have* $u_j = a \wedge b$, *where* $a, b > u_j$, *then* $a \nleqq u$, $b \nleqq u$;
(5) *if* $j, k < n$, $p, q \in R$, $p \leqq q$, $u_j \in L_p$, $u_k \in L_q$, *then* $p = q$ *and* $j = k$.

Proof. Let $u \in W(Q)$. If $u \in Q$ or u is a $\wedge$-polynomial, we put $n = 1$ and $u_0 = u$. If u is a $\vee$-polynomial, we take its minimal representation $u = u_0 \vee \ldots \vee u_{n-1}$, $n > 1$ (theorem 2.1).

Theorem 3.1. *Let* L *be a poproduct of the bounded lattices* L_r; $r \in R$. *For each* $r \in R$ *let* K_r *be a sublattice of* L_r *and let* K *be a sublattice of* L *such that for each* $a \in K$, $r \in R$ *if* a_r *exists, then* $a_r \in K_r$ *and if* a^r *exists, then* $a^r \in K^r$. *Let* $n \in \{1, 2, 3\}$. *If, for all* $r \in R$, *the sublattice* K_r *satisfies* (Fn), *then the sublattice* K *satisfies* (Fn).

Proof.

1. $n = 1$. Let all K_r satisfy $(F1)$. We shall show that K also satisfies $(F1)$. Let $a, b, c, d \in K$, $a \wedge b \leqq c \vee d$. By lemma 3.1 one of the conditions (1), (2), (3) holds. If (2)

or (3) holds, the proof is accomplished. If (1) holds, then $a^p \wedge b^p \leqq c_q \vee d_q$ for some $p, q \in R$, $p \leqq q$. There are two cases:

First, if $p < q$, then $a \leqq a^p < c_q \leqq c$, $a \leqq a^p < d_q \leqq d$, $b \leqq b^p < c_q \leqq c$, $b \leqq b^p < d_q \leqq d$, this implies that $a \wedge b \leqq c$, $a \wedge b \leqq d$, $a \leqq c \vee d$, $b \leqq c \vee d$.
Second by, if $p = q$, then $a^p \wedge b^p \leqq c_p \vee d_p$ in $K_p \sqsubseteq L_p$ and because K_p satisfies $(F1)$, at least one of the following holds:

$a \leqq a^p \leqq c_p \vee d_p \leqq c \vee d$, $\quad b \leqq b^p \leqq c_p \vee d_p \leqq c \vee d$, $\quad a \wedge b \leqq a^p \wedge b^p \leqq c_p \leqq c$, $\quad a \wedge b \leqq a^p \wedge b^p \leqq d_p \leqq d$.

2. $n = 2$. Let all K_r satisfy $(F2)$. We shall show that K also satisfies $(F2)$. Let x, y, z, $u \in K$, let $x \vee y = x \vee z = u$. It is enough to prove that $u \leqq x \vee (y \wedge z)$. By lemma 3.2 the element u can be written in the form $u = u_0 \vee \ldots \vee u_{n-1}$, $n \geqq 1$, where u_j, $j < n$ satisfy the conditions (1)—(5) of lemma 3.2. We shall show that for each $j < n$ there holds $u_j \leqq x \vee (y \wedge z)$. There are two possibilities: $u_j \notin Q$ or $u_j \in L_p$ for some $p \in R$.

Let $j < n$, $u_j \in L_p$, $p \in R$. Then $u_j \in T_p(u) = (u_p]$, $u_j \leqq u_p$.
For $u = x \vee y$ there are three possibilities:

$$u_p = \begin{cases} x_p & \text{if} \quad T_p(x) \neq \emptyset, \; T_p(y) = \emptyset; \\ y_p & \text{if} \quad T_p(x) = \emptyset, \; T_p(y) \neq \emptyset; \\ x_p \vee y_p & \text{if} \quad T_p(x) \neq \emptyset, \; T_p(y) \neq \emptyset. \end{cases}$$

For $u = x \vee z$ there are three possibilities:

$$u_p = \begin{cases} x_p & \text{if} \quad T_p(x) \neq \emptyset, \; T_p(z) = \emptyset; \\ z_p & \text{if} \quad T_p(x) = \emptyset, \; T_p(z) \neq \emptyset; \\ x_p \vee z_p & \text{if} \quad T_p(x) \neq \emptyset, \; T_p(z) \neq \emptyset. \end{cases}$$

If $u_p = x_p$, then $u_j \leqq u_p = x_p \leqq x \leqq x \vee (y \wedge z)$.

If $u_p = y_p$, then $T_p(x) = \emptyset$, therefore $T_p(z) \neq \emptyset$ and $u_p = z_p$. It implies $u_j \leqq u_p = y_p \leqq y$, $u_j \leqq u_p = z_p \leqq z$, hence $u_j \leqq y \wedge z = x \vee (y \wedge z)$.

If $u_p = z_p$, then, similarly to the preceding case, we get $u_j \leqq x \vee (y \wedge z)$.

If $u_p = x_p \vee y_p$, then $T_p(x) \neq \emptyset$ and there holds either $u_p = x_p$, which implies $u_j \leqq x \vee (y \wedge z)$, or $u_p = x_p \vee z_p$, which implies $u_p = x_p \vee y_p = x_p \vee z_p$ in K_p and by the assumption concerning K_p we have now $u_j \leqq u_p \leqq x_p \vee (y_p \wedge z_p) \leqq x \vee (y \wedge z)$.

If $u_p = x_p \vee z_p$, then, similarly to the preceding case, we get $u_j \leqq x(y \wedge z)$.

Let $j < n$, $u_j \notin Q$. Applying lemma 3.2 we conclude that there exist $a, b \in L$ such that $a > u_j$, $b > u_j$ and $a \wedge b = u_j$. Now $a \wedge b = u_j \leqq x \vee y$, hence $a \wedge b \leqq x \vee y$ and by lemma 3.1 there are three possibilities:

If $a \leqq x \vee y$ or $b \leqq x \vee y$, we get a contradiction to (4) of lemma 3.2

If $a^p \wedge b^p \leqq x_q \vee y_q$, then $M(u_j) \supseteqq T^p(u_j) = [a^p \wedge b^p)$, $J(u) \supseteqq T_q(u) = (x_q \vee y_q]$, $p \leqq q$, hence $T^p(u_j) \cap T_q(u) \neq \emptyset$, therefore $M(u_j) \cap J(u) \neq \emptyset$, a contradiction to (3) of lemma 3.2.

Therefore the third case must hold, that is, either $a \wedge b \leqq y$ or $a \wedge b \leqq x$. Then either $u_j \leqq x$ or $u_j \leqq y$ and, similarly, either $u_j \leqq x$ or $u_j \leqq z$ Hence $u_j \leqq x \vee (y \wedge z)$

We have shown that for each $j < n$ there holds $u_j \leqq x \vee (y \wedge z)$ and therefo e $u \leqq x \vee (y \wedge z)$.

3. $n = 3$. The case of $(F3)$ is dual to the case of $(F2)$. The theorem is proved.

Corollary 1. *The poproduct of bounded distributive lattices satisfies $(F2)$ a d $(F3)$.*

Proof. Any distributive lattice satisfies $(F2)$ and $(F3)$.

Corollary 2. *Let L be a poproduct $P(L_r; r \in R)$. If K is a sublattice of the poproduct L such that for each $a \in K$ and each $r \in R$, if a_r exists, then $a = o_r$ and if a^r exists, then $a^r = i_r$, then K satisfies all $(F\text{n})$.*

Proof. For each $r \in R$ there is $K_r \subseteqq \{o_r, i_r\}$ and therefore satisfies all (Fn)

4. Poproduct decomposition of a lattice

In this section we generalise for the case of poproduct the results of [7] about a common refinement of any two representations of a lattice as a free $\mathcal{K}$-product.

Susspose that the equational class $\mathcal{K}$ of lattices satisfies the following property:

(J) *If L is a K-poproduct of the lattices $(L_r, r \in R)$, A_r is a sublattice of the lattice L_r for each $r \in R$ and A is the sublattice of L generated by $\cup(A_r; r \in R)$, then A is the $\mathcal{K}$-poproduct of lattices $(A_r, r \in R)$*

Let R, S be partially ordered sets. Let $(A_r, r \in R)$, $(B_s, s \in S)$ be two systems of pairwise disjoint lattices. Let $L = P_{\mathcal{K}}(A_r; r \in R) = P_{\mathcal{K}}(B_s; s \in S)$. We shall show that in such a case $L = P_{\mathcal{K}}(A_r \cap B_s; \langle r, s \rangle \in R \times S)$, where the set $R \times S$ is partially ordered in the following way: $\langle r_1, s_1 \rangle \leqq \langle r_2, s_2 \rangle$ is and only if $r_1 \leqq r_2$ and $s_1 \leqq s_2$. Since $(A_r, r \in R)$, $(B_s, s \in S)$ can be considered as a family of pairwise disjoint sublattices of the lattice L respectively, this ordering is well defined.

We introduce some notations. If p is a lattice polynomial symbol, then denote by $\tilde{p}$ the polynomial symbol arising from p in the way that the symbols $\vee$, $\wedge$ will be replaced by ∇, Δ, respectively (∇, Δ denote the lattice operations in the lattice of ideals and dual ideals, respectively). If L_1, L_2 are two subsets of the $\mathcal{K}$-poproduct, then $L_1 < L_2$ will denote that for the ideals $(L_1]$,$(L_2]$,$(L_1] \subseteqq (L_2]$ holds. Especially, $L_1 \leqq L_2$ denotes that $l_1 \leqq l_2$ for each pair $l_1 \in L_1$, $l_2 \in L_2$.

Theorem 4.1. *Let $L=P_{\mathcal{K}}(A_r\,;\,r\in R) = P_{\mathcal{K}}(B_s\,;\,s\in S)$ and let $\mathcal{K}$ satisfy the condition (J). Then $L=P_{\mathcal{K}}(A_r\cap B_s\,;\,\langle r,s\rangle \in R\times S)$. Moreover, for $r\in R$, $A_r= P_{\mathcal{K}}(A_r\cap B_s\,;\,s\in S)$ and for $s\in S$, there is $B_s = P_{\mathcal{K}}(A_r\cap B_s\,;\,r\in R)$.*

To prove the theorem, two lemmas will be needed.

Lemma 1. *If $a\in A_r$ and the lower cover a_s of a in B_s is proper, then $a_s\in A_r\cap B_s$.*

Proof. Let $a\in A_r$, let a_s be proper. As L is generated by the set $\cup(B_s\,;\,s\in S)$, the element a can be written in the form

(1) $a=p(b_{s_1,1},\ \dots,\ b_{s_1,n_1},\ \dots,\ b_{s_k,1},\ \dots,\ b_{s_k,n_k})$, where p is an $(n_1+\dots+n_k)$-ary polynomial, $s_1,\ \dots,\ s_k\in S$ and $b_{s_h,m}\in B_{s_h}$ for $h=1,\ \dots,\ k$, $1\leqq m\leqq n_h$. Therefore

(2) $T_r(a)\cong p(T_r(b_{s_1,1}),\ \dots,\ T_r(b_{s_k,n_k}))$, where $T_r(a)=J(a)\cap A_r$.

Without loss of generality we can suppose that $s=s_1$, then from (1) we get

(3) $T_s(a_{(s)}) = T_s(a) = \tilde{p}(T_s(b_{s_1,1}),\ \dots,\ T_s(b_{s_1,n_1}),\ T_s(b_{s_2,1}),\ \dots,\ T_s(b_{s_k,n_k}))$, where $b_{s_1,1},\ \dots,\ b_{s_1,n_1}\in B_s$. Let us consider now the expressions $T_s(b_{s_i,m})$ for $s_i\neq s(=s_1)$. The following holds:

(i) If $s\nleqq s_i$ in S, hence if $B_s\nleqq B_{s_i}$, then $T_s(b_{s_i,m})=\emptyset$,

(ii) If $s\leqq s_i$ in S, hence if $B_s\leqq B_{s_i}$, then $T_s(b_{s_i,m})=B_s$.

(Note that in both cases $b_{s_i,m}\notin B_s$.)

If $U\subseteq L$, denote $T_s(U)=\cup(T_s(u)\,;\,u\in U)$.

As $T_s(b)$ is an isotone function of its argument b, there holds that $T_s(a) = T_s(T_r(a))$ and from (2) we get

(4) $T_s(a_{(s)}) = T_s(T_r(a))=\tilde{p}(T_s(T_r(b_{s_1,1})),\ \dots,\ T_s(T_r(b_{s_k,n_k})))$.

Moreover, the following holds:

(iii) If $s\nleqq s_i$ in S, hence if $B_s\nleqq B_{s_i}$, then $B_s\nless T_r(b_{s_i,m})$.

Since if there were $B_s<T_r(b_{s_i,m})$, it would imply $T_r(b_{s_i,m})<B_{s_i}$ and hence by transitivity $B_s<B_{s_i}$, a contradiction.)

Now from $B_s\nless T_r(b_{s_i,m})$ it follows that $T_s(T_r(b_{s_i,m}))=\emptyset$ (because $b\in T_s(T_r(B_{s_i,m}))$ would imply $b\leqq b_{s_i,m}$, $b\in B_s$, contradicting $B_s\nleqq B_{s_i}$).

(iv) $T_r(b_{s_i,m})<T_{s_i}(b_{s_i,m})$ and $s\leqq s_i$ in S yield

$T_s(T_r(b_{s_i,m}))<T_s(T_{s_i}(b_{s_i,m}))=T_s(b_{s_i,m}))$.

Now the following inequalities hold:

$$
(5)\qquad
\begin{aligned}
T_{s_i}(b_{s_i,m})>T_r(b_{s_l,m})>T_s(T_r(b_{s_1,m}))\\
T_s(b_{s_i,m})>X_{s_i,m}>T_s(T_r(b_{s_i,m}))\ \text{ for }\ s_i\neq s_1,
\end{aligned}
$$

where $X_{s_i,m}$ will be suitable defined as follows:

(v) $X_{s_i,m}=\emptyset$ if $T_s(b_{s_i,m}))=\emptyset$.

See (i). Evidently $T_r(b_{s_i,m})<T_{s_i}(b_{s_i,m})$ infers $T_s(T_r(b_{s_i,m}))) < T_s(b_{s_i,m}))$ and (5) is correct.

(vi) $X_{s_i,m}=B_s$ for $T_s(b_{s_i,m})=B_s$.

Now from (3) and (4), using (5), we get

$$(6)\qquad \begin{aligned} T_s(a_{(s)}) &= \tilde{p}(T_s(b_{s_1,1}), \ldots, T_s(b_{s_1,n_1}), T_s(b_{s_2,1}), \ldots, T_s(b_{s_k,n_k})) > \\ &> \tilde{p}(T_r(b_{s_1,1}), \ldots, T_r(b_{s_l,n_l}), X_{s_2,1}, \ldots, X_{s_k,n_k}) > \\ &> \tilde{p}(T_s(T_r(b_{s_1,1}), \ldots, T_s(T_r(b_{s_k,n_k}))) = T_s(a_{(s)}). \end{aligned}$$

From (6) we obtain

$$(7)\qquad T_s(a_{(s)}) = \tilde{p}(T_t(b_{s_1,1}), \ldots, T_r(b_{s_l,n_l}), X_{s_2,1}, \ldots, X_{s_k,n_k}),$$

where $X_{s_i,m}$ is either $\emptyset$ or B_s.
By proposition 1.2 there follows the existence of the polynomial symbol q such that

$$(8)\qquad a_{(s)} = q((b_{s_{l_1}})_{(r)}, \ldots, (b_{s_{l_v}})_{(r)}),$$

where $b_{s_{l_1}}, \ldots, b_{s_{l_v}}$, $l_v \leqq n_l$ are such from among the $b_{s_1,1}, \ldots, b_{s_1,n_1}$ for which there exist their lower covers in the lattice A_r. Since $b_{s_{l_t}} \in A_r$ for $t = 1, \ldots, v$ by (8) there is also $a_{(s)} \in A_r$ and because by definition of $a_{(s)}$, $a_{(s)} \in B_s$, we now have $a_{(s)} \in A_r \cap B_s$. Lemma 1 is proved.

Lemma 2 *(the „associativity" of the poproduct). Let S be a partially ordered set. Let there be R_s a partially ordered set for each $s \in S$. Assume the sets R_s to be pairwise disjoint. Denote by $R = \cup(R_s\,;\, s \in S)$ a partially ordered set with the following ordering: $r_1 \leqq r_2$ holds for $r_1, r_2 \in R$ if at least one of the following conditions hold:*
1. there is an $s \in S$ such that $r_1, r_2 \in R_s$ and $r_1 \leqq r_2$ in R_s;
2. there are $s_1, s_2 \in S$ such that $r_1 \in R_{s_1}$, $r_2 \in R_{s_2}$ and $s_1 < s_2$.
Let for each $r \in R$ be $L_r \in \mathcal{K}$ and let L_r, $r \in R$ be pairwise disjoint lattices. Then the poproducts $P_{\mathcal{K}}(L_r\,;\, r \in R)$ and $P_{\mathcal{K}}(P_{\mathcal{K}}(L_r\,;\, r \in R_s)\,;\, s \in S)$ are isomorphic.

Proof. The idea how to prove the lemma is as follows:
Let Q denote the skeleton of $P_{\mathcal{K}}(L_r\,;\, r \in R)$. We show first that Q can be embedded into $P_{\mathcal{K}}(P_{\mathcal{K}}(L_r\,;\, r \in R_s)\,;\, s \in S)$. Then, in the second step, let $M \in \mathcal{K}$ and let $f\colon Q \to M$ be an isotone mapping. We prove that there exists a unique homomorphism $h\colon P_{\mathcal{K}}(P_{\mathcal{K}}(L_r\,;\, r \in R_s)\,;\, s \in S) \to M$ such that $f = h/Q$. Therefore, $P_{\mathcal{K}}(L_r\,;\, r \in R) \cong P_{\mathcal{K}}(P_{\mathcal{K}}(L_r\,;\, r \in R_s)\,;\, s \in S)$, because the poproduct is defined uniquely (up to isomorphism).

Let $Q_s = \cup(L_r\,;\, r \in R_s)$ be a skeleton of $P_{\mathcal{K}}(L_r\colon r \in R_s)$, $s \in S$. Clearly, $Q = \cup(Q_s\,;\, s \in S)$ is a skeleton of $P_{\mathcal{K}}(L_r\,;\, r \in R)$. Suppose $Q' = \cup(P_{\mathcal{K}}(L_r\,;\, r \in R_s)\,;\, s \in S)$ to be a skeleton of $P_{\mathcal{K}}(P_{\mathcal{K}}(L_r\,;\, r \in R_s)\,;\, s \in S)$. Since Q_s can be embedded into $P_{\mathcal{K}}(L_r\,;\, r \in R_r)$, we see that Q can be embedded into $P_{\mathcal{K}}(P_{\mathcal{K}}(L_r\,;\, r \in R_s)\,;\, s \in S)$. Now let $f\colon Q \to M$ be an isotone mapping such that $f_r\colon L_r \to M$ is a homomorphism. It is easy to see that there exists a unique homomorphism $g_s\colon P_{\mathcal{K}}(L_r\,;\, r \in R_s) \to M$ such that $g_s/Q_s = f/Q_s$ for every $s \in S$. Evidently, $s_1 < s_2$ in S implies $t_1 \leqq t_2$ for $t_i \in f(Q_{s_i})$, $i = 1, 2$ by definition of the mapping f. Therefore, $t_1 \leqq t_2$ for $t_i \in [f(Q_{s_i})]_M$,

$i=1,2$, where $[A]_M$ means the sublattice of M generated by $A \subseteqq M$. Hence, $g\colon Q' \to M$ is an isotone mapping such that $g/P_{\mathscr{K}}(L_r;\, r \in R_s) = g_s$. Thus there exists a unique homomorphism $h\colon P_{\mathscr{K}}(P_{\mathscr{K}}(L_r;\, r \in R_s);\, s \in S) \to M$ such that $h/Q' = g$ and consequently $h/Q = f$ and the proof of lemma is complete.

Proof of theorem 4.1. Let $L = P_{\mathscr{K}}(A_r;\, r \in R) = P_{\mathscr{K}}(B_s;\, s \in S)$. Using the definition of lower covers and the fact that $a \in A_r$, i.e. $a = a_{(r)}$ we get from (2) the existence of the polynomial symbol o such that

$$a = o((b_{f1})_{(r)}, \ldots, b_{f_m})_{(r)}), \tag{9}$$

where $b_{f_i} \in B_{f_i}$ for $i=1, \ldots, m$; $m \leqq n_1 + \ldots + n_k$ and b_{f_i} are such from among the $b_{s_1,1}, \ldots, b_{s_k,n_k}$, for which there exist $(b_{f_i})_{(r)}$. By lemma 1 $(b_{f_i})_{(r)} \in B_{f_i} \cap A_r$ holds for $i=1, \ldots, m$. From (9) it follows now that a belongs to the sublattice of L generated by the set $\cup(A_r \cap B_s;\, s \in S)$ for each $a \in A_r$. Therefore A_r is generated by the set $\cup(A_r \cap B_s;\, s \in S)$. Since $(A_r \cap B_s,\, s \in S)$ are sublattices of A_r, using the property (J) we get that $A_r = P_{\mathscr{K}}(A_r \cap B_s;\, s \in S)$. Then by lemma 2 it follows that $L = P_{\mathscr{K}}(A_r \cap B_s;\, \langle r, s \rangle \in R \times S)$ and theorem 1 is proved.

5. Poproduct and direct (inverse) limits of lattices

The aim of the present section is to investigate the interchangeability of the operators of poproduct and of the limit of lattices. In the case of the direct limit these operators are interchangeable, in the case of the inverse limit a weaker result is obtained.
(For the definition of direct and inverse limit see [4]).

Poproduct and direct limit

Let R be a partially ordered set, J a directed partially ordered set and let for each pair $r \in R$, $j \in J$ A_{rj} be a lattice. Let the lattices A_{rj} be pairwise disjoint. Denote by Q_j the poset $\cup(A_{rj};\, r \in R)$, where $a \leqq b$ in Q_j iff $a \leqq b$ in some A_{rj} or $a \in A_{rj}$, $b \in A_{sj}$ and $r < s$ in R. The system $\{Q_j,\, j \in J\}$ becomes directed if there are isotone mappings $\varphi_{ij}\colon Q_i \to Q_j$ for any $i \leqq j$ such that φ_{ij}/A_{ri} are homomorphisms, $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ and $\varphi_{ii} = \mathrm{id}$ for every $i \in J$. Now we can define $P(A_{rj};\, r \in R)$ for every $j \in J$ and this system becomes directed because any $\varphi_{ij}\colon Q_i \to Q_j$ can be uniquely prolonged into a homomorphism $\bar{\varphi}_{ij}\colon P(A_{ri};\, r \in R) \to P(A_{rj};\, r \in R)$ (see definition 1.1). It is not difficult to verify that $\bar{\varphi}_{jk} \circ \bar{\varphi}_{ij} = \bar{\varphi}_{ik}$ if $i \leqq j \leqq k$ and $\bar{\varphi}_{ii} = \mathrm{id}$ for every $i \in J$. This leads to a direct limit $L_{\rightarrow}P = L_{\rightarrow}(P(A_{rj};\, r \in R);\, j \in J)$.

By [4] there exists for each $r \in R$ the direct limit $A_r = L_{\rightarrow}(A_{rj};\, j \in J)$. Since the lattices A_r are pairwise disjoint, wo can define a poset $Q = \cup(A_r;\, r \in R)$ such that $a \leqq b$ in Q iff $a \leqq b$ in some A_r or $a \in A_r$ and $b \in A_s$ for some $r < s$ in R. This enables us to define $L_{\rightarrow} = P(L_{\rightarrow}(A_{ri};\, i \in J);\, r \in R)$.

Before studying the connections between the lattices $PL_\to$ and $L_\to P$ we need the following observation (using the notation from [4]).

Lemma 5.1. *Let $(A_i\,;\ \varphi_{ij},\ i\in J)$ and $(B_i\,;\ \psi_{ij},\ i\in J)$ be directed families of algebras of the same type. Let for every $i\in J$, $h_i\colon\ A_i\to B_i$ be a homomorphism such that any diagram*

$$\begin{tikzcd} A_i \arrow[r, "h_i"] \arrow[d, "\varphi_{ij}"'] & B_i \arrow[d, "\psi_{ij}"] \\ A_j \arrow[r, "h_j"'] & B_j \end{tikzcd}$$

commutes for $i\leqq j$. Then there exists a homomorphism $h\colon\ L_\to A_i\to L_\to B_i$ defined by the rule $h(\bar a)\ =\ \overline{(h_i(a))}$, where $a\in\bar a$ and $a\in A_i$. Moreover, the diagram

$$\begin{tikzcd} A_i \arrow[r, "h_i"] \arrow[d, "\varphi_{i\infty}"'] & B_i \arrow[d, "\psi_{i\infty}"] \\ L_\to A_i \arrow[r, "h"'] & L_\to B_i \end{tikzcd}$$

commutes for every $i\in J$.

The proof is straightforward (see [4]).

Theorem 5.1. *Let $\mathcal{K}$ be an arbitrary equational class of lattices. Then the lattices $P_K(L_\to(A_{rj}\,;\ j\in J)\,;\ r\in R)$ and $L_\to(P_k(Ar_j\,;\ r\in R)\,;\ j\in J)$ are isomorphic.*

Proof. Suppose $i_j\colon\ Q_j\to P(A_{rj}\,;\ r\in R)$ to be a canonical embedding for $j\in J$. By lemma 5.1 there exists an isotone mapping $h\colon\ Q\to L_\to P$ such that h/A_r is a homomorphism. The diagram

$$\begin{tikzcd} Q_i \arrow[r] \arrow[d, "\varphi_{ij}"'] & P(A_{ri}\,;\ r\in R) \arrow[d, "\psi_{ij}"] \\ Q_j \arrow[r] & P(A_{rj}\,;\ r\in R) \end{tikzcd}$$

commutes for $i\leqq j$, hence we can apply lemma 5.1.

Now the upper part of the diagram 1

$$\begin{tikzcd} Q_j \arrow[r, "i_j"] \arrow[d, "\varphi_{j\infty}"'] & P(A_{rj}\,;\ r\in R) \arrow[r, "g_j"] \arrow[d, dash, "\bar\varphi_{j\infty}"] & PL_\to \arrow[d, "\mathrm{id}"] \\ Q \arrow[r, "h"] \arrow[d, "\mathrm{i}"'] & L_\to P \arrow[r, "\bar g"] \arrow[d, dash, "\mathrm{id}"'] & PL_\to \arrow[d, "\mathrm{id}"'] \\ PL_\to \arrow[r, "\bar h"] & L_\to P \arrow[r, "\bar g"] & PL_\to \end{tikzcd}$$

diagram 1

commutes. This fact and the definition 1.1 imply the existence of a homomorphism $\bar{h}\colon PL_{\rightarrow} \rightarrow L_{\rightarrow}P$ such that the lower part of the diagram 1 commutes. Conversely, starting with

$$Qj \xrightarrow{\varphi_{j\infty}} Q \xrightarrow{\ \mathrm{i}\ } [Q] = PL_{\rightarrow},$$

we obtain by definition 1.1 and lemma 3.1 a commuting diagram

$$\begin{tikzcd}
Q_j \arrow[r, "i \circ \varphi_{j\infty}"] \arrow[d, "i_j"'] & PL_{\rightarrow} \arrow[d, "\mathrm{id}"] \\
P(A_{rj}\,;\, r \in R) \arrow[r, "g_j"] \arrow[d, "\bar{\varphi}_{j\infty}"'] & PL_{\rightarrow} \arrow[d, "\mathrm{id}"] \\
L_{\rightarrow}P \arrow[r, "\bar{g}"] & PL_{\rightarrow}
\end{tikzcd}$$

diagram 2

which says that there exists a homomorphism $\bar{g}\colon L_{\rightarrow}P \rightarrow PL_{\rightarrow}$. It remains to be proved that $\bar{h}\circ\bar{g} = \mathrm{id}$ and $\bar{g}\circ\bar{h} = \mathrm{id}$. In order to show that $\bar{g}\circ\bar{h} = \mathrm{id}$ is enough to establish $\bar{g}\circ\bar{h}(i(Q)) = \mathrm{id}$. It is known that for every $\bar{a} \in Q$ there exists $a \in Q_j$ for some $j \in J$ such that $\varphi_{j\infty}(a) = \bar{a}$. Now, $\bar{g}\circ\bar{h}\circ\mathrm{i}\circ\varphi_{j\infty}(a) \; = \; g_j \circ i_j(a)$ (see diagram 1). Since $i \circ \varphi_{j\infty} = g_j \circ i_j$ (see diagram 2), we get $\bar{g}\circ\bar{h}(i(Q)) = \mathrm{id}$ and whence $\bar{g}\circ\bar{h} = \mathrm{id}$.

Take $\bar{a} \in L_{\rightarrow}P$. There exists $a \in Q_j$ for some $j \in J$ such that $\bar{h}\circ\mathrm{i}\circ\varphi_{j\infty}(a) = h\circ\varphi_{j\infty}(a) = \bar{a}$. However $\bar{\varphi}_{j\infty}\circ i_j \; = \; \bar{h}\circ\mathrm{i}\circ\varphi_{j\infty}$. Since $\mathrm{i}\circ\varphi_{j\infty} \; = \; g_j\circ i_j$, we have $\bar{h}\circ\mathrm{i}\circ\varphi_{j\infty} = \bar{h}\circ g_j\circ i_j \; = \; \bar{\varphi}_{j\infty}\circ i_j$. Therefore, $\bar{\varphi}_{j\infty} = \bar{h}\circ g_j$ by definition 1.1. On the other hand $\bar{h}\circ g_j\circ i_j \; = \; \bar{h}\circ \bar{g}\circ\bar{\varphi}_{j\infty}\circ i_j$. Hence $\bar{h}\circ\bar{g}(\bar{\varphi}_{j\infty}(i_j(a))) \; = \; (\bar{\varphi}_{j\infty}\circ i_j(a)) \; = \; \bar{a}$. Thus $(\bar{h}\circ\bar{g})\,(\bar{a}) = \bar{a}$ and $\bar{h}\circ\bar{g} = \mathrm{id}$. The proof is complete.

Corollary. *The operators of the free product of lattices and of the direct limit are interchangeable* (in the sense of theorem 3.1).

Poproduct and inverse limit

Suppose as above $(J, \leqq)$ to be a directed poset, $(R, \leqq)$ a poset and $\{A_{rj}\,;\, r \in R, j \in J\}$ a family of pairwise disjoint lattices. We can form the poset $Q_j = \cup(A_{rj}\,;\, r \in R)$ for every $j \in J$. The family $\{Q_j, j \in J\}$ will become inverse if there exist isotone mappings $\varphi_j^i\colon Q_i \rightarrow Q_j$ for any $i \geqq j$ such that φ_j^i / A_{ri} are homomorphisms, $\varphi_{ii} = \mathrm{id}$ for every $i \in J$ and $\varphi_j^i \circ \varphi_k^j = \varphi_k^i$ if $i \geqq j \geqq k$. We can define $A_r = L_{\leftarrow}(A_{rj}\,;\, j \in J)$ for every $r \in R$. Since A_r may be empty, we adapt slightly the definition 1.1. Forming the poproduct $P(A_r\,;\, r \in R)$ we permit also the void lattices A_r. If $A_r = \emptyset$ for all $r \in R$, then we put $P(A_r\,;\, r \in R) = \emptyset$. Otherwise, the A_r are pairwise disjoint lattices and the set $Q = \cup(A_r\,;\, r \in R)$ can be ordered in the standard way. Now we define $PL_{\leftarrow} = P(A_r\,;\, r\in R) = P(L_{\leftarrow}(A_{rj}\,;\, j\in J)\,;\, r\in R)$ as well as in definition 1.1 admitting that f_r is a void homomorphism iff $A_r=\emptyset$.

Having an inverse family $(A_{rj}\,;\,\varphi_i,\, j\in J)$ we can prolong uniquely the mappings $\varphi_j^i\colon Q_i\to Q_j$ to $\bar\varphi_j^i\colon P(A_{ri}\,;\, r\in R)\to P(A_{rj}\,;\, r\in R)$ such that $(P(A_{rj}\,;\, r\in R)\,;\,\bar\varphi_i,\, i\in J)$ becomes an inverse family. Hence we can form $L_{\leftarrow}P = L_{\leftarrow}(P(A_{rj}\,;\, r\in R)\,;\, j\in J)$.

Theorem 5.2. *There exists a monomorphism* $h\colon P(L_{\leftarrow}(A_{rj}\,;\, j\in J)\,;\, r\in R)\to L_{\leftarrow}(P(A_{rj}\,;\, r\in R)\,;\, j\in J)$, *which need not be an epimorphism.*

Proof. We have the canonical embedding $i_j\colon Q_j\to P(A_{rj}\,;\, r\in R)$ for every $j\in J$. Let $x\in Q$. Clearly, $x=(x(i))_{i\in J}$, $x(i)\in Q_i$. Put $g(x) = (i_j(x(j)))_{j\in J}\in \Pi(P(A_{rj}\,;\, r\in R)\,;\, j\in J)$. Since the diagram

$$\begin{tikzcd} Q_i \arrow[r, "i_j"] \arrow[d, "\bar\varphi_k^i"'] & P(A_{rj}\,;\, r\in R) \arrow[d, "\bar\varphi_k^i"] \\ Q_k \arrow[r, "i_k"] & P(A_{rk}\,;\, r\in R) \end{tikzcd}$$

commutes, there is $g(x)\in L_{\leftarrow}P$. By easy computation we see that $g\colon Q\to L_{\leftarrow}P$ is an injective mapping such that g/A_r is a homomorphism. Therefore, by definition 1.1, there exists a unique homomorphism $h\colon PL_{\leftarrow}\to L_{\leftarrow}P$ such that $h/Q=g$.

Now we prove that h is injective. Let $x, y\in PL_{\leftarrow}$. Suppose we have $h(x)=h(y)$, i.e. $h(x)\leqq h(y)$ and $h(x)\geqq h(y)$. There are $a_1,\dots,a_n, b_1,\dots,b_m\in Q$ and lattice polynomials $p, q\in W(Q)$ such that $p(a_1,\dots,a_n)=x$, $q(b_1,\dots,b_m)=y$. First consider $h(x)\leqq h(y)$. Two cases can arise:

(i) There exists $j\in J$ such that $(h(x))(j)\leqq(h(y))(j)$ follows from theorem 1.1, condition (5).

(ii) The property (i) is true for no $j\in J$.

In the first case there exist $r, s\in R$ with $r\leqq s$, polynomials p', q' and elements $c_1,\dots,c_k\in\{a_1,\dots,a_n\}$, $d_1,\dots,d_l\in\{b_1,\dots,b_m\}$ (Proposition 1.2) such that $((h(x))(j))^{(r)} = p(i_j(a_1(j)),\dots,i_j(a_n(j)))^{(r)} = p'(i_j(c_1(j)),\dots,i_j(c_k(j)))\leqq ((h(y))(j))_{(s)} = q(i_j(b_1(j)),\dots,i_j(b_m(j)))_{(s)} = q'(i_j(d_1(j)),\dots,i_j(d_l(j)))$. It is not difficult to check that $p(a_1,\dots,a_n)^{(r)}$ is proper and $p(a_1,\dots,a_n)^{(r)} = p'(c_1,\dots,c_k)$. Analogously, $g(b_1,\dots,b_m)_{(s)} = q'(d_1,\dots,d_l)$. Since $p'(c_1,\dots,c_k)$, $q'(d_1,\dots,d_l)\in Q$ and $h/Q=g$ is injective, we have $p'(c_1,\dots,c_k)\leqq q'(d_1,\dots,d_l)$. Therefore, by theorem 1.1, $x\leqq y$.

In the second case $(h(x))(j)\leqq(h(y))(j)$ follows from theorem 1.1, conditions (1)—(4), for every $j\in J$. By easy computations we obtain $x\leqq y$. In a dual manner one can establish $x\geqq y$, and the proof is complete.

Therefore h is a homomorphism. We shall show that h need not be an epimorphism. It will follow from the following example.

Example. Let R be a two-element antichain, $R=\{s, t\}$, let $J=N$ (the set of all natural numbers). Let $A_{sn}=\{x_1^n, \dots, x_n^n\}$ be for each $n \geqq 1$ the n-element chain with the following ordering: $x_1^n < x_n^n < x_{n-1}^n < \dots < x_2^n$. Let $A_{t1}=\{y_1^1\}$ and let $A_{tn}=\{y_1^n, \dots, y_{n-1}^n\}$ be for each $n>1$ the $(n-1)$-element chain with the following ordering: $y_1^n < \dots < y_{n-1}^n$. For $n=1, 2, \dots$ define the mappings $\varphi_{n+1,n}^s$, $\varphi_{n+1,n}^t$ as follows:

$$\varphi_{n+1,n}^s(x_i^{n+1}) = x_i^n \text{ for } i<n+1,$$
$$\varphi_{n+1,n}^s(x_{n+1}^{n+1}) = x_1^n,$$
$$\varphi_{n+1,n}^t(y_i^{n+1}) = y_i^n \text{ for } i<n,$$
$$\varphi_{n+1,n}^t(y_n^{n+1}) = y_{n-1}^n.$$

Any element $x \in L_{\leftarrow}(A_{rn}\,;\, n \in N)$ can be written in the form of a sequence, i.e. $x=(x_n)$. For each $n \in N$, $P(A_{rn}\,;\, r \in R)$ is the free product of two chains. Let

$w_1 \equiv x_1^1 \in P(A_{r1}\,;\, r \in R)$,
$w_2 \equiv (x_1^2 \vee y_1^2) \wedge x_2^2 \in P(A_{r2}\,;\, r \in R)$,
$w_3 \equiv (((x_1^3 \vee y_2^3) \wedge x_3^3) \vee y_1^3) \wedge x_2^3 \in P(A_{r3}\,;\, r \in R)$,
$w_4 \equiv (((((x_1^4 \vee y_3^4) \wedge x_4^4) \vee y_2^4) \wedge x_3^4) \vee y_1^4) \wedge x_2^4 \in P(A_{r4}\,;\, r \in R)$,
$w_n \equiv (\dots(x_1^n \vee y_{n-1}^n) \wedge x_n^n) \vee y_{n-2}^n) \wedge x_{n-1}^n) \vee \dots \vee y_1^n) \wedge x_2^n \in P(A_{rn}\,;\, r \in R)$.

Let k be a function $N \to \cup(P(A_{rn}\,;\, r \in R);\ n \in N)$ such that for each $n \in N$, $k(n)=w_n$. (In fact, k is a sequence $\{w_n\}_{n \in N}$.) Now $\bar{\varphi}_{n+1,n}(w_{n+1})=w_n$ holds. Therefore $k \in L_{\leftarrow}P$.

We shall show that there is no $w \in PL_{\leftarrow}$ such that $h(w)=k$. To prove this we shall show that for each word $\bar{w}_n$ such that $\bar{w}_n = w_n$ there is $l(\bar{w}_n) \geqq 2n-1$. We shall prove that in each such word $\bar{w}_n$ there must occur all the $2n-1$ elements $x_1^n, \dots, x_n^n$, $y_1^n, \dots, y_{n-1}^n$ of the set $A_{sn} \cup A_{tn}$.

Suppose that there exists a word $\bar{w}_n$ such that $\bar{w}_n = w_n$ and let $z \in A_{sn} \cup A_{tn}$ not occur in $\bar{w}_n$. Let $S=\{a, b, c\}$ be a three-element chain, $a<b<c$. We shall define a mapping $g\colon A_{sn} \cup A_{tn} \to S$ as follows:

1. $g(z)=b$,
2. if $x \in A_{sn}$, then
   $g(x)=a$ if x occurs to the left of z in the word w_n,
   $g(x)=c$ if x occurs to the right of z in the word w_n;
3. if $y \in A_{tn}$, then
   $g(y)=c$ if y occurs to the left of z in the word w_n,
   $g(y)=a$ if y occurs to the right of z in the word w_n.

From the definition of $P(A_{rn}\,;\, r \in R)$ it follows that the isotone mapping g can be extended to the lattice homomorphism $e\colon P(A_{rn}\,;\, r \in R) \to S$. Then $e(\bar{w}_n) = e(w_n) = b = g(z)$, but in the expression of $e(\bar{w}_n)$ $g(z)$ does not occur — a contradiction. Therefore for each word $\bar{w}_n$ such that $\bar{w}_n = w_n$ there holds $l(\bar{w}_n) \geqslant 2n-1$.

Suppose that there exists $w \in PL_{\leftarrow}$ such that $h(w)=k$, $w=w(x_{r_1}, \ldots, x_{r_m})$, where for $1 \leqq i \leqq m$ there holds $x_{r_i} \in L_{\leftarrow}(A_{sn}; n \in \mathrm{N}) \cup L_{\leftarrow}(A_{tn}; n \in \mathrm{N})$, $l(w)=m$; then $w_n = k(n) = (h(w))(n) = (h(w(x_{r_1}, \ldots, x_{r_m})))(n) = w(x_{r_1}(n), \ldots, x_{r_m}(n))$, but $l(w(x_{r_1}(n), \ldots, x_{r_m}(n))) \leqq m$, $m>1$ and $l(w_n) \geqq 2n-1$. Now if we take $n=\left[\frac{m+1}{2}\right]+1$, then $2n-1>m$, a contradiction.

Therefore $k \in L_{\leftarrow}P$ has no preimage in $PL_{\leftarrow}$. This proves that h is not an epimorphism.

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Matematický ústav SAV
Obrancov mieru 49
814 73 Bratislava

ПОПРОДУКТ СТРУКТУР

Зузана Ладзианска

Резюме

В работе изучаются свойства попродукта. Попродукт является обобщением свободного произведения и ординальной суммы структур.