Mathematica Slovaca

András Szücs

On the torsion groups of the cobordism groups of immersions

Mathematica Slovaca, Vol. 43 (1993), No. 3, 341--344

Persistent URL: http://dml.cz/dmlcz/133051

Mathematica Slovaca
©1993
Mathematical Institute
Slovak Academy of Sciences

Math. Slovaca, 43 (1993), No. 3, 341–344

ON THE TORSION GROUPS OF THE COBORDISM GROUPS OF IMMERSIONS

ANDRÁS SZÜCS [1)]

(*Communicated by Július Korbaš*)

ABSTRACT. In most of the earlier known cases the cobordism groups of immersions had torsion elements only of order 2 and 4. Here we show that some of these groups actually have elements of order 128.

The cobordism group of immersions of unoriented n-manifolds in $\mathbb{R}^{n+k}$ will be denoted by $Imm(n,k)$. Its rank is well known:

$$\operatorname{rank} Imm(n,k) = \operatorname{rank} H^{n+k}\big(MO(k);Q\big) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \operatorname{rank} \Omega_{n-k} & \text{if } k \text{ is even} \end{cases}$$

(see [Burlet], [Wells], [Szücs]).

The computation of its torsion group is more difficult. There are results about this torsion group for the cases when the codimension k is equal to 1 and when k is near to n. In the first case (when $k = 1$) these groups are the stable homotopy groups of the infinite dimensional projective space $P_\infty\mathbb{R}$ and these groups are known to have arbitrarily high 2-torsion elements. (The first few of these groups were found by Liulevicius.)

On the other hand, in the second case (when k is n, or $n-1$, or $n-2$) these groups have torsion elements only of order 2 or 4 (see [Wells] and [Koschorke]).

What can be said about these torsion groups for the other codimensions?

Here we show that the groups $Imm(n,k)$ for $k = n-3$, or $k = n-4$, and $n < 2k-1$ also may have torsion elements only of order 2 and 4. On the other hand, if $k = n - 13$ and $n > 29$, then at least one of the groups $Imm(n,k)$ and $Imm(n-1,k-1)$ has an element of order 128.

AMS Subject Classification (1991): Primary 57R42.
Key words: Torsion group, Cobordism group of immersions.
1) Supported by the A. v. Humboldt Foundation .

Remark. It is known that for k odd the groups $Imm(n,k)$ are finite 2-primary groups (see [Wells, Theorem 4].) But Wells's Theorem 6 claiming that $Imm(n,k)$ has p torsion if k is even and $n = k + 2p^2 - 2p - 1$ is false. Actually the groups $Imm(n,k)$ have only 2-torsion if $n \le 3k$, (see [Szücs]).

THEOREM 1. *For any $n > 29$ at least one of the groups $Imm(n, n-13)$ and $Imm(n-1, n-14)$ has an element of order 128.*

Proof of Theorem 1. If $n < 2k-1$, then the following sequence is exact:

$$\Omega^{2\gamma}_{n-k} \xrightarrow{\partial_{n,k}} Imm(n,k) \xrightarrow{e_*} Imm(n,k+1) \xrightarrow{\sigma_{n,k+1}} \Omega^{2\gamma}_{n-k-1} \xrightarrow{\partial_{n-1,k}} Imm(n-1,k) \to \quad (*)$$

(see [Salomonsen], [Koschorke], [Pastor (2.3)]).

Here the map e_* is induced by the inclusion $\mathbb{R}^{n+k} \subset \mathbb{R}^{n+k+1}$ and $\Omega^{2\gamma}_i$ is the cobordism group of those i-dimensional manifolds the stable normal bundle of which is decomposed into the direct sum of two isomorphic bundles. These groups $\Omega^{2\gamma}_i$ have been computed by Golubjatnikov for $i \le 13$. In particular $\Omega^{2\gamma}_{13} \approx Z_{256} \oplus G$, where G is a finite 2-primary group of exponent 64.

We shall use two observations due to Koschorke (see [Korschorke, 7.15, and page 112 after 10.2]):

(1) $2e_* = 0$,

(2) $\partial_{n,k} \circ \sigma_{n,k} = 2 \cdot \mathrm{id}$.

Let us write instead of the indices $\{n, k+1\}$ simply 1 and instead of $\{n-1, k\}$ just 2. Then we have

$$\begin{aligned} \partial_1 \circ \sigma_1 &= 2 \cdot \mathrm{id}, \\ \partial_2 \circ \sigma_2 &= 2 \cdot \mathrm{id}, \\ \mathrm{im}\, \sigma_1 &= \ker \partial_2 . \end{aligned}$$

LEMMA. *For any $a \in \Omega^{2\gamma}_{n-k-1}$ there exist elements x, y such that*

$$2a = \sigma_1(x) + \sigma_2(y).$$

Proof. Let us denote by b the element $2a - \sigma_2 \circ \partial_2(a)$. Then $\partial_2(b) = 0$ and so there exists an element x such that $b = \sigma_1(x)$. Now putting $y = \partial_2(a)$ we obtain $2a = \sigma_1(x) + \sigma_2(y)$. □

Notice that at least one of the elements x and y has finite order because at least one of the groups $Imm(n,k)$ and $Imm(n,k+1)$ is finite.

Notice also that no elements of infinite order are mapped by σ_1 or σ_2 into elements of finite order (use (1)).

Now let a be an element of order 256 in $\Omega_{13}^{2\gamma}$. Then

$$2a = \sigma_1(x) + \sigma_2(y)$$

for some $x \in Imm(n, n-13)$ and $y \in Imm(n-1, n-14)$.

Here the left side has order 128. Further at least one of the summands on the right side has finite order and so both of them have finite order. These orders are powers of 2 therefore at least one of them has order at least 128. Then x or y has at least order 128 too. □

The next theorem shows that when the dimension and the codimension are close to each other then there are no high order torsion elements.

THEOREM 2. *If $n \leq k+4$ and $n < 2k-1$ then the torsion groups of the groups $Imm(n,k)$ have only Z_2 and Z_4 factors, and the number of the Z_4 factors is at most one.*

Moreover, two consecutive groups $Imm(n,k)$ and $Imm(n-1,k-1)$ together may have at most one Z_4 factor unless $n-k=4$ and k is even.

Remark. For $n \leq k+2$ this follows from Koschorke's results, which show also that Z_4 factors do occur.

Proof of Theorem 2. The groups $\Omega_i^{2\gamma}$ are the following for $2 \leq i \leq 4$:

$$\Omega_2^{2\gamma} \approx \Omega_3^{2\gamma} \approx Z_2, \qquad \Omega_4^{2\gamma} \approx Z \oplus Z_2,$$

(see [Golubjatnikov]).

For $k = n-2$ or $n-3$ from $(*)$ we get the following exact sequence

$$0 \to \operatorname{im} e_* \to Imm(n,k) \to Z_2 \to$$

and this implies the claims for $k = n-2$ or $n-3$. For $k = n-4$ we can write a similar exact sequence if we replace $Imm(n,k)$ by its torsion group. It remains to show the last claim of the theorem.

Suppose first that $0 < n-k < 4$. Then the groups $Imm(n,k)$ are finite. If $Imm(n,k)$ has a factor Z_4 then it must be mapped by $\sigma_{n,k}$ onto Z_2. Now $\sigma_{n,k}$ is epimorphic and so $\partial_{n-1,k-1}$ is zero, i.e. $Imm(n-1,k-1)$ is mapped monomorphically into $\operatorname{im} e_*$ and so $2 \cdot Imm(n-1,k-1) = 0$.

Finally let k be odd and equal to $n-4$. Then, if $Imm(n,k)$ has a Z_4 factor, then it must be mapped onto the Z_2 factor of $\Omega_4^{2\gamma} \approx Z \oplus Z_2$ by $\sigma_{n,k}$ since $2 \cdot \operatorname{im} e_* = 0$. Then the torsion group of $Imm(n-1,k-1)$ is mapped monomorphically in $\operatorname{im} e_*$ and so it contains no elements of order 4.

REFERENCES

BURLET, O.: *Cobordismes de plongements et produits homotopiques*, Comment. Math. Helv. **46** (1971), 277–288.

GOLUBJATNIKOV, V.: *Bordism rings with split normal bundles*, Russian Math. Surveys **34** (1979), 172–176.

KOSCHORKE, U.: *Vector Fields and Other Vector Bundle Morphisms – a Singularity Approach. Lecture Notes in Math.* 847, Springer-Verlag, Berlin-New York, 1981.

LIULEVICIUS, A.: *A theorem in homological algebra and stable homotopy of projective spaces*, Trans. Amer. Math. Soc. **109** (1963), 540–552.

PASTOR, G.: *On bordism groups of immersions*, Trans. Amer. Math. Soc. **283** (1984), 295–301.

SALOMONSEN, H. A.: *On the homotopy groups of Thom complexes and unstable bordism.* In: Proc. Adv. Study Inst. Alg. Top. August 10–23, 1970, Aarhus, Denmark.

SZÜCS, A.: *On the cobordism groups of immersions and embeddings*, Math. Proc. Cambridge Philos. Soc. **109** (1991), 343-349.

WELLS, R.: *Cobordism groups of immersions*, Topology **5** (1966), 281-294.

Received September 1, 1992

Eötvös Loránd University
Department of Analysis
H-1088 Budapest
Hungary