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MAXIMAL DEDEKIND COMPLETION OF A HALF LATTICE ORDERED GROUP

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ABSTRACT. In this note there is defined and studied the maximal Dedekind completion $M_h(G)$ of a half lattice ordered group G. Further there is studied the maximal Dedekind completion of the small direct product of half lattice ordered groups.

Let G be an Abelian lattice ordered group. C. J. Everett [3] introduced the notion of the maximal Dedekind completion $M(G)$ of G by means of Dedekind cuts of the lattice $(G, \leq)$. $M(G)$ was studied by J. Jakubík [6]. The notion of the maximal Dedekind completion was generalized to an arbitrary lattice ordered group G in [1]. L. Fuchs [4] investigated the maximal Dedekind completion $M(G)$ of a partially ordered group G.

M. Giraudet and F. Lucas [5] defined and studied the notion of a half partially (lattice) ordered group. It is a generalization of the notion of a partially (lattice) ordered group.

In [2] there is extended the notion of the maximal Dedekind completion $\mathfrak{M}_h(G)$ for the case of a half partially ordered group G.

Especially, if G is a half lattice ordered group then the maximal Dedekind completion of G can be constructed by a simpler method than in [2]. The second order elements of G are applied and the method is presented in this paper.

Further there is studied the maximal Dedekind completion of the small direct product of half lattice ordered groups. The notion of the small direct product of half lattice ordered groups was introduced by J. Jakubík [7].

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1. Preliminaries

We recall a construction of the maximal Dedekind completion of a lattice ordered group (see [1] and [3], [4] for an Abelian lattice ordered group).

Let H be a lattice. For a subset X of H we denote

$$U(X) = \{h \in H: \ h \geqslant x \text{ for each } x \in X\},$$
$$L(X) = \{h \in H: \ h \leq x \text{ for each } x \in X\}.$$

If $U(X) \neq \emptyset$ $(L(X) \neq \emptyset)$ then X is called an *upper* (*lower*) *bounded subset* of H. Assume that $X \neq \emptyset$ and $U(X) \neq \emptyset$. Then the set $L(U(X))$ is said to be a *Dedekind cut* of H. The set of all Dedekind cuts of H will be denoted by $H^{\#}$. $H^{\#}$ is partially ordered by inclusion. Then $H^{\#}$ is a conditionally complete lattice. The lattice operations in $H^{\#}$ are denoted by $\wedge$, $\vee$.

Assume that $Z_i \in H^{\#}$ $(i \in I)$ and that there exists $Z_1 \in H^{\#}$ with $Z_i \subseteq Z_1$ $(i \in I)$. Then $\bigvee Z_i (i \in I) = L\big(U\big(\bigcup Z_i\big)\big)(i \in I)$. If there exists $Z_2 \in H^{\#}$ with $Z_2 \subseteq Z_i$ $(i \in I)$ then $\bigwedge Z_i (i \in I) = \bigcap Z_i (i \in I)$.

The mapping $\varphi \colon H \to H^{\#}$ defined by $\varphi(h) = L(h)$ for each $h \in H$ is an injection and it preserves all greatest lower bounds and least upper bounds existing in H. We shall identify h and $\varphi(h)$ in the sequel. Then H is a sublattice of $H^{\#}$ and the following conditions are valid:

(i) for each nonempty upper bounded subset X of H there exists an element $z \in H^{\#}$ such that $z = \sup X$ in $H^{\#}$

and dually,

(ii) for each element $z \in H^{\#}$ there exist nonempty subsets X, Y of H such that X is upper bounded in H, Y is lower bounded in H and $z = \sup X = \inf Y$ in $H^{\#}$.

Let H be a lattice ordered group.

Assume that $z_1, z_2 \in H^{\#}$. By (ii) there are nonempty and upper bounded subsets X_1 and X_2 of H with $z_1 = \sup X_1$, $z_2 = \sup X_2$. Hence the set $\{x_1 + x_2 : \ x_1 \in X_1, \ x_2 \in X_2\}$ is also nonempty and upper bounded in H. According to (i) there is an element $z \in H^{\#}$ such that $z = \sup\{x_1 + x_2 : x_1 \in X_1, \ x_2 \in X_2\}$. The element z does not depend on the choices of X_1 and X_2 (see [1; Lemma 1.1]).

Then we can define the operation $+$ in $H^{\#}$ as follows: we put $z_1 + z_2 = z$. Then $(H^{\#}, +)$ is a semigroup. Let us remark that, in general, $(H^{\#}, +)$ need not be a group (cf. [3]). We denote by $M(H)$ the set of all elements of $H^{\#}$ having an inverse in $H^{\#}$. Then $M(H)$ is a group and a sublattice of $H^{\#}$. In [1] it is proved that $M(H)$ is an l-group and H is an l-subgroup of $M(H)$. $M(H)$ is called the *maximal Dedekind completion* of H. If $M(H) = H$ then H is said to be *M-complete.*

Let $z \in H^{\#}$. We form the sets $U(z) = \{h \in H : h \geqslant z\}$, $L(z) = \{h \in H : h \leq z\}$.

We have $z = \sup L(z) = \inf U(z)$ in $H^{\#}$.

1.1. THEOREM. (cf. [1; Lemma 1.3]) *Let $z \in H^{\#}$, $X, Y \subseteq H$, $z = \sup X = \inf Y$ in $H^{\#}$. Then $z \in M(H)$ if and only if the following conditions are satisfied in H:*

(c_1) $\inf\{y - x : x \in X, y \in Y\} = 0$,

(c_2) $\inf\{-x + y : x \in X, y \in Y\} = 0$.

The same procedure as in the proof of "if" part of 1.1 in [1] can be applied to prove the following lemma.

1.2. LEMMA. *Let $X, Y \neq \emptyset$, $X, Y \subseteq H$, $x \leq y$ for each $x \in X$, $y \in Y$ and let $z \in H^{\#}$, $z = \sup X$ in $H^{\#}$. Assume that the conditions (c_1) and (c_2) are satisfied in H. Then $z \in M(H)$.*

P r o o f. We have $-y \leq -x$ for each $x \in X$, $y \in Y$. Hence the set $-Y = \{-y \in H : y \in Y\}$ is nonempty and upper bounded in H. Thus there exists $z' \in H^{\#}$, $z' = \sup(-Y)$. We have $z + z' = \sup\{x + y : x \in X, y \in -Y\} = \sup\{x - y : x \in X, y \in Y\} = \inf\{y - x : x \in X, y \in Y\}$ in $H^{\#}$. The condition (c_1) implies that $\inf\{y - x : x \in X, y \in Y\} = 0$ in H and so $\inf\{y - x : x \in X, y \in Y\} = 0$ in $H^{\#}$ as well. Hence $z + z' = 0$.

Analogously by applying (c_2) we show that $z' + z = 0$ holds. Hence z' is an inverse to z in $H^{\#}$ and so $z \in M(H)$. □

1.3. LEMMA. ([6; Lemma 1.8]) *Let $h \in H$, $X, Y \subseteq H$, $h = \sup X = \inf Y$. Then* $\inf\{y - x : x \in X, y \in Y\} = 0$.

1.4. Remark. Under the assumptions in 1.3 it can be proved in a similar way that $\inf\{-x + y : x \in X, y \in Y\} = 0$.

The following two results were obtained by J a k u b í k [6; Lemma 1.9 and Theorem 1.10] under assumption that H is an Abelian lattice ordered group. The same method can be applied to extend J a k u b í k's results to an arbitrary lattice ordered group.

1.5. LEMMA. *$M(H)$ is M-complete.*

1.6. THEOREM. *Let H be a lattice ordered group and let G be a lattice ordered group such that the following conditions are satisfied:*

(α) *G is M-complete.*

(β) *H is an l-subgroup of G.*

(γ) *For each $g \in G$ there exist subsets X, Y of H such that $g = \sup X = \inf Y$ in G.*

Then there exists an isomorphism of $M(H)$ onto G such that $\psi(h) = h$ for each $h \in H$.

Now we recall the definition of a half lattice ordered group (cf. [5]) and some results that will be applied in the next.

Let G be a group and, moreover, a partially ordered set. A partial order $\leq$ on the set G is called *compatible from the right* if $y, z \in G$, $y \leq z$, imply that $y + x \leq z + x$ for each $x \in G$. Denote by $G\uparrow$ ($G\downarrow$) the set of all elements $x \in G$ such that whenever $y, z \in G$, $y \leq z$, then $x + y \leq x + z$ ($x + y \geqslant x + z$). The set $G\uparrow$ is called an *increasing part* of G.

G is said to be a *half lattice ordered group* if the following conditions are satisfied:

(I) $\leq$ is a non-trivial partial order on G,
(II) $\leq$ is compatible from the right,
(III) $G = G\uparrow \cup G\downarrow$,
(IV) $G\uparrow$ is a lattice.

If G is a half lattice ordered group then $G\uparrow$ is a lattice ordered group.

1.7. PROPOSITION. (cf. [5; Proposition I.1.3]) *Let G be a half lattice ordered group such that $G\downarrow \neq \emptyset$. Then*

(i) *$G\uparrow$ is a subgroup of G having the index 2,*
(ii) *partially ordered sets $G\uparrow$ and $G\downarrow$ are isomorphic and, at the same time, dually isomorphic,*
(iii) *if $x \in G\uparrow$ and $y \in G\downarrow$ then x and y are incomparable.*

1.8. PROPOSITION. ([5; Proposition I.3.1]) *Let G be a half lattice ordered group such that $G\downarrow \neq \emptyset$. Then the set $A = \{a \in G : a \neq 0 \text{ and } 2a = 0\} \neq \emptyset$.*

Evidently, that $A \subseteq G\downarrow$.

2. The maximal Dedekind completion of a half lattice ordered group

In the whole section we assume that G is a half lattice ordered group such that $G\downarrow \neq \emptyset$. Then G fails to be a lattice ordered group.

Let G' be a half lattice ordered group such that

(i) the group G is a subgroup of the group G',
(ii) $G\uparrow$ is a sublattice of $G'\uparrow$ and $G\downarrow$ is a sublattice of $G'\downarrow$.

Then we say that G is an *hl-subgroup* of G'.

We shall use the notation $H = G\uparrow$ and $K = G\downarrow$.

Since $K \neq \emptyset$, 1.8 implies that there exists an element $a \in A$.

Assume that $z_1, z_2 \in H^{\#}$. Hence $z_i = \sup L(z_i) = \inf U(z_i)$ $(i = 1,2)$. Then $v_1 \leq u_1$, $v_2 \leq u_2$ for each $u_i \in U(z_i)$, $v_i \in L(z_i)$ $(i = 1,2)$ yield $a+v_1 \geqslant a+u_1$, $a+v_2 \geqslant a+u_2$. Therefore $a+u_1+a+v_2 \leq a+v_1+a+u_2$ for each $u_i \in U(z_i)$, $v_i \in L(z_i)$ $(i = 1,2)$. Thus there exists $\sup\{a+u_1+a+v_2 : u_1 \in U(z_1), v_2 \in L(z_2)\}$ in $H^{\#}$.

2.1. Lemma. *Let $z_1, z_2 \in M(H)$, $z \in H^{\#}$, $z = \sup\{a+u_1+a+v_2 : u_1 \in U(z_1),\ v_2 \in L(z_2)\}$. Then $z \in M(H)$.*

Proof. Denote $B = \{a+u_1+a+v_2 : u_1 \in U(z_1),\ v_2 \in L(z_2)\}$, $C = \{a+v_1+a+u_2 : u_2 \in U(z_2),\ v_1 \in L(z_1)\}$. We have seen above that $b \leq c$ for each $b \in B$, $c \in C$. Hence $c-b \geqslant 0$ for each $b \in B$, $c \in C$. Let $h \in H$, $h \leq c-b$. Then $h \leq a+v_1+a+u_2-(a+u_1+a+v_2)$ for each $u_i \in U(z_i)$, $v_i \in L(z_i)$ $(i = 1,2)$. Then $a-v_1+a+h+a+u_1+a \leq u_2-v_2$ for each $u_2 \in U(z_2)$, $v_2 \in L(z_2)$. Since $z_2 \in M(H)$, according to 1.1 we obtain $\inf\{u_2-v_2 : u_2 \in U(z_2),\ v_2 \in L(z_2)\} = 0$ in H and so $a-v_1+a+h+a+u_1+a \leq 0$, $a-h+a \leq u_1-v_1$ for each $u_1 \in U(z_1)$, $v_1 \in L(z_1)$. Because of $z_1 \in M(H)$, again by 1.1 we get $a-h+a \leq 0$, $h \leq 0$. Finally, we have proved that $\inf\{c-b : b \in B,\ c \in C\} = 0$ in H. Analogously we obtain that $\inf\{-b+c : b \in B,\ c \in C\} = 0$ in H. From 1.2 we infer that $z \in M(H)$. □

Assume that $z_1, z_2 \in M(H)$, $X_i, Y_i \subseteq H$, $z_i = \sup X_i = \inf Y_i$ $(i = 1,2)$ in $H^{\#}$. Analogously as above we obtain that there exists $\sup\{a+y_1+a+x_2 : x_2 \in X_2,\ y_1 \in Y_1\}$ in $H^{\#}$.

2.2. Lemma. *Let $z_1, z_2 \in M(H)$, $X_i, Y_i \subseteq H$, $z_i = \sup X_i = \inf Y_i$ $(i = 1,2)$ in $H^{\#}$ and let z be as in 2.1. Then $z = \sup\{a+y_1+a+x_2 : x_2 \in X_2,\ y_1 \in Y_1\}$ in $H^{\#}$.*

Proof. Denote $z' = \sup\{a+y_1+a+x_2 : x_2 \in X_2,\ y_1 \in Y_1\}$ in $H^{\#}$. Since $Y_1 \subseteq U(z_1)$ and $X_2 \subseteq L(z_2)$, $z' \leq z$. We want to show that $z \leq z'$, i.e., $U(z') \subseteq U(z)$. Let $h \in H$, $h \geqslant z'$. Hence $h \geqslant a+y_1+a+x_2$, $a-y_1+a+h \geqslant x_2$, for each $x_2 \in X_2$. This yields that $a-y_1+a+h \geqslant z_2 \geqslant v_2$, $a+h-v_2+a \leq y_1$, for each $y_1 \in Y_1$. Whence $a+h-v_2+a \leq z_1 \leq u_1$, $h \geqslant a+u_1+a+v_2$. Therefore $h \in U(z)$ and so $U(z') \subseteq U(z)$. □

Suppose that $z \in M(H)$. From $v \leq u$ for each $v \in L(z)$, $u \in U(z)$ it follows that $a+u \leq a+v$ for each $u \in U(z)$, $v \in L(z)$. Hence there exists $\sup\{a+u : u \in U(z)\}$ in $K^{\#}$. We put

$$a+z = \sup\{a+u : u \in U(z)\} \qquad \text{in} \quad K^{\#}.$$

Let us remark that if $z \in H$ then $a+z$ coincides with the group operation in G.

Let us denote

$$a + M(H) = \{a + z : \ z \in M(H)\}$$

and

$$M_h(G) = M(H) \cup (a + M(H)) . \tag{1}$$

Let $X, Y \subseteq H$, $z = \sup X = \inf Y$ in H. There exists $\sup\{a + y : \ y \in Y\}$ in $K^{\#}$. We can verify that

$$a + z = \sup\{a + y : \ y \in Y\} \qquad \text{in} \quad K^{\#} . \tag{2}$$

Let $z_1, z_2 \in M(H)$. It is easily seen that $a + z_1 \le a + z_2$ if and only if $z_1 \geqslant z_2$. All elements of $M(H)$ and $a + M(H)$ are considered incomparable. Therefore $M_h(G)$ is a partially ordered set. Then we can form the sets $M_h(G)\uparrow$ and $M_h(G)\downarrow$.

We shall define a binary operation $+$ on $M_h(G)$ coinciding with the group operation on $M(H)$ and G.

Let $z_1, z_2 \in M(H)$, $X_i, Y_i \subseteq H$, $z_i = \sup X_i = \inf Y_i$ $(i = 1, 2)$ in $H^{\#}$. We put

$$z_1 + z_2 = \sup\{x_1 + x_2 : \ x_1 \in X_1 , \ x_2 \in X_2\} \qquad \text{in} \quad H^{\#} ,$$
$$(a + z_1) + (a + z_2) = \sup\{a + y_1 + a + x_2 : \ y_1 \in Y_1 , \ x_2 \in X_2\} \qquad \text{in} \quad H^{\#} ,$$

$$z_1 + (a + z_2) = a + ((a + z_1) + (a + z_2)) ,$$
$$(a + z_1) + z_2 = a + (z_1 + z_2) .$$

With respect to 2.1 and 2.2 we have $(a + z_1) + (a + z_2) \in M(H)$.

2.3. Lemma. *Let $z_1, z_2, z \in M(H)$. Then*

(i) *The relation $\le$ on $M_h(G)$ is compatible from the right.*
(ii) *$z_1 \le z_2$ implies $z + z_1 \le z + z_2$ and $(a + z) + z_2 \le (a + z) + z_1$.*
(iii) *$a + z_1 \le a + z_2$ implies $z + (a + z_1) \le z + (a + z_2)$ and $(a + z) + (a + z_2) \le (a + z) + (a + z_1)$.*

Proof. We shall consider only some cases of (i) and (iii). Proofs of the remaining cases are analogous.

(i) Assume that $z_1 \le z_2$. We intend to verify that $z_1 + (a + z) \le z_2 + (a + z)$. From $U(z_2) \subseteq U(z_1)$ we infer that $\{u_2 : \ u_2 \in U(z_2)\} \subseteq \{u_1 : \ u_1 \in U(z_1)\}$, $\{a + u_2 + a + v : \ u_2 \in U(z_2) , \ v \in L(z)\} \subseteq \{a + u_1 + a + v : \ u_1 \in U(z_1) , \ v \in L(z)\}$. Therefore $\sup\{a + u_2 + a + v : \ u_2 \in U(z_2) , \ v \in L(z)\} \le \sup\{a + u_1 + a + v : \ u_1 \in U(z_1) , \ v \in L(z)\}$. Hence $(a + z_2) + (a + z) \le (a + z_1) + (a + z)$ and so $a + ((a + z_1) + (a + z)) \le a + ((a + z_2) + (a + z))$, i.e., $z_1 + (a + z) \le z_2 + (a + z)$.

(iii) Assume that $a+z_1 \le a+z_2$. We want to show that $z+(a+z_1) \le z+(a+z_2)$. We have $z_2 \le z_1$. Since $L(z_2) \subseteq L(z_1)$, $\{v_2 : v_2 \in L(z_2)\} \subseteq \{v_1 : v_1 \in L(z_1)\}$ and so $\{a+u+a+v_2 : u \in U(z),\ v_2 \in L(z_2)\} \subseteq \{a+u+a+v_1 : u \in U(z),\ v_1 \in L(z_1)\}$. This implies that $\sup\{a+u+a+v_2 : u \in U(z),\ v_2 \in L(z_2)\} \le \sup\{a+u+a+v_1 : u \in U(z),\ v_1 \in L(z_1)\}$. Hence $(a+z)+(a+z_2) \le (a+z)+(a+z_1)$ and thus $a+\big((a+z)+(a+z_1)\big) \le a+\big((a+z)+(a+z_2)\big)$, i.e., $z+(a+z_1) \le z+(a+z_2)$. □

2.4. Lemma. *Let $z_1, z_2 \in M(H)$, $X_i, Y_i \subseteq H$, $z_i = \sup X_i = \inf Y_i$ $(i=1,2)$. Then*

(i) $z_1+z_2 = \inf\{y_1+y_2 : y_1 \in Y_1,\ y_2 \in Y_2\}$ *in* $H^{\#}$,
(ii) $(a+z_1)+(a+z_2) = \inf\{a+x_1+a+y_2 : x_1 \in X_1,\ y_2 \in Y_2\}$ *in* $H^{\#}$.

Proof. We prove only (ii). The proof of (i) is similar.

(ii) We have $a+y_1+a+x_2 \le a+x_1+a+y_2$ for each $x_i \in X_i$, $y_i \in Y_i$ $(i=1,2)$.

Let us denote $z = \sup\{a+y_1+a+x_2 : x_2 \in X_2,\ y_1 \in Y_1\}$, $z' = \inf\{a+x_1+a+y_2 : x_1 \in X_1,\ y_2 \in Y_2\}$ in $H^{\#}$. We have to show that $z = z'$ is valid. Since $z \le z'$, it suffices to prove that $z' \le z$, i.e., $L(z') \subseteq L(z)$. Let $h \in L(z')$. Hence $h \le z'$, $h \le a+x_1+a+y_2$, $a+h-y_2+a \geqslant x_1$ for each $x_1 \in X_1$. Then $a+h-y_2+a \geqslant z_1$. Applying 2.3(ii), (iii) we obtain $a-z_1+a+h \le y_2$ for each $y_2 \in Y_2$. Thus $a-z_1+a+h \le z_2$. Again by using 2.3(ii) we get $h \le (a+z_1)+(a+z_2) = z$. Therefore $L(z') \subseteq L(z)$. □

2.5. Lemma. *$\big(M_h(G),+\big)$ is a group.*

Proof. At first we prove that the operation $+$ on $M_h(G)$ is associative. Only some cases will be verified. The rest is analogous.

Let $z_i \in M(H)$, $z_i = \sup X_i = \inf Y_i$, $X_i, Y_i \subseteq H$ $(i=1,2,3)$.

We denote $(a+z_1)+(a+z_2) = z$. With respect to 2.4(ii) we get $z = \inf\{a+x_1+a+y_2 : x_1 \in X_1,\ y_2 \in Y_2\}$.

We have $\big((a+z_1)+(a+z_2)\big)+(a+z_3) = z+(a+z_3) = a+\big((a+z)+(a+z_3)\big) = a+\sup\{a+a+x_1+a+y_2+a+x_3 : x_1 \in X_1,\ y_2 \in Y_2,\ x_3 \in X_3\} = a+\sup\{x_1+a+y_2+a+x_3 : x_1 \in X_1,\ y_2 \in Y_2,\ x_3 \in X_3\}$.

Now we denote $(a+z_2)+(a+z_3) = z$. Hence $z = \sup\{a+y_2+a+x_3 : x_3 \in X_3,\ y_2 \in Y_2\}$. Then $(a+z_1)+\big((a+z_2)+(a+z_3)\big) = (a+z_1)+z = a+(z_1+z) = a+\sup\{x_1+a+y_2+a+x_3 : x_1 \in X_1,\ y_2 \in Y_2,\ x_3 \in X_3\}$.

Therefore $\big((a+z_1)+(a+z_2)\big)+(a+z_3) = (a+z_1)+\big((a+z_2)+(a+z_3)\big)$.

We intend to show that $\big((a+z_1)+z_2\big)+(a+z_3) = (a+z_1)+\big(z_2+(a+z_3)\big)$.

We denote $z = z_1+z_2$. Then with respect to 2.4(i) we get $z = \inf\{y_1+y_2 : y_1 \in Y_1,\ y_2 \in Y_2\}$.

We have $((a+z_1)+z_2)+(a+z_3) = (a+(z_1+z_2))+(a+z_3) = (a+z)+(a+z_3) = \sup\{a+y_1+y_2+a+x_3 : y_1 \in Y_1, y_2 \in Y_2, x_3 \in X_3\}$.

We use the notation $(a+z_2)+(a+z_3) = z$ and we get $(a+z_1)+(z_2+(a+z_3)) = (a+z_1)+(a+((a+z_2)+(a+z_3))) = (a+z_1)+(a+z) = \sup\{a+y_1+a+a+y_2+a+x_3 : y_1 \in Y_1, y_2 \in Y_2, x_3 \in X_3\} = \sup\{a+y_1+y_2+a+x_3 : y_1 \in Y_1, y_2 \in Y_2, x_3 \in X_3\}$.

To prove that every element of $M_h(G)$ has an inverse it suffices to show that each element of $a+M(H)$ has an inverse. Any element of $a+M(H)$ has the form $a+z$, $z \in M(H)$. Then $a+(a-z+a) \in a+M(H)$ and it is an inverse to $a+z$.

We conclude that $(M_h(G),+)$ is a group. □

Since $\leq$ is a non-trivial partial order on G, it is a non-trivial partial order on $M_h(G)$ as well.

As an immediate consequence of 2.3, 2.5 and (1) we obtain:

2.6. THEOREM. *Let G be a half lattice ordered group such that $G\downarrow \neq \emptyset$. Then $M_h(G)$ is a half lattice ordered group with $M_h(G)\uparrow = M(H)$, $M_h(G)\downarrow = a+M(H)$.*

A half lattice ordered group $M_h(G)$ is said to be the *maximal Dedekind completion* of a half lattice ordered group G. If $M_h(G) = G$ then G will be called M_h-complete.

It is easy to verify that the following lemma holds:

2.7. LEMMA. *G is M_h-complete if and only if its increasing part H is M-complete.*

Since $M_h(G)\uparrow = M(H)$, from 1.5 and 2.7 we infer that $M_h(G)$ is M_h-complete.

Let $X \subseteq H$. Since elements of $M(H)$ and $a+M(H)$ are incomparable, $\sup X$ ($\inf X$) exists in $M(H)$ if and only if $\sup X$ ($\inf X$) exists in $M_h(G)$ and they are equal.

Now we show that every element of $M_h(G)$ is a supremum and an infimum of some subsets of G. This assertion is valid for elements of $M(H)$. We choose an arbitrary element of $a+M(H)$. It has the form $a+z$, $z \in M(H)$. There are subsets X and Y of H such that $z = \sup X = \inf Y$ in $M(H)$. Then $a+z = \sup\{a+y : y \in Y\} = \inf\{a+x : x \in X\}$ in $a+M(H)$.

By summarizing the above results we obtain the following theorem.

2.8. THEOREM. *A half lattice ordered group $M_h(G)$ has the following properties:*

(a) *$M_h(G)$ is M_h-complete.*
(b) *G is an hl-subgroup of $M_h(G)$.*
(c) *For each element $w \in M_h(G)$ there exist subsets X and Y of G such that $w = \sup X = \inf Y$ in $M_h(G)$.*

2.9. THEOREM. *Let G' be a half lattice ordered group satisfying conditions (a)–(c) (G' instead of $M_h(G)$). Then there exists an isomorphism ϕ of a half lattice ordered group $M_h(G)$ onto G' such that $\phi(g) = g$ for each $g \in G$.*

Proof. Let $z \in M(H)$, $X = L(z)$, $Y = U(z)$, $Z = \{y - x : x \in X, y \in Y\}$, $Z_1 = \{-x + y : x \in X, y \in Y\}$.

Then $z = \sup X = \inf Y$ in $M(H)$. According to 1.3 and 1.4 we get $\inf Z = \inf Z_1 = 0$ in $M(H)$ and also in H. Let us denote $Y' = \{g' \in G' : g' \geqslant x$ for each $x \in X\}$, $X' = \{g' \in G' : g' \leq y'$ for each $y' \in Y'\}$, $Z' = \{y' - x' : x' \in X', y' \in Y'\}$, $Z_1' = \{-x' + y' : x' \in X', y \in Y'\}$, $H' = G'\uparrow$. Then X', Y', Z', Z_1' are subsets of H'. There exists an element $z' \in (H')^{\#}$ such that $z' = \sup X' = \inf Y'$ in $(H')^{\#}$. Since $\inf Z = \inf Z' = 0$ in H', 1.1 implies that $z' \in M(H')$. In view of 2.7 we get $M(H') = H'$. Thus $z' \in H'$.

Define the mapping ϕ of $M_h(G)$ into G' by the rule $\phi(z) = z'$, $\phi(a + z) = a + \phi(z)$. Then ϕ is an isomorphism of $M_h(G)$ onto G' leaving fixed all elements of G. □

$M_h(G)$ is a half lattice ordered group corresponding to an element $a \in A$. Assume that there exists $a' \in A$, $a' \neq a$. It can be constructed a half lattice ordered group $M_h'(G)$ corresponding to a'. The operation $+'$ and the partial order $\leq'$ in $M_h'(G)$ are defined formally in the same way as $+$ and $\leq$ in $M_h(G)$. Hence the operations $+$ and $+'$ coincide on G and $M(H)$.

2.10 THEOREM. *A half lattice ordered group $M_h(G) = M_h'(G)$.*

Proof. We have $M_h(G) = M(H) \cup \big(a + M(H)\big)$, $M_h'(G) = M(H) \cup \big(a' +' M(H)\big)$.

Let $a + z \in a + M(H)$, $z \in M(H)$, $z = \sup X = \inf Y$. We have $a' + a \in H$. With respect to 2.4(i) we get $(a' + a) + z = \inf\{a' + a + y : y \in Y\}$ in $H^{\#}$ and

$$a' +' (a' + a + z) = \sup\{a' + a' + a + y : y \in Y\} = \sup\{a + y : y \in Y\} = a + z .$$

Hence $a + M(H) \subseteq a' +' M(H)$. Analogously we get $a' +' M(H) \subseteq a + M(H)$. Then $a + M(H) = a' +' M(H)$ and thus the set $M_h(G) = M_h'(G)$.

It is clear that $a + z_1 \leq a + z_2$ if and only if $a' +' (a' + a + z_1) \leq' a' +' (a' + a + z_2)$ for each $z_1, z_2 \in M(H)$.

Now we verify that the group operations in $M_h(G)$ and $M'_h(G)$ coincide.

Let $z_1, z_2 \in M(H)$, $X_i, Y_i \subseteq H$, $z_i = \sup X_i = \inf Y_i$ $(i = 1, 2)$ in $H^\#$.

$(a + z_1) + (a + z_2) = \sup\{a + y_1 + a + x_2 :\ y_1 \in Y_1,\ x_2 \in X_2\} = \sup\{a' + (a' + a + y_1) + a' + (a' + a + x_2) :\ y_1 \in Y_1,\ x_2 \in X_2\} = \big(a' +' (a' + a + z_1)\big) +' \big(a' +' (a' + a + z_2)\big)$.

$z_1 + (a + z_2) = a + \big((a + z_1) + (a + z_2)\big) = a' +' \big(a' + a + \big((a + z_1) + (a + z_2)\big)\big) = a' +' \sup\{a' + a + a + y_1 + a + x_2 :\ y_1 \in Y_1,\ x_2 \in X_2\} = a' +' \sup\{a' + y_1 + a' + (a' + a + x_2) :\ y_1 \in Y_1,\ x_2 \in X_2\} = a' +' \big((a' +' z_1) +' \big(a' +' (a' + a + z_2)\big)\big) = z_1 +' \big(a' +' (a' + a + z_2)\big)$.

$(a + z_1) + z_2 = a + (z_1 + z_2) = a' +' \big(a' + a + (z_1 + z_2)\big) = a' +' \big((a' + a + z_1) + z_2\big) = \big(a' +' (a' + a + z_1)\big) +' z_2$. □

2.11. Remark. Half lattice ordered groups with the same increasing part need not be isomorphic, in general (see [5]).

3. $\mathfrak{M}_h(G)$ and $M_h(G)$ of a half lattice ordered group G

Let G be a group and also a partially ordered set such that the conditions (I) – (III) are satisfied. Then G is called a *half partially ordered group* (cf. [5]).

We continue with the notations $G\uparrow = H$, $G\downarrow = K$. If G is a half partially ordered group then H is a partially ordered group.

Let G be a half partially ordered group. We describe the construction from [2] of the maximal Dedekind completion $\mathfrak{M}_h(G)$ of G. The method for partially ordered groups (see [4]) is applied for G.

For the partially ordered set G we construct the set $G^\#$ of all Dedekind cuts of G (in the same way as $H^\#$ for the lattice H in the Section 1). $G^\#$ is a conditionally complete conditional lattice. Symbols $H^\#, K^\#$ have an analogous meaning for partially ordered sets H, K. Then $G^\#$ is a disjoint union of $H^\#$ and $K^\#$.

Let $w_1, w_2 \in G^\#$. We define a binary operation $w_1 + w_2$ in $G^\#$ in the following way:

(a) Let $w_1, w_2 \in H^\#$, $X_i \subseteq H$, $w_i = \sup X_i$ $(i = 1, 2)$ in $H^\#$. Then $w_1 + w_2 = \sup\{x_1 + x_2 :\ x_1 \in X_1,\ x_2 \in X_2\}$ in $H^\#$.

(b) Let $w_1, w_2 \in K^\#$, $X_1 \subseteq K$, $Y_2 \subseteq K$, $w_1 = \sup X_1$, $w_2 = \inf Y_2$ in $K^\#$. Then $w_1 + w_2 = \sup\{x_1 + y_2 :\ x_1 \in X_1,\ y_2 \in Y_2\}$ in $H^\#$.

(c) Let $w_1 \in H^\#$, $w_2 \in K^\#$, $X_1 \subseteq H$, $X_2 \subseteq K$, $w_1 = \sup X_1$ in $H^\#$, $w_2 = \sup X_2$ in $K^\#$. Then $w_1 + w_2 = \sup\{x_1 + x_2 :\ x_1 \in X_1,\ x_2 \in X_2\}$ in $K^\#$.

(d) Let $w_1 \in K^{\#}$, $w_2 \in H^{\#}$, $X_1 \subseteq K$, $Y_2 \subseteq H$, $w_1 = \sup X_1$ in $K^{\#}$, $w_2 = \inf Y_2$ in $H^{\#}$. Then $w_1 + w_2 = \sup\{x_1 + y_2 :\ x_1 \in X_1,\ y_2 \in Y_2\}$ in $K^{\#}$.

Then $(G^{\#}, +)$ need not be a group, not even a semigroup, in general. If $w \in G^{\#}$ then w need not have an inverse in $G^{\#}$. The set of all elements of $G^{\#}$ ($K^{\#}$) having an inverse in $G^{\#}$ ($K^{\#}$) will be denoted by $\mathfrak{M}_h(G)$ ($I(K^{\#})$). The set of all elements of $H^{\#}$ possessing an inverse in $H^{\#}$ is the maximal Dedekind completion $M(H)$ of a partially ordered group H (see [3]).

3.1. THEOREM. ([2; Theorem 2.15]) *$\mathfrak{M}_h(G)$ is a half partially ordered group and $\mathfrak{M}_h(G)\uparrow = M(H)$, $\mathfrak{M}_h(G)\downarrow = I(K^{\#})$.*

Assume, again, that G is a half lattice ordered group such that $K \neq \emptyset$. Then H is a lattice ordered group. Since $M(H)$ is also a lattice ordered group, $\mathfrak{M}_h(G)$ is a half lattice ordered group.

Let a be a fixed element of K. Then $a \in I(K^{\#})$. The mapping φ: $M(H) \to I(K^{\#})$ defined by $\varphi(z) = a + z$ for each $z \in M(H)$ is a dual isomorphism of a lattice $M(H)$ onto $I(K^{\#})$. Hence $I(K^{\#}) = \{a + z :\ z \in M(H)\}$. Remark that $a + z$ is the group operation in $\mathfrak{M}_h(G)$. According to (d) we get in $\mathfrak{M}_h(G)$ that $a + z = \sup\{a + y :\ y \in Y\}$ in $K^{\#}$. Then $I(K^{\#}) = a + M(H)$. This is a consequence of (2). Therefore the set $\mathfrak{M}_h(G) = M_h(G)$.

Evidently, the partial order in $\mathfrak{M}_h(G)$ is equal to the one in $M_h(G)$. It can be easily verified that the group operations in $\mathfrak{M}_h(G)$ and in $M_h(G)$ coincide. Hence we have arrived at the following result.

3.2. THEOREM. *Let G be a half lattice ordered group. Then $\mathfrak{M}_h(G) = M_h(G)$.*

4. Maximal Dedekind completion of the small direct product of half lattice ordered groups

Let I be a nonempty set and let G_i be a lattice ordered group for each $i \in I$. The direct product

$$G^1 = \prod G_i (i \in I)$$

is defined in the usual way, i.e., the group operation and the partial order in G^1 are defined componentwise. Then G^1 is a lattice ordered group. An element $g \in G^1$ is expressed in the form $g = (\dots, g_i, \dots)_{i \in I}$; g_i is a component of g in G_i.

Let φ be an isomorphism of a lattice ordered group G onto $\prod G_i (i \in I)$, $\varphi(g) = (\dots, g_i, \dots)_{i \in I}$. We shall identify g and $\varphi(g)$. Further every element $g^i \in G_i$ will be identified with an element $g \in G$ such that $g_i = g^i$ and $g_j = 0$ for each $j \in I$, $j \neq i$. Then we shall write $G = \prod G_i (i \in I)$.

4.1. THEOREM. ([6; Proposition 2.2, Theorem 2.7]) *Let G be an abelian lattice ordered group,*

$$G = \prod G_i (i \in I).$$

Then

(i) *G is M-complete if and only if all G_i are M-complete.*
(ii) *$M(G) = \prod M(G_i)(i \in I)$.*

4.2. Remark. If G is not assumed to be abelian, it can be checked that 4.1 is also valid.

Now, let G_i be a half lattice ordered group for each $i \in I$. Then the direct product G^1 need not be a half lattice ordered group, in general (see [7]). This was the reason to introduce the notion of a small direct product of half lattice ordered groups by J. Jakubík [7]. This notion is defined in the following way.

Let G_i be a half lattice ordered group for each $i \in I$. Denote

$$G_\circ = \{g \in G^1 : \text{ either } g_i \in G_i\!\uparrow \text{ for each } i \in I \text{ or } g_i \in G_i\!\downarrow \text{ for each } i \in I\}.$$

$G_\circ$ is a subgroup of G^1 and a partially ordered set with the inherited partial order from G^1. Moreover, $G_\circ$ is a half lattice ordered group; $G_\circ\!\uparrow = \{g \in G_\circ : g_i \in G_i\!\uparrow \text{ for each } i \in I\}$, $G_\circ\!\downarrow = \{g \in G_\circ : g_i \in G_i\!\downarrow \text{ for each } i \in I\}$. The half lattice ordered group $G_\circ$ is called the *small direct product* of half lattice ordered groups G_i $(i \in I)$ and denoted by $(s)\prod G_i(i \in I)$.

If all G_i are lattice ordered groups such that $G_i \neq \{0\}$ for each $i \in I$ then $(s)\prod G_i(i \in I) = \prod G_i(i \in I)$.

In the next G and G_i are half lattice ordered groups for each $i \in I$, $G\!\downarrow \neq \emptyset$. We apply the notations $H = G\!\uparrow$, $K = G\!\downarrow$ as above and $H_i = G_i\!\uparrow$, $K_i = G_i\!\downarrow$; a is a fixed element of K.

Let ψ be an isomorphism of G onto $(s)\prod G_i(i \in I)$. Identifying analogous elements as above, we write $G = (s)\prod G_i(i \in I)$. For $B \subseteq G$ we denote $B_i = \{x_i \in G_i : x \in B\}$.

4.3. THEOREM. *Let*

$$G = (s)\prod G_i(i \in I).$$

Then G is M_h-complete if and only if G_i is M_h-complete for each $i \in I$.

Proof. Let $G = (s)\prod G_i(i \in I)$. The definition of the small direct product implies that for the lattice ordered group H we get $H = \prod H_i(i \in I)$. According to 4.1(i) and 4.2, H is M-complete if and only if H_i is M-complete for each $i \in I$. With respect to 2.7. a half lattice ordered group is M_h-complete if and only if its increasing part is M-complete and the proof is finished. □

4.4. THEOREM. *Let*

$$G = (s)\prod G_i(i \in I). \tag{3}$$

Then

$$M_h(G) = (s)\prod M_h(G_i)(i \in I).$$

Proof. For the lattice ordered group H and for the lattice K we have

$$H = \prod H_i(i \in I), \tag{4}$$
$$K = \prod K_i(i \in I).$$

This is a consequence of (3).

The last relation implies that for $a = (\dots, a_i, \dots)_{i\in I}$ we have $a_i \in K_i$ for each $i \in I$.

Let $z \in M(H)$, $z = \sup X = \inf Y$, $X, Y \subseteq H$. If for each component z_i of z in $M(H_i)$ we put $z_i = \sup X_i = \inf Y_i$ then

$$M(H) = \prod M(H_i)(i \in I). \tag{5}$$

This follows from (4), 4.1 (ii) and 4.2.

Further we put $(a+z)_i = a_i + z_i$ for each $i \in I$. Then from (5) we infer that for the lattice $a + M(H)$ the following relation is satisfied

$$a + M(H) = \prod\big(a_i + M(H_i)\big)(i \in I). \tag{6}$$

By 2.6 we have $M(H) = M_h(G)\uparrow$, $a + M(H) = M_h(G)\downarrow$. Because of (5) and (6) the partial order in $M_h(G)$ is performed componentwise. It remains to verify that the group operation in $M_h(G)$ is also performed componentwise.

Let $z_1, z_2 \in M(H)$, $z_i = \sup X_i = \inf Y_i$ $(i = 1, 2)$.

With respect to (5) we get

$$(z_1 + z_2)_i = z_{1i} + z_{2i}.$$

Applying (5) and (3) we obtain
$\big((a + z_1) + (a + z_2)\big)_i = \big(\sup\{a + y_1 + a + x_2 :\ y_1 \in Y_1,\ x_2 \in X_2\}\big)_i = \sup\{a + y_1 + a + x_2 :\ y_1 \in Y_1,\ x_2 \in X_2\}_i = \sup\{a_i + y_{1i} + a_i + x_{2i} :\ y_{1i} \in Y_{1i},\ x_{2i} \in X_{2i}\} = (a_i + z_{1i}) + (a_i + z_{2i}) = (a + z_1)_i + (a + z_2)_i$.
$\big(z_1 + (a + z_2)\big)_i = \big(a + ((a + z_1) + (a + z_2))\big)_i = a_i + \big((a + z_1) + (a + z_2)\big)_i = a_i + \big((a + z_1)_i + (a + z_2)_i\big) = a_i + \big((a_i + z_{1i}) + (a_i + z_2)_i\big) = z_{1i} + (a_i + z_{2i}) = z_{1i} + (a + z_2)_i$.
$\big((a + z_1) + z_2\big)_i = \big(a + (z_1 + z_2)\big)_i = a_i + (z_1 + z_2)_i = a_i + (z_{1i} + z_{2i}) = (a_i + z_{1i}) + z_{2i} = (a + z_1)_i + z_{2i}$. □

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