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ON TRANSFORMATIONS $z(t) = L(t)y(\varphi(t))$ OF FUNCTIONAL-DIFFERENTIAL EQUATIONS

VÁCLAV TRYHUK

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ABSTRACT. The paper describes the general form of an ordinary differential equation of the order $n+1$ ($n \geq 1$) with m ($m \geq 1$) delays which allows a nontrivial global transformations consisting of a change of the independent variable and of a nonvanishing factor. A functional equation of the form

$$f(s, W\vec{v}, W_{(1)}\vec{v}_{(1)}, \dots, W_{(m)}\vec{v}_{(m)}) = \sum_{i=0}^{n} w_{n+1i}v_i + w_{n+1n+1} f(x, \vec{v}, \vec{v}_{(1)}, \dots, \vec{v}_{(m)}),$$

$s, x \in \mathbb{R}$; $W, W_{(1)}, \dots, W_{(m)}$ are real valued $n+1$ by $n+1$ matrices, $\vec{v}, \vec{v}_{(j)} \in \mathbb{R}^{n+1}$; $w_{ij} = a_{ij}(x_1, \dots, x_{i-j+1}, u, u_1, \dots, u_{i-j})$ for the given functions a_{ij} is solved on $\mathbb{R}$, $u \neq 0$.

1. Introduction

The theory of global pointwise transformations of homogeneous linear differential equations was developed in the monograph of F. Neuman [8] (see historical remarks, definitions, results and applications). The most general form of global pointwise transformations for homogeneous linear differential equations of the nth order ($n \geq 2$) is

$$z(t) = L(t)y(\varphi(t)),$$

where φ is a bijection of an interval J onto an interval I ($J \subseteq \mathbb{R}$, $I \subseteq \mathbb{R}$) and $L(t)$ is a nonvanishing function on J, i.e. this global transformation consists of a change of the independent variable and of a nonvanishing factor L. The form of the most general pointwise transformations of homogeneous linear differential

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equations with deviating arguments was derived in [4], [7], [9], [10], [11]. This form coincides for an arbitrary order with the form considered for linear differential equations of the nth $(n \geq 2)$ order without deviation. An interesting problem is solved by J. Aczél [2] by means of functional equations, eliminating regularity conditions from [5]. In [2] the ordinary differential equation of second order in the explicit form

$$y''(x) = f\big(x, y(x), y'(x)\big) \tag{1}$$

is considered together with solutions $y(x)$ and $y\big(h(x)\big)$ of the equation (1), where h satisfies a differential equation

$$h''(x) = g\big(x, h(x), h'(x)\big)\,.$$

We can also formulate Aczél's problem by using transformation $z(t) = L(t)y\big(\varphi(t)\big)$ with the factor $L \equiv 1$ under the conditions $\varphi(I) = I$, $\varphi''(x) = g\big(x, \varphi(x), \varphi'(x)\big)$, $x \in I$, such that the transformation converts any equation (1) into itself, i.e. by using a nontrivial stationary transformation.

Moreover, a general form

$$y''(x) = b\big(y(x)\big)y'(x)^2 + p(x)y'(x)$$

where φ satisfies a differential equation $\varphi''(x) = p(x)\varphi'(x) - p\big(\varphi(x)\big)\varphi'(x)^2$ and b, p are arbitrary functions, was derived by J. Aczél [2], Moór–Pintér [5] for the equation (1). This general form is generally nonlinear second order differential equation and allows a transformation $z(t) = y\big(\varphi(t)\big)$ such that transforms the equation into itself on the whole interval of definition. Aczél's result is generalized in [13] for transformations $z(t) = L(t)y\big(\varphi(t)\big)$ of the second order differential equations, in [15], [16] for ordinary differential equations of the order $n+1$ $(n \geq 1)$, in [14] for functional-differential equations of the first order with m $(m \geq 1)$ delays.

A general form

$$y'(x) = \sum_{i=1}^{k} a_i(x) b_i\big(y(x)\big) \prod_{j=1}^{m} \delta_{ij}\Big(y\big(\xi_j(x)\big)\Big) + q(x)y(x)$$

where b_i, δ_{ij} are continuous (at a point) solutions of Cauchy's power functional equation $b(xy) = b(x)b(y)$, $b\colon \mathbb{R}-\{0\} \to \mathbb{R}$; $\xi_j\big(\varphi(x)\big) = \varphi\big(\xi_j(x)\big)$, $x \in I = \varphi(I)$, φ satisfies a differential equation

$$\begin{aligned}\varphi'(x) &= g\Big(x, \varphi(x), L(x), L\big(\xi_1(x)\big), \dots, L\big(\xi_m(x)\big)\Big) \\ &= \frac{a_i(x) b_i\big(L(x)\big) \prod\limits_{j=1}^{m} \delta_{ij}\Big(L\big(\xi_j(x)\big)\Big)}{a_i\big(\varphi(x)\big) L(x)}\,, \qquad i = 1, \dots, k\,,\end{aligned}$$

L satisfies a differential equation

$$\begin{aligned} L'(x) &= h\Big(x, \varphi(x), L(x), L\big(\xi_1(x)\big), \dots, L\big(\xi_m(x)\big)\Big) \\ &= \Big(q(x) - q\big(\varphi(x)\big)\varphi'(x)\Big) L(x)\,, \qquad x \in I \end{aligned}$$

and $a_i \neq 0$, q are arbitrary functions, was derived in [14]. This form allows a transformation $z(x) = L(x)y\big(\varphi(x)\big)$ such that transforms the equation into itself on I.

In this paper we derive a general form of functional-differential equations of the order $n+1$ $(n \geq 1)$ with m $(m \geq 1)$ delays which allows transformations $z(t) = L(t)y\big(\varphi(t)\big)$ that transform the equation into itself on the whole interval of definition. Further on we assume that solutions vanishes at some points on I. We prove that the most general functional-differential equation of the order $n+1$ $(n \geq 1)$ of the above property, defined for $y \in \mathbb{R}$, is a linear functional-differential equation.

2. Notation

Let $\mathbf{V}_{n+1}$ denote an $(n+1)$-dimensional vector space, $\vec{c} = [c_0, \dots, c_n]^T = [c_i]_{i=0}^n \in \mathbf{V}_{n+1}$ being a vector of the space written in the column form; ^{T} means the transposition. Denote $\vec{o} = [0, \dots, 0]^T$ the origin of $\mathbf{V}_{n+1}$ and $\vec{e}_0, \dots, \vec{e}_n$ an orthonormal basis in $\mathbf{V}_{n+1}$. Let $\mathbf{V}_{n+1}$ be equipped with the scalar product $(\vec{p}, \vec{q}) = \sum\limits_{i=0}^{n} p_i q_i$ for any pair $\vec{p}$, $\vec{q}$ of its vectors.

Let $\vec{p}_1, \dots, \vec{p}_m$ be m vectors from $\mathbf{V}_{n+1}$. Notation $P = [\vec{p}_1, \dots, \vec{p}_m] = [p_{ij}]_{j=1,\dots,m}^{i=0,\dots,n}$ denotes a matrix and $(P, Q) = \sum\limits_{j}^{i} p_{ij} q_{ij}$ the scalar product of two matrices of the same type, PQ or $P\vec{p}$ denotes the matrix multiplication. We denote $O = [\vec{o}, \dots, \vec{o}]_{j=0,\dots,n}$ the zero matrix, $E = [\vec{e}_0, \dots, \vec{e}_n]$ the unit matrix, $E_{ij} = [\vec{o}, \dots, \vec{o}, \vec{e}_i, \vec{o}, \dots, \vec{o}]$ with $\vec{e}_i \in \mathbf{V}_{n+1}$ in the jth column.

Consider real functions $y \in C^{n+1}(I)$, $I \subseteq \mathbb{R}$ being an interval, $\xi_1, \xi_2, \dots, \xi_m \in C^n(I)$, $\xi_j \colon I \to I$, $\xi_0 = \mathrm{id}_I$, $\xi_j \neq \xi_k$ for $j \neq k$; $j, k \in \{0, \dots, m\}$; $m, n \in \mathbb{N} = \{1, 2, \dots\}$. We denote $\big(y\big(\xi_j(x)\big)\big)^{(i)} = \mathrm{d}^i y\big(\xi_j(x)\big) / \mathrm{d}x^i$, $y^{(i)}\big(\xi_j(x)\big) = \mathrm{d}^i y\big(\xi_j(x)\big) / \mathrm{d}\xi_j(x)^i$, $x \in I$ and $y_i(x) = y^{(i)}(x)$, $y_{ij}(x) = y^{(i)}\big(\xi_j(x)\big)$. Then $\vec{y}(x) = \big[y_0(x), y_1(x), \dots, y_n(x)\big]^T = \big[y(x), y'(x), \dots, y^{(n)}(x)\big]^T \in \mathbf{V}_{n+1}$ for each $x \in I$ and we denote $Y(x) = \big[\vec{y}\big(\xi_l(x)\big), \dots, \vec{y}\big(\xi_m(x)\big)\big]$, $x \in I$.

3. Definitions, preliminary results

Denote by (f) and (f^*) the ordinary differential equations

$$y^{(n+1)}(x) = f\Big(x, y(x), \dots, y^{(n)}(x), y\big(\xi_1(x)\big), \dots, y^{(n)}\big(\xi_1(x)\big), \dots \\ \dots, y\big(\xi_m(x)\big), \dots, y^{(n)}\big(\xi_m(x)\big)\Big), \qquad x \in I \subseteq \mathbb{R},$$

$$z^{(n+1)}(t) = f^*\Big(t, z(t), \dots, z^{(n)}(t), z\big(\eta_1(t)\big), \dots, z^{(n)}\big(\eta_1(t)\big), \dots \\ \dots, z\big(\eta_m(t)\big), \dots, z^{(n)}\big(\eta_m(t)\big)\Big), \qquad t \in J \subseteq \mathbb{R},$$

of the order $n+1$ $(n \geq 1)$ with m $(m \geq 1)$ delays. Here $y \in C^{n+1}(I)$, $I \subseteq \mathbb{R}$ being an interval, $\xi_1, \xi_2, \dots, \xi_m \in C^n(I)$, $\xi_j \colon I \to I$, $\xi_0 = \mathrm{id}_I$, $\xi_j \neq \xi_k$ for $j \neq k$; $j, k \in \{0, \dots, m\}$; $m, n \in \mathbb{N}$, for (f). Similar assumptions we consider for (f^*).

To obtain the functional-differential equations we suppose that $f(x, 0, \dots, 0, \alpha_{01}, \dots, \alpha_{nm}) \neq 0$ for $\sum\limits_{i=0,\dots,n}^{j=1,\dots,m} \alpha_{ij}^2 \neq 0$.

Definition. (See [8; p. 25–26].) We say that (f) is *globally transformable* into (f^*) with respect to the transformation $z(t) = L(t)y\big(\varphi(t)\big)$ if there exist two functions L, φ such that

- the function L is of the class $C^{n+1}(J)$ and is nonvanishing on J,
- the function φ is a C^{n+1} diffeomorphism of the interval J onto the interval I

and the function

$$z(t) = L(t)y\big(\varphi(t)\big), \qquad t \in J, \tag{2}$$

is a solution of the equation (f^*) whenever y is a solution of the equation (f).

If (f) is globally transformable into (f^*), then we say that (f), (f^*) are *equivalent equations*. We say that (2) is a *stationary transformation* if it globally transforms the equation (f) into itself on I, i.e. if $L \in C^{n+1}(I)$, $L(x) \neq 0$ on I, φ is a C^{n+1} diffeomorphism of I onto $I = \varphi(I)$ and the function $z(x) = L(x)y\big(\varphi(x)\big)$ is a solution of (f) whenever $y(x)$ is a solution of (f).

If (f), (f^*) are equivalent equations then (see [4], [7], [9], [11])

$$\xi_j\big(\varphi(t)\big) = \varphi\big(\eta_j(t)\big)$$

is satisfied on J for deviations ξ_j, η_j, $j = 1, \dots, m$.

For stationary transformations we get

$$\xi_j\big(\varphi(t)\big) = \varphi\big(\xi_j(t)\big)$$

on I, $j = 1, \dots, m$. Such commutable functions were investigated in [17], [18].

Proposition 1. ([12; Lemma 1]) *Let $n \in \mathbb{N}$ and let the relation*

$$z(t) = L(t)y\bigl(\varphi(t)\bigr)$$

be satisfied where real functions $y\colon I \to \mathbb{R}$, $z\colon J \to \mathbb{R}$ belong to classes $C^{n+1}(I)$, $C^{n+1}(J)$ respectively, and $L\colon J \to \mathbb{R}$, $L \in C^r(J)$, $L(t) \neq 0$ on J, and φ is a C^r diffeomorphism of J onto I, for some integer $r \geq n+1$. Then

$$\begin{aligned} z^{(i)}(t) &= \sum_{j=0}^{i} a_{ij}(t)y^{(j)}\bigl(\varphi(t)\bigr) \\ &= a_{i0}(t)y\bigl(\varphi(t)\bigr) + a_{i1}(t)y'\bigl(\varphi(t)\bigr) + \cdots + a_{ii}(t)y^{(i)}\bigl(\varphi(t)\bigr)\,, \\ &\qquad\qquad i \in \{0,1,\dots,n+1\}\,, \end{aligned}$$

where

$$\begin{aligned} a_{00}(t) &= L(t), \dots, a_{i0}(t) = a'_{i-10}(t)\,, && i \geq 1\,; \\ a_{ij}(t) &= a'_{i-1j}(t) + a_{i-1j-1}(t)\varphi'(t)\,, && i > j > 1\,; \\ a_{ii}(t) &= a_{i-1i-1}(t)\varphi'(t)\,, && i \in \{0,1,\dots,n+1\}\,, \end{aligned}$$

are real functions, $a_{ij}(t) \in C^{r-(i-j)-1}(J)$ for $j > 0$, and $a_{i0}(t) \in C^{r-i}(J)$. Moreover,

$$a_{i0}(t) = L^{(i)}(t)\,, \qquad i \geq 0\,;$$

$$a_{i1}(t) = \bigl(L(t)\varphi(t)\bigr)^{(i)} - L^{(i)}(t)\varphi(t) = \sum_{j=0}^{i-1} \binom{i}{j} L^{(j)}(t)\varphi^{(i-j)}(t)\,, \qquad i \geq 1\,;$$

$$\vdots$$

$$\begin{aligned} a_{ij}(t) &= \binom{i}{j} L^{(i-j)}(t)\varphi'(t)^j + \binom{i}{j-1} L(t)\varphi'(t)^{j-1}\varphi^{(i-j+1)}(t) \\ &\quad + r_{ij}\bigl(L,\dots,L^{(i-j-1)},\varphi',\dots,\varphi^{(i-j)}\bigr)(t)\,, \qquad i > j > 1\,; \end{aligned}$$

$$\vdots$$

$$\begin{aligned} a_{ii-2}(t) &= \binom{i}{2} L''(t)\varphi'(t)^{i-2} + \binom{i}{3}\bigl(L(t)\varphi'''(t) + 3L'(t)\varphi''(t)\bigr)\varphi'(t)^{i-3} \\ &\quad + 3\binom{i}{4} L(t)\varphi'(t)^{i-4}\varphi''(t)^2\,, \qquad i \geq 2\,; \end{aligned}$$

$$a_{ii\ 1}(t) = \binom{i}{1} L'(t)\varphi'(t)^{i-1} + \binom{i}{2} L(t)\varphi'(t)^{i-2}\varphi''(t)\,, \qquad i \geq 2\,;$$

$$a_{ii}(t) = L(t)\varphi'(t)^i\,, \qquad i \geq 0$$

and

$$a_{i0}(t) = a_{i0}\bigl(L^{(i)}\bigr)(t)\,, \qquad i \geq 0\,;$$

$$a_{ij}(t) = a_{ij}\bigl(L,\dots,L^{(i-j)},\varphi',\dots,\varphi^{(i-j+1)}\bigr)(t)\,, \qquad i \geq j > 0\,,\ i \in \{0,1,\dots,n+1\}\,.$$

Remark 1. Let the assumptions of Proposition 1 be satisfied. Then

$$\vec{z}(t) = A(t)\vec{y}(\varphi(t))$$

is true on J for $A(t) = \left[a_{ij}(t)\right]_{j=1,\dots,m}^{i=0,\dots,n}$, where $a_{ij}(t) = 0$ for $j > i$. Moreover, from (f), (f^*) and Proposition 1 we get

$$\begin{aligned} z_{n+1}(t) &= f^*\Big(t, \vec{z}(t), \vec{z}(\eta_1(t)), \dots, \vec{z}(\eta_m(t))\Big) \\ &= f^*\Big(t, A(t)\vec{y}(\varphi(t)), A(\eta_1(t))\vec{y}(\varphi(\eta_1(t))), \dots, A(\eta_m(t))\vec{y}(\varphi(\eta_m(t)))\Big) \end{aligned}$$

and

$$\begin{aligned} &z_{n+1}(t) = \\ &= \sum_{i=0}^{n+1} a_{n+1i}(t) y^{(i)}(\varphi(t)) \\ &= (\vec{a}_{n+1}(t), \vec{y}(\varphi(t))) + a_{n+1n+1}(t) y^{(n+1)}(\varphi(t)) \\ &= (\vec{a}_{n+1}(t), \vec{y}(\varphi(t))) + a_{n+1n+1}(t) f\Big(\varphi(t), \vec{y}(\varphi(t)), \vec{y}(\xi_1(\varphi(t))), \dots, \vec{y}(\xi_m(\varphi(t)))\Big) \\ &= (\vec{a}_{n+1}(t), \vec{y}(\varphi(t))) + a_{n+1n+1}(t) f\Big(\varphi(t), \vec{y}(\varphi(t)), \vec{y}(\varphi(\eta_1(t))), \dots, \vec{y}(\varphi(\eta_m(t)))\Big) \end{aligned}$$

is satisfied on J for transformations (2). Thus (f), (f^*) are equivalent equations if and only if functions L, φ satisfy the assumptions of Proposition 1 and

$$\begin{aligned} &f^*\Big(t, A(t)\vec{y}(\varphi(t)), A(\eta_1(t))\vec{y}(\varphi(\eta_1(t))), \dots, A(\eta_m(t))\vec{y}(\varphi(\eta_m(t)))\Big) \\ &= (\vec{a}_{n+1}(t), \vec{y}(\varphi(t))) + a_{n+1n+1}(t) f\Big(\varphi(t), \vec{y}(\varphi(t)), \vec{y}(\varphi(\eta_1(t))), \dots, \vec{y}(\varphi(\eta_m(t)))\Big) \end{aligned}$$

holds on J for functions f, f^*.

4. Results

LEMMA 1. *Let $n, r \in \mathbb{N}$ and $r \geq n+1$. Let φ satisfy the assumptions of Proposition 1. Then (2) is a stationary transformation of the equation (f) if and only if $\varphi(I) = I$ and the real function f satisfies the functional equation*

$$\begin{gathered} f\Big(s, W\vec{v}, \left[W_{(j)}\vec{v}_{(j)}\right]\Big) = (\vec{w}_{n+1}, \vec{v}) + w_{n+1n+1} f\Big(x, \vec{v}, \left[\vec{v}_{(j)}\right]\Big), \\ f(x, \vec{o}, V) \neq 0 \qquad \text{for} \quad V \neq O, \end{gathered} \tag{3}$$

where $W = [\vec{w}_j]_{j=0,\dots,n} = [w_{ij}]_{j=0,\dots,n}^{i=0,\dots,n}$, $\vec{w}_{n+1} = [w_{n+10}, w_{n+11}, \dots, w_{n+1n}]^T$, $\vec{v} = [v_0, v_1, \dots, v_n]^T$ and $w_{i0} = a_{i0}(u_i)$, $w_{ij} = a_{ij}(x_1, x_2, \dots, x_{i-j+1}, u, u_1, \dots$

$\dots, u_{i-j})$ *for* $j > 0$, *are defined by*

$$
\begin{aligned}
w_{i0} &= u_i\,, && 1 \le i \le n\,;\\
w_{n+10} &= h(s, x, x_1, u, u_1, \dots, u_n)\,;\\
w_{i1} &= \tbinom{i}{0} u x_i + \tbinom{i}{1} u_1 x_{i-1} + \dots + \tbinom{i}{i-1} u_{i-1} x_1\,, && 1 \le i \le n\,;\\
w_{n+11} &= (n+1) u g(s, x, x_1, \dots, x_n, u, u_1, \dots, u_n) + \sum_{j=1}^{n} \tbinom{n}{j} u_j x_{n-j}\,;\\
&\ \vdots\\
w_{ij} &= \tbinom{i}{j} u_{i-j} x_1^j + \tbinom{i}{j-1} u x_1^{j-1} x_{i-j+1}\\
&\qquad + r_{ij}(x_1, \dots, x_{i-j}, u_1, \dots, u_{i-j-1})\,, && 1 < j < i\,;\\
&\ \vdots\\
w_{ii-2} &= \tbinom{i}{2} u_2 x_1^{i-2} + \tbinom{i}{3} (u x_3 + 3 u_1 x_2) x_1^{i-3} + 3\tbinom{i}{4} u x_1^{i-4} x_2^2\,, && i \ge 2\,;\\
w_{ii-1} &= \tbinom{i}{1} u_1 x_1^{i-1} + \tbinom{i}{2} u x_1^{i-2} x_2\,, && i \ge 2\,;\\
w_{ii} &= u x_1^i\,, && i \ge 0\,;
\end{aligned}
$$

where s, $x{=}x_0$, x_i, $v{=}v_0$, v_i, $u{=}u_0, \dots, u_i \in \mathbb{R}$, $u \ne 0$; a_{ij}, r_{ij} *are real functions,* $n \in \mathbb{N}$. *Here* $\vec{v}, \vec{v}_{(1)}, \dots, \vec{v}_{(m)} \in \mathbf{V}_{n+1}$, $W, W_{(1)}, \dots, W_{(m)}$ *are matrices defined similarly to* W.

P r o o f. Let the assumptions of Lemma 1 be satisfied. The transformation (2) is a global stationary transformation of the equation (f) if and only if $\varphi(I) = I$ and the real function f satisfies

$$
\begin{aligned}
&f\Big(t, A(t)\vec{y}(\varphi(t)), A(\xi_1(t))\vec{y}(\varphi(\xi_1(t))), \dots, A(\eta_m(t))\vec{y}(\varphi(\eta_m(t)))\Big)\\
&= \big(\vec{a}_{n+1}(t), \vec{y}(\varphi(t))\big)\\
&\qquad + a_{n+1n+1}(t) f\Big(\varphi(t), \vec{y}(\varphi(t)), \vec{y}(\varphi(\xi_1(t))), \dots, \vec{y}(\varphi(\xi_m(t)))\Big)\,, \qquad t \in I\,.
\end{aligned}
$$

We denote $s = t$, $x = \varphi(t)$, $x_{(j)} = \varphi(\xi_j(t))$, $x_{i(j)} = \varphi^{(i)}(\xi_j(t))$, $u_{i(j)} = L^{(i)}(\xi_j(t))$, $w_{i0} = u_i$, $w_{i0(j)} = u_{i(j)}$; $w_{ik} = a_{ik}(x_1, \dots, x_{i-k+1}, u, u_1, \dots, u_{i-k})$, $w_{ik(j)} = a_{ik}(x_{1(j)}, \dots, x_{i-k+1(j)}, u_{(j)}, u_{1(j)}, \dots, u_{i-k(j)})$ $(j = 1, \dots, m)$ for $i \ge k \ge 1$. Using definitions of functions a_{ik} we obtain the assertion of Lemma 1. □

Lemma 2. *Consider arbitrary matrices* $W_{(k)} = [\vec{w}_{j(k)}]_{j=0,\dots,n}$; $V = [\vec{v}_{(j)}]_{j=1,\dots,m}$; $H = [h_{ij}]_{j=1,\dots,m}^{i=0,\dots,n} = [\vec{h}_j]_{j=0,\dots,n}$; *where* $\vec{w}_{j(k)}, \vec{v}_{(j)}, \vec{h}_j \in \mathbf{V}_{n+1}$, $h_{ij} = h_{ij}(V)$, $k = 1, \dots, m$; $m, n \in \mathbb{N}$.

The general continuous solution, in the class of functions continuous at a point, of the matrix functional equation

$$H\big(W_{(1)}\vec{v}_{(1)},\dots,W_{(m)}\vec{v}_{(m)}\big)$$
$$=\sum_{i=0}^{n} H\big(W_{(1)}\vec{e}_i,\vec{o},\dots,\vec{o}\big)h_{i1}(V)+\dots+\sum_{i=0}^{n} H\big(\vec{o},\vec{o},\dots,W_{(m)}\vec{e}_i\big)h_{im}(V)\,, \tag{4}$$
$$H(O)=O\,,\qquad H(E_{ij})=E_{ij}\,,$$

is given by

$$H(V)=V\,.$$

P r o o f. Let $k\in\{1,\dots,m\}$ be fixed and $\vec{v}_{(j)}=\vec{o}$ for $j\neq k$. Then

$$H\big(\vec{o},\dots,\vec{o},W_{(k)}\vec{v}_{(k)},\vec{o},\dots,\vec{o}\big)$$
$$=\sum_{i=0}^{n} H\big(W_{(1)}\vec{e}_i,\vec{o},\dots,\vec{o}\big)h_{i1}\big(\vec{o},\dots,\vec{o},\vec{v}_{(k)},\vec{o},\dots,\vec{o}\big)+\cdots$$
$$\cdots+\sum_{i=0}^{n} H\big(\vec{o},\dots,\vec{o},W_{(k)}\vec{e}_i,\vec{o},\dots,\vec{o}\big)h_{ik}\big(\vec{o},\dots,\vec{o},\vec{v}_{(k)},\vec{o},\dots,\vec{o}\big)+\cdots$$
$$\cdots+\sum_{i=0}^{n} H\big(\vec{o},\dots,\vec{o},W_{(m)}\vec{e}_i\big)h_{im}\big(\vec{o},\dots,\vec{o},\vec{v}_{(k)},\vec{o},\dots,\vec{o}\big)$$

where $W_{(k)}=\big[\vec{w}_{j(k)}\big]$, $\vec{w}_{j(k)},\vec{v}_{(k)}\in\mathbf{V}_{n+1}$. We have

$$h_{ij}\big(\vec{o},\dots,\vec{o},\vec{v}_{(k)},\vec{o},\dots,\vec{o}\big)=0\qquad\text{for}\quad j\neq k \tag{5}$$

because the left hand side of the above equation is independent of $W_{(j)}$, $j\neq k$. Hence

$$H^*\big(W_{(k)}\vec{v}_{(k)}\big)=\sum_{i=0}^{n} H^*\big(W_{(k)}\vec{e}_i\big)h^*_{ik}\big(\vec{v}_{(k)}\big)\,, \tag{6}$$

$H^*\big(\vec{v}_{(k)}\big):=H\big(\vec{o},\dots,\vec{o},\vec{v}_{(k)},\vec{o},\dots,\vec{o}\big)$.

Using $\vec{v}_{(k)}=\vec{e}_0+\vec{e}_1$ and $H^*(\vec{o})=H(O)=O$ for $W_{(k)}\vec{v}_{(i)}=\vec{w}_{i(k)}=\vec{o}$ $(i\geq 2)$ we get

$$H^*(\vec{x}+\vec{y})=\alpha H^*(\vec{x})+\beta H^*(\vec{y})\,,\qquad \alpha,\beta\in\mathbb{R},\quad \vec{x},\vec{y}\in\mathbf{V}_{n+1}, \tag{7}$$

where $\alpha=h^*_{0k}(\vec{e}_0+\vec{e}_1)$, $\beta=h^*_{1k}(\vec{e}_0+\vec{e}_1)$, $\vec{x}=W_{(k)}\vec{e}_0$, $\vec{y}=W_{(k)}\vec{e}_1$.

For $\vec{x}=\vec{o}$ we have $H^*(\vec{y})=\alpha H^*(\vec{o})+\beta H^*(\vec{y})$ and $\beta=1$. Similarly $\vec{y}=\vec{o}$ gives $\alpha=1$ and (7) becomes

$$H^*(\vec{x}+\vec{y})=H^*(\vec{x})+H^*(\vec{y})\,,\qquad \vec{x},\vec{y}\in\mathbf{V}_{n+1}\,. \tag{8}$$

Breaking up (8) into columns $\vec{h}_j^*$ $(j = 1, \dots, m)$ we obtain the analogue of Cauchy's functional equation

$$\vec{h}_j^*(\vec{x} + \vec{y}) = \vec{h}_j^*(\vec{x}) + \vec{h}_j^*(\vec{y}), \qquad \vec{x}, \vec{y} \in \mathbf{V}_{n+1}, \tag{9}$$

for every $j \in \{1, \dots, m\}$. The general solution of (9) in the class of functions $\vec{h}^* : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ continuous at a point is given by

$$\vec{h}_j^*(\vec{x}) = C_j \vec{x}, \qquad \vec{x} \in \mathbf{V}_{n+1}, \tag{10}$$

where C_j is a constant $n+1$ by $n+1$ matrix, $j \in \{1, \dots, m\}$ (see Aczél [1]).

For a fixed column k, the condition $H(E_{ik}) = E_{ik}$ $(i, k \in \{0, \dots, n\})$ is equivalent to the conditions

$$\vec{h}_j^*(\vec{e}_i) = \vec{h}_j(\vec{o}, \dots, \vec{o}, \vec{e}_i, \vec{o}, \dots, \vec{o}) = \begin{cases} \vec{o} & \text{for } j \neq k, \\ \vec{e}_i & \text{for } j = k, \end{cases} \qquad i = 0, \dots, n.$$

We get

$$C_k = E, \quad C_j = O \qquad \text{for} \quad j \neq k,$$

and

$$H^*(\vec{v}_{(k)}) = H(\vec{o}, \dots, \vec{o}, \vec{v}_{(k)}, \vec{o}, \dots, \vec{o}) = [\vec{o}, \dots, \vec{o}, \vec{v}_{(k)}, \vec{o}, \dots, \vec{o}] \tag{11}$$

for every fixed $k \in \{1, \dots, m\}$ using (10).

Putting $\vec{v}_{(j)} = \vec{e}_0$ $(j = 1, \dots, m)$ into (4) we get

$$\begin{aligned} &H(\vec{w}_{0(1)}, \dots, \vec{w}_{0(m)}) \\ &= \alpha_{01} H(\vec{w}_{0(1)}, \vec{o}, \dots, \vec{o}) + \dots + \alpha_{0m} H(\vec{o}, \dots, \vec{w}_{0(m)}) \\ &\qquad + \sum_{i=1}^{n} \alpha_{i1} H(\vec{w}_{i(1)}, \vec{o}, \dots, \vec{o}) + \dots + \sum_{i=1}^{n} \alpha_{im} H(\vec{o}, \dots, \vec{w}_{i(m)}), \end{aligned}$$

where $\vec{w}_{i(j)} = W_{(j)} \vec{e}_i$ and $\alpha_{ij} = h_{ij}(\vec{e}_0, \dots, \vec{e}_0)$. Here $\alpha_{ij} = 0$ for $i > 0$ because the left hand side of the equation is independent of $\vec{w}_{i(j)}$, $i > 0$.

Thus, using (11),

$$\begin{aligned} H(\vec{w}_{0(1)}, \dots, \vec{w}_{0(m)}) &= \alpha_{01} H(\vec{w}_{0(1)}, \vec{o}, \dots, \vec{o}) + \dots + \alpha_{0m} H(\vec{o}, \dots, \vec{o}, \vec{w}_{0(m)}) \\ &= \alpha_{01} [\vec{w}_{0(1)}, \vec{o}, \dots, \vec{o}] + \dots + \alpha_{0m} [\vec{o}, \dots, \vec{o}, \vec{w}_{0(m)}] \\ &= [\alpha_{01} \vec{w}_{0(1)}, \dots, \alpha_{0m} \vec{w}_{0(m)}] \end{aligned}$$

and we have

$$H(\vec{v}_{(1)}, \dots, \vec{v}_{(m)}) = [\alpha_{01} \vec{v}_{(1)}, \dots, \alpha_{0m} \vec{v}_{(m)}], \qquad \vec{v}_{(j)} \in \mathbf{V}_{n+1}.$$

The conditions $H(E_{ij}) = E_{ij}$ are fulfilled if and only if $\alpha_{0j} = 1$, $i \in \{0, \dots, n\}$, $j \in \{1, \dots, m\}$.

Hence

$$H(V) = V \quad (\iff h_{ij}(V) = v_{ij})$$

is the general solution of the matrix functional equation (4) in the class of functions continuous at a point. □

THEOREM 1. *The general real valued solutions of* (3) *continuous at a point are given by*

$$f(x,\vec{v},V) = \big(\vec{p}(x),\vec{v}\big) + q(x)\,, \qquad q(x) \neq 0\,, \tag{12}$$

and

$$f(x,\vec{v},V) = \big(\vec{p}(x),\vec{v}\big) + \big(Q(x),V\big)\,, \tag{13}$$

where p_i, q, q_{ij} *are arbitrary functions,* $Q = [q_{ij}] = [\vec{q}_j]$, $\vec{p},\vec{v},\vec{q}_j \in \mathbf{V}_{n+1}$, $i \in \{0,\dots,n\}$, $j \in \{1,\dots,m\}$, $m,n \in \mathbb{N}$.

P r o o f . Consider the functional equation (3)

$$f\Big(s, W\vec{v}, \big[W_{(j)}\vec{v}_{(j)}\big]\Big) = (\vec{w}_{n+1},\vec{v}\,) + w_{n+1n+1} f\Big(x,\vec{v},\big[\vec{v}_{(j)}\big]\Big)\,,$$
$$\tilde{f}(x,V) = f(x,\vec{o},V) \neq 0 \qquad \text{for} \quad V \neq O\,.$$

Using (3) and $\vec{v} = \vec{o}$ we have

$$\tilde{f}\Big(s,\big[W_{(j)}\vec{v}_{(j)}\big]\Big) = w_{n+1n+1}\tilde{f}\Big(x,\big[\vec{v}_{(j)}\big]\Big)\,. \tag{14}$$

Define the functions $p_i(x) = f(x,\vec{e}_i,O)$. Then (3) together with $V = \big[\vec{v}_{(j)}\big] = O$ and $\vec{v} = \vec{e}_i$, $i \in \{0,\dots,n\}$, gives

$$w_{n+1i} = f(s,W\vec{e}_i,O) - w_{n+1n+1}p_i(x)\,, \qquad i \in \{0,\dots,n\}\,. \tag{15}$$

Substituting (14), (15) into (3) we obtain

$$\begin{aligned} h(\vec{v},[\vec{v}_{(j)}]) &:= \frac{f\Big(s,W\vec{v},\big[W_{(j)}\vec{v}_{(j)}\big]\Big) - \sum\limits_{i=0}^{n} f(s,W\vec{e}_i,O)v_i}{\tilde{f}\Big(s,\big[W_{(j)}\vec{v}_{(j)}\big]\Big)} \\ &= \frac{f\big(x,\vec{v},[\vec{v}_{(j)}]\big) - \big(p(\vec{x}),\vec{v}\big)}{\tilde{f}\big(s,[\vec{v}_{(j)}]\big)}\,. \end{aligned} \tag{16}$$

The function f is given by

$$f(x,\vec{v},V) = \big(\vec{p}(x),\vec{v}\big) + \tilde{f}(x,V)h(\vec{v},V)\,, \quad h(\vec{o},V) = 1 \qquad \text{for} \quad V \neq O \tag{17}$$

because $\tilde{f}(x,V) = f(x,\vec{o},V) = \tilde{f}(x,V)h(\vec{o},V)$ and $\tilde{f}(x,V) \neq 0$ for arbitrary $n+1$ by m matrix $V \neq O$. Moreover,

$$f\Big(s,W\vec{v},\big[W_{(j)}\vec{v}_{(j)}\big]\Big) = \sum_{i=0}^{n} f(s,W\vec{e}_i,O)v_i + \tilde{f}\Big(s,\big[W_{(j)}\vec{v}_{(j)}\big]\Big)h\Big(\vec{v},\big[\vec{v}_{(j)}\big]\Big)\,. \tag{18}$$

We denote $q(s) = \tilde{f}(s,O)$ and $\delta(\vec{v}) = h(\vec{v},O)$. If we combine (17) with (18) it follows

$$\begin{aligned}\tilde{f}\Big(s,[W_{(j)}\vec{v}_{(j)}]\Big)h\Big(W\vec{v},[W_{(j)}\vec{v}_{(j)}]\Big)\\ = q(s)\sum_{i=0}^{n}\delta(W\vec{e}_i)v_i + \tilde{f}\Big(s,[W_{(j)}\vec{v}_{(j)}]\Big)h\Big(\vec{v},[\vec{v}_{(j)}]\Big)\end{aligned} \tag{19}$$

and using $[\vec{v}_{(j)}] = O$ we get

$$q(s)\delta(W\vec{v}) = q(s)\left(\sum_{i=0}^{n}\delta(W\vec{e}_i)v_i + \delta(\vec{v})\right), \qquad \delta(\vec{o}) = h(\vec{o},O) = 1\,. \tag{20}$$

because

$$q(s) = q(s)h(\vec{o},O)$$

with respect to (17).

First we consider $q(s) = \tilde{f}(s,O) \neq 0$. Then $\delta(\vec{o}) = h(\vec{o},O) = 1$,

$$\delta(W\vec{v}) = \sum_{i=0}^{n}\delta(W\vec{e}_i)v_i + \delta(\vec{v}) \tag{21}$$

and $\delta(\vec{e}_i) = \delta(W\vec{e}_i) - \delta(W\vec{e}_i) = 0$ for $i = 0,\dots,n$. Choosing $\vec{v} = \sum\limits_{i=0}^{n}\vec{e}_i = (1,\dots,1)^T = \vec{1} \in \mathbf{V}_{n+1}$ we obtain

$$\delta(W\vec{1}) = \sum_{i=0}^{n}\delta(W\vec{e}_i) + K\,, \qquad K = \delta(\vec{1})$$

and

$$\delta^*(W\vec{1}) = \delta^*\left(\sum_{i=0}^{n}W\vec{e}_i\right) = \sum_{i=0}^{n}\delta^*(W\vec{e}_i) \tag{22}$$

for $\delta^*(\vec{u}) = \delta(\vec{u}) + \frac{K}{n}$. The general solution of (22) continuous at a point is of the form

$$\delta^*(\vec{u}) = \sum_{i=0}^{n}c_i u_i = (\vec{c},\vec{u})\,, \qquad c_i \in \mathbb{R}, \quad \vec{u} \in \mathbf{V}_{n+1} \tag{23}$$

(see A c z é l [1]). Moreover, $1 = \delta(\vec{o}) = \delta^*(\vec{o}) - \frac{K}{n} = -\frac{K}{n}$. Hence

$$\delta(\vec{u}) = (\vec{c},\vec{u}) + 1 \tag{24}$$

by means of (22), (23). We have $0 = \delta(\vec{e}_i) = (\vec{c},\vec{e}_i)+1 = c_i+1$ for $i \in \{0,\dots,n\}$. Thus $\vec{c} = -\vec{1}$ and

$$\delta(\vec{u}) = 1 - (\vec{1},\vec{u})\,, \qquad \vec{u} \in \mathbf{V}_{n+1}\,. \tag{25}$$

We get $\sum_{i=0}^{n} \delta(W\vec{e}_i) = (\vec{1}, (E-W)\vec{v})$ and (21) is satisfied for all $\vec{v} \in \mathbf{V}_{n+1}$. Using (11) and (25) we obtain

$$\tilde{f}\Big(s, [W_{(j)}\vec{v}_{(j)}]\Big)\Big(h\Big(W\vec{v}, [W_{(j)}\vec{v}_{(j)}]\Big) - h(\vec{v}, V)\Big) = q(s)\big(\vec{1}, (E-W)\vec{v}\big)\,. \quad (26)$$

We have

$$\tilde{f}(s,V)\big(h(W\vec{v},V) - h(\vec{v},V)\big) = q(s)\big(\vec{1},(E-W)\vec{v}\big)\,, \qquad V = [\vec{v}_{(j)}] \quad (27)$$

for $W_{(j)}\vec{v}_{(j)} = E$ $(j = 1, \dots, m)$. Then (27) together with $V = E$, $\tilde{q}(s) = \tilde{f}(s,E)$, $\tilde{h}(\vec{v}) = h(\vec{v}, E)$ gives

$$\tilde{q}(s)\big(\tilde{h}(W\vec{v}) - \tilde{h}(\vec{v})\big) = q(s)\big(\vec{1},(E-W)\vec{v}\big)\,, \quad (28)$$

i.e.

$$\frac{\tilde{h}(W\vec{v}) - \tilde{h}(\vec{v})}{\big(\vec{1},(E-W)\vec{v}\big)} = \frac{q(s)}{\tilde{q}(s)} = r \in \mathbb{R} - \{0\}$$

for $W \neq E$, $\vec{v} \neq \vec{o}$. It follows

$$\tilde{q}(s) = \frac{1}{r} q(s) \quad (29)$$

and we get

$$\tilde{h}(W\vec{v}) = \tilde{h}(\vec{v}) + r \cdot \big(\vec{1},(E-W)\vec{v}\big)\,.$$

For $\vec{v} = \vec{e}_0$ we have $\tilde{h}(\vec{w}_0) = \tilde{h}(\vec{e}_0) + r\big(\vec{1},\vec{e}_0\big) - r\big(\vec{1},\vec{w}_0\big)$, i.e.

$$\tilde{h}(\vec{v}) = a - r\big(\vec{1},\vec{v}\big)\,, \qquad a \in \mathbb{R}, \quad r \in \mathbb{R} - \{0\}\,. \quad (30)$$

The comparison of (27) and (28) gives

$$\tilde{f}(s,V)\big(h(W\vec{v},V) - h(\vec{v},V)\big) = \tilde{q}(s)\big(\tilde{h}(W\vec{v}) - \tilde{h}(\vec{v})\big) = q(s)\big(\vec{1},(E-W)\vec{v}\big)$$

and

$$z(V) := \frac{h(W\vec{v},V) - h(\vec{v},V)}{\big(\vec{1},(E-W)\vec{v}\big)} = \frac{q(s)}{\tilde{f}(s,V)} \neq 0$$

for $W \neq E$, $\vec{v} \neq \vec{o}$ by means of (29), (30). Thus

$$\tilde{f}(s,V) = \frac{q(s)}{z(V)} \quad (31)$$

and

$$h(W\vec{v},V) = h(\vec{v},V) + z(V)\big(\vec{1},(E-W)\vec{v}\big)\,.$$

For $\vec{v} = \vec{e}_0$ we have

$$h(\vec{w}_0,V) = h(\vec{e}_0,V) + z(V)\big(\vec{1},\vec{e}_0\big) - z(V)\big(\vec{1},\vec{w}_0\big)$$

and $h(\vec{v}, V)$ is given by

$$h(\vec{v}, V) = \gamma(V) - z(V)(\vec{1}, \vec{v}). \tag{32}$$

Using (26) we obtain

$$\gamma([W_{(j)}\vec{v}_{(j)}]) - \gamma([\vec{v}_{(j)}]) = \Big(z([W_{(j)}\vec{v}_{(j)}]) - z([\vec{v}_{(j)}])\Big)(\vec{1}, \vec{v}) \tag{33}$$

and with $\vec{v} = \vec{o}$, $\vec{v}_{(j)} = \vec{e}_0$ $(j = 1, \ldots, m)$ we have

$$\gamma([\vec{w}_{0(j)}]) = \gamma \in \mathbb{R},$$

i.e.

$$\gamma(V) = \gamma \in \mathbb{R}. \tag{34}$$

Similarly, from (33) and (34) we get

$$z(V) = z \in \mathbb{R} - \{0\}. \tag{35}$$

Hence

$$h(\vec{v}, V) = \gamma - z(\vec{1}, \vec{v}), \qquad \gamma \in \mathbb{R}, \quad z \in \mathbb{R} - \{0\} \tag{36}$$

in accordance with (32), (34), (35). We compare (36) with (25),

$$h(\vec{v}, V) = \gamma - z(\vec{1}, \vec{v}) = h(\vec{v}, O) = \delta(\vec{v}) = 1 - (\vec{1}, \vec{v})$$

and we obtain

$$h(\vec{v}, V) = 1 - (\vec{1}, \vec{v}), \qquad \gamma(V) = 1, \qquad z(V) = 1, \tag{37}$$

$V = [\vec{v}_{(j)}]$; $\vec{v}, \vec{v}_{(j)} \in \mathbf{V}_{n+1}$. Combined (17) with (31) and (37) it follows

$$f(x, \vec{v}, V) = (\vec{p}(x), \vec{v}) + q(x)(1 - (\vec{1}, \vec{v})),$$

i.e.

$$f(x, \vec{v}, V) = (\vec{p}*(x), \vec{v}) + q(x), \qquad q(x) \neq 0, \tag{38}$$

where $p_0^*, p_1^*, \ldots, p_n^*$ are arbitrary functions and the form (12) of Theorem 1 is derived.

In the case $q(s) = \tilde{f}(s, O) = 0$, using (19), we have

$$h\Big(W\vec{v}, [W_{(j)}\vec{v}_{(j)}]\Big) = h\Big(\vec{v}, [\vec{v}_{(j)}]\Big). \tag{39}$$

Choosing $\vec{v} = \vec{v}_{(j)} = \vec{e}_0$ $(j = 1, \ldots, m)$ we obtain $h(\vec{w}_0, [\vec{w}_{0(j)}]) = h \in \mathbb{R}$. Hence

$$h(\vec{v}, V) = 1 \tag{40}$$

because $h(\vec{o}, V) = 1$ by (17). From (17) and (40) we have

$$f(x, \vec{v}, V) = (\vec{p}(x), \vec{v}) + \tilde{f}(x, V), \tag{41}$$

where $\tilde{f}(x,O) = q(x) = 0$. Consider (14)

$$\tilde{f}\Big(s, [W_{(j)}\vec{v}_{(j)}]\Big) = w_{n+1n+1}\tilde{f}\Big(x, [\vec{v}_{(j)}]\Big).$$

Define the functions $q_{ij} = \tilde{f}(x, E_{ij})$, $i \in \{0,\dots,n\}$, $j \in \{1,\dots,m\}$. Then

$$\tilde{f}(s, W_{ij}) = w_{n+1n+1}\tilde{f}(s, E_{ij}) = w_{n+1n+1}q_{ij}(x), \tag{42}$$

where

$$W_{ij} = [\vec{o},\dots,\vec{o}, W_{(j)}\vec{e}_i, \vec{o},\dots,\vec{o}],$$

$W_{(j)}\vec{e}_i = \vec{w}_{i(j)}$ being the ith column of $W_{(j)}$. Using (14), (42) we have

$$m(n+1)h_{ij}(V) := \frac{\tilde{f}(x,V)}{q_{ij}(x)} = \frac{\tilde{f}(s, [W_{(k)}\vec{v}_{(k)}])}{\tilde{f}(s, W_{ij})}, \tag{43}$$

$V = [\vec{v}_{(k)}]$, $h_{ij}(V) \neq 0$, $i \in \{0,\dots,n\}$, $j \in \{1,\dots,m\}$. Thus

$$\tilde{f}(x,V) = m(n+1)q_{ij}(x)h_{ij}(V)$$

and the sum for all i, j gives

$$\tilde{f}(x,V) = \big(Q(x), H(V)\big), \qquad Q = [q_{ij}], \quad H = [h_{ij}]. \tag{44}$$

In accordance with $q(x) = \tilde{f}(x,O) = \big(Q(x), H(O)\big) = 0$ and $q_{ij}(x) = \tilde{f}(x, E_{ij}) = \big(Q(x), H(E_{ij})\big)$ we get

$$H(O) = O \quad \text{and} \quad H(E_{ij}) = E_{ij}, \quad i \in \{0,\dots,n\}, \ j \in \{1,\dots,m\}. \tag{45}$$

Similarly, using (43),

$$\tilde{f}\Big(s, [W_{(k)}\vec{v}_{(k)}]\Big) = \sum_i^j \tilde{f}(s, W_{ij})h_{ij}([\vec{v}_{(k)}]). \tag{46}$$

Substituting (44) into (46) we obtain

$$\begin{aligned}\Big(Q(s), H\big([W_{(k)}\vec{v}_{(k)}]\big)\Big) &= \sum_i^j \Big(Q(s), H\big(W_{ij}\big)\Big)h_{ij}(V)\\ &= \Big(Q(s), \sum_i^j H(W_{ij})h_{ij}(V)\Big)\end{aligned}$$

and we need to solve the matrix equation

$$\begin{aligned}&H\big(W_{(1)}\vec{v}_{(1)},\dots,W_{(m)}\vec{v}_{(m)}\big)\\ &= \sum_{i=0}^{n} H\big(W_{(1)}\vec{e}_i, \vec{o},\dots,\vec{o}\big)h_{i1}(V) + \dots + \sum_{i=0}^{n} H\big(\vec{o},\vec{o},\dots,W_{(m)}\vec{e}_i\big)h_{im}(V),\\ &\qquad H(O) = O, \qquad H(E_{ij}) = E_{ij}, \qquad V = [\vec{v}_{(k)}],\end{aligned}$$

in accordance with the definition of W_{ij} and (45). The general continuous solution of (47), in the class of functions continuous at a point, is given by

$$H(V) = V$$

due to Lemma 2. Combined (41), (44) and (47) we obtain (13) and the assertion of Theorem 1 is proved. □

THEOREM 2. *If (2) is the stationary transformation of the equation (f) then (f) is a linear functional-differential equation*

$$y^{(n+1)}(x) = y_{n+1}(x) = f\big(x, \vec{y}(x), Y(x)\big) = \big(\vec{p}(x), \vec{y}(x)\big) + \big(Q(x), Y(x)\big)\,, \quad (47)$$

where $p_i(x)$, $q_{ij}(x)$ $(i = 0, \dots, n;\ j = 1, \dots, m)$ are arbitrary functions, $\vec{y}(x) = \big(y(x), y'(x), \dots, y^{(n)}(x)\big)^T$, $Y(x) = \big[\vec{y}(\xi_1(x)), \dots, \vec{y}(\xi_m(x))\big]$, $x \in I$.

Proof. The assertion of Theorem 2 follows from Lemma 1 and Theorem 1. The transformation (2) is a stationary transformation of (f) if and only if $\varphi(I) = I$ and the real function (f) satisfies the functional equation (3). The solution of (3) corresponding to the functional-differential equation (f) is given by (13)

$$f(x, \vec{v}, V) = \big(\vec{p}(x), \vec{v}\big) + \big(Q(x), V\big)$$

and (f) becomes (48). □

Remark 2. The criterion of global equivalence of the second order linear differential equations was published by O. Borůvka [3], of the third and higher order equations by F. Neuman [8]. Some criterion of global equivalence of the second and higher orders linear functional-differential equations with m $(m \geq 1)$ delays is derived in [12].

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