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ON THE EXISTENCE OF A SOLUTION FOR NONLINEAR OPERATOR EQUATIONS IN FRÉCHET SPACES

MÁRIA KEČKEMÉTYOVÁ

ABSTRACT. There is proved a theorem on the existence of a solution for operator equation $Lx = Nx$ in Fréchet space in this paper, where L is a linear operator and N is generally nonlinear and also the existence of a continuous solution for a system of nonlinear differential equations with linear boundary conditions is proved.

Introduction

The aim of this paper is to prove some theorems which assure the existence of a solution for the equation

$$Lx = Nx, \tag{1.1}$$

where L is a linear operator, N is an operator, generally nonlinear, both of them are defined in a Fréchet space. In the first section we shall prove the continuation theorem by using the theorem which states equivalence between the set of solutions for (1.1) and the set of fixed points of the operator M defined by (1.6) and using Schaefer's theorem [5]. This theorem is a modification of the continuation theorem which was proved by P. L. Zezza [6] in Banach space on Fréchet space . In the second section we shall transform by the method of M. Cecchi, M. Marini, P. L. Zezza [1] the nonlinear system

$$\dot{x}(t) - A(t)x(t) = f\big(t, x(t)\big)$$

with linear boundary conditions

$$Tx = r$$

into the form of (1.1) and using the equivalence theorem and Tichonov's fixed point theorem [3] we shall prove a theorem which assures the existence of a continuous solution, generally unbounded, for this boundary-value problem.

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1. Let X be a Fréchet space, Y be a locally convex space and let $L\colon \operatorname{dom} L \subset X \to Y$ be a linear operator such that: $\dim(\operatorname{Ker} L) < \infty$. Then there exists [4] a linear continuous projection P from X into X such that:

$$\begin{aligned} &\operatorname{Im} P = \operatorname{Ker} L && \text{(and simultaneously} \\ &LPx = 0 && \text{for each } x \in X\text{).} \end{aligned} \tag{1.2}$$

Then the space X can be expressed as a topological direct sum

$$X = X_P \oplus X_{I-P}, \tag{1.3}$$

where $X_P = \operatorname{Im} P$, $X_{I-P} = \operatorname{Im}(I-P)$ and $I\colon X \to X$ is the identity mapping. Clearly $L\big|_{(\operatorname{dom} L) \cap X_{I-P}}$ is invertible and it is onto $\operatorname{Im} L$.

Let K_P be its inverse operator:

$$K_P\colon \operatorname{Im} L \to (\operatorname{dom} L) \cap X_{I-P}. \tag{1.4}$$

Let N be an operator generally nonlinear, $N\colon \operatorname{dom} N \subset X \to Y$. The following theorem holds for the operators L, N and for the equation

$$Lx = Nx. \tag{1.1}$$

THEOREM 1.1. *Let* $A = \{x \in X\colon Nx \in \operatorname{Im} L\} = N^{-1}(\operatorname{Im} L) \neq \emptyset$. *The equation* (1.1) *is then equivalent to the equation*

$$x = Px + K_P N x \qquad \textit{with} \quad x \in A. \tag{1.5}$$

For demonstration see [6].

We can write the equation (1.5) in the form

$$x = Mx \tag{1.6}$$

with $M\colon \operatorname{dom} M \subset X \to X$, $\operatorname{dom} M = A$, where $Mx = Px + K_P N x$.

COROLLARY 1.1. *Let* $A \neq \emptyset$. *Then the equation*

$$x = K_P N x \tag{1.7}$$

is equivalent to the equations $Lx = Nx$, $Px = 0$.

Remark 1.1. If $A = \emptyset$, then $\operatorname{Im} L \cap \operatorname{Im} N = \emptyset$ and equation (1.5) has no solution.

Remark 1.2. If the operator N is completely continuous and K_P is continuous, then $K_P N\colon \operatorname{dom} M \subset X \to Y$ is completely continuous. The operator M is also completely continuous since P is continuous and its range is finite dimensional. We shall obtain a similar result if the operator K_P is completely continuous and N is continuous and bounded, it means that N maps bounded sets into bounded sets.

Further we shall use the following theorem which is the extension of Leray-Schauder's theorem to locally convex spaces. First, we shall introduce the following definition.

DEFINITION 1.1. *Let X be a real locally convex space. The mapping $\psi : X \to X$ is called strictly completely continuous if and only if it is continuous and such that $\psi(n\overline{U})$ is a relatively compact set for each natural number n and neighbourhood U of 0 in X.*

THEOREM 1.2. *(Schaefer's theorem [5]). Let X be a real complete locally convex space. Let $\psi\colon X \to X$ be strictly completely continuous. Then either there exist a solution of the equation $x = \lambda\psi(x)$ for each $\lambda \in \langle 0;1\rangle$ or the set of all possible solutions of the equation $x = \lambda\psi(x)$ $\{x\colon x = \lambda\psi(x);\ \lambda \in (0;1)\}$ is not bounded in X.*

Remark 1.3. Because a bounded set is absorbed by each neighbourhood of 0, a strictly completely continuous mapping maps each bounded set into a relatively compact set and therefore the following implication holds:
If a mapping is strictly completely continuous, then it is completely continuous.

LEMMA 1.1. *Let the operators L and N be such that N is defined in the whole space X and*

$$N(X) \subset \operatorname{Im} L. \tag{1.8}$$

Let either

(1.9) *N be strictly completely continuous and K_P be continuous*
or

(1.10) *N be a continuous mapping with the property:*
For each neighbourhood $U_1(N(0))$ of the point $N(0)$ there exists such a neighbourhood $V(0)$ of 0 that for each natural number n there exists a natural number k for which we have: $N\big(n\overline{V(0)}\big) \subset k\overline{U_1\big(N(0)\big)}$ and K_P be strictly completely continuous.

Then the mapping $K_P N\colon X \to X$ is strictly completely continuous.

Proof. Since $Nx \in \operatorname{Im} L$ for each $x \in X$, $A = X$. Further, the mapping $K_P N$ is continuous in both cases (1.9), (1.10). Suppose that (1.9) holds.

Let n be a natural number, let $U(0)$ be a neighbourhood of 0 such that $\overline{N(n\overline{U(0)})}$ is a relatively compact set. Then $\overline{N(n\overline{U(0)})}$ is a compact set and the set $K_P(\overline{N(n\overline{U(0)})})$ is also compact since the mapping K_P is continuous. Then the set $K_P(N(n\overline{U(0)}))$ is also relatively compact.

Now let (1.10) hold and n be an arbitrary, but fixed natural number. Then there exists such a neighbourhood $U(0)$ of 0 that $K_P(n\overline{U(0)})$ is a relatively compact set. Let us consider the neighbourhood $U_1(N(0)) = N(0) + U(0)$ of the point $N(0)$. Then there exists a neighbourhood $V(0)$ of 0 such that for each natural number n there exists a natural number k with the property: $N(n\overline{V(0)}) \subset k\overline{U_1(N(0))}$. So we have:

$$K_P N(n\overline{V(0)}) \subset K_P(k\overline{U_1(N(0))}) =$$
$$= K_P(k(\overline{N(0) + U(0)})) = kK_P(N(0)) + K_P(k\overline{U(0)})$$

and $K_P(k(\overline{U_1(N(0))}))$ is a relatively compact set and hence its subset $K_P N(n(\overline{V(0)}))$ is also relatively compact.

Consequently, the mapping $K_P N$ is in both cases strictly completely continuous.

COROLLARY 1.2. *Let X be a real Fréchet space, the topology of which is determined by the system of seminorms $\{p_m\}_{m=1}^{\infty}$. Let the mapping $M_0 \colon X \to X$ be strictly completely continuous. If the set $\{x \colon x = \lambda M_0(x);\ 0 < \lambda < 1\}$ is bounded (that means: For each natural number m there exists $c_m > 0$ such that if $x = \lambda M_0(x)$, then $p_m(x) \le c_m$), then there exists at least one fixed point of M_0.*

Using Corollaries 1.1, 1.2 and Lemma 1.1 we shall prove the following theorem.

THEOREM 1.3. (*Continuation theorem*). *Let X be a real Fréchet space the topology of which is determined by the nondecreasing system of seminorms $\{p_m\}_{m=1}^{\infty}$. Let there exist $c_m > 0$ for each natural number m such that the following implication holds:*
If x is an arbitrary possible solution of the equation $Lx = \lambda Nx$ for each λ, $0 < \lambda < 1$, then $p_m(x) \le c_m$.
Let L and N satisfy all hypotheses of Lemma 1.1. Then equation $Lx = Nx$ has at least one solution.

Proof. Let $M_0 = K_P N$. By Lemma 1.1 $M_0 \colon X \to X$ is strictly completely continuous. Now let us prove

$$\{x \colon x = \lambda M_0(x);\ 0 < \lambda < 1\} \subset \{x \colon Lx = \lambda Nx;\ 0 < \lambda < 1\}. \tag{1.11}$$

Consider an arbitrary element $x \in X$ such that $x = \lambda M_0(x)$. Applying operator L to both sides of the last equality we obtain $Lx = L(\lambda M_0(x))$, where $L(\lambda M_0(x)) = \lambda L(K_P N x)$ and $LK_P Nx = Nx$. Therefore $Lx = \lambda Nx$, it means that (1.11) holds. Hence the set $\{x\colon x = \lambda M_0(x);\ 0 < \lambda < 1\}$ is also bounded and by Corollary 1.2 there exists at least one fixed point of M_0. The assertion of the theorem follows from Corollary 1.1.

By the Banach fixed point theorem and Corollary 1.1 we shall prove the following theorem.

THEOREM 1.4. *Let X and Y be two Banach spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Y$, respectively. Let the mapping $K_P\colon \operatorname{Im} L \subset Y \to X$ be continuous with the norm $\|K_P\|$. Let $N\colon X \to Y$ satisfy condition* (1.8) *and*

$$\|Nx - Ny\|_Y \le q\|x - y\|_X \qquad \textit{for each} \quad x, y \in X, \tag{1.12}$$

where $q\|K_P\| < 1$.

Then there exists a unique solution x of the equation (1.1) *which satisfies $Px = 0$.*

P r o o f. It suffices to prove that the mapping $M_0 = K_P N$ is contractive in X. First of all, from (1.8) it follows that M_0 is defined in X. Further we have for any two elements $x, y \in X$:

$$\|M_0 x - M_0 y\|_X \le \|K_P\| \cdot \|Nx - Ny\|_Y \le q\|K_P\| \cdot \|x - y\|_X$$

wherefrom the result follows.

2. Let a be a real number and let $C = C(\langle a;\infty),\mathbb{R}^n)$ be a real locally convex space of continuous functions from $\langle a;\infty)$ into $\mathbb{R}^n$, the topology of which is given by the system of seminorms: $p_m(x) = \sup\{\|x(t)\|;\ t \in \langle a; a+m\rangle\}$ for each $x \in C$, where $\|\cdot\|$ is a norm in $\mathbb{R}^n$. The space C with this system of seminorms is a Fréchet space, which means that it is locally convex, metrizable and complete.

If $\mathbf{A} = (a_{ij})_{i,j=1}^n$ is an $n \times n$ matrix, then the matrix norm

$$\|\mathbf{A}\| = \Big(\sum_{i,j=1}^{n} a_{i,j}^2\Big)^{1/2}$$

is compatible with the vector norm [2], it means that it satisfies the following conditions:

$\|\mathbf{A}\| \le \|\mathbf{A}\| \cdot \|\mathbf{x}\|$ for an arbitrary $n \times n$ matrix and for each vector $\mathbf{x} \in \mathbb{R}^n$.

$\|\mathbf{AB}\| \le \|\mathbf{A}\| \cdot \|\mathbf{B}\|$ for arbitrary two $n \times n$ matrices $\mathbf{A}, \mathbf{B}$.

In this section we shall investigate the existence of a solution for the system

$$\dot{x}(t) - A(t)x(t) = f\big(t, x(t)\big) \tag{2.1}$$

which satisfies the boundary conditions:

$$Tx = r \qquad r \in \mathbb{R}^m \quad (1 \le m \le n), \tag{2.2}$$

where $A(t)$ is an $n \times n$ matrix, continuous on $\langle a; \infty)$.

Let $f\colon \langle a;\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function and let $T\colon \operatorname{dom} T \subset C \to \mathbb{R}^m$ $(1 \le m \le n)$ be a linear continuous operator. This means that there exists such a $\gamma > 0$ and natural number m_0 that:
$\|Tx\| \le \gamma p_{m_0}(x)$ for each $x \in \operatorname{dom} T$. Let D be a space of all possible solutions of the linear system

$$\dot{y}(t) - A(t)y(t) = 0 \tag{2.3}$$

from $C\big(\langle a;\infty), \mathbb{R}^n\big)$. Let us assume that T satisfies the condition:

$$D \subset \operatorname{dom} T \qquad \text{and} \quad T(D) = \mathbb{R}^m. \tag{2.4}$$

Now let us transform (2.1)–(2.2) into the form of the equation (1.1).

Let $L\colon \operatorname{dom} L \subset C \to C \times \mathbb{R}^m$ be the linear operator defined by the relation: $x(\cdot) \mapsto \big(\dot{x}(\cdot) - A(\cdot)x(\cdot), Tx\big)$, where $\operatorname{dom} L = \big(C^1\langle a;\infty), \mathbb{R}^n\big) \cap \operatorname{dom} T$ and let $N\colon \operatorname{dom} N \subset C \to C \times \mathbb{R}^m$ be the operator which is determined by the relation: $x(\cdot) \mapsto \big(f\big(\cdot, x(\cdot)\big), r\big)$. Then the system (2.1)–(2.2) is equivalent to the equation of the form (1.1).

Now we shall construct the operator M which is defined by (1.6).

Let $k = \dim(\operatorname{Ker} L) = n - m$ ($k \ne 0$ if $m < n$). Let $\varphi_1, \dots, \varphi_k$ be a basis for $\operatorname{Ker} L$. This basis can be extended to a basis of D:

$$\varphi_1, \dots, \varphi_k, \varphi_{k+1}, \dots, \varphi_n, \qquad \varphi_i \in C \quad i = 1, \dots, n.$$

Then $X(t) = \big(\varphi_1(t), \dots, \varphi_n(t)\big)$ is the fundamental matrix for equation (2.3). System $\{\varphi_i\}_{i=1}^n$ is bounded in C, so the following assertion holds:
For any $m \in \mathbb{N}$ there exists $H_m > 0$ such that

$$\sup\big\{\|X(t)\|;\ t \in \langle a; a+m\rangle\big\} \le H_m.$$

Further, let

$$P_1 : C \to D \qquad\qquad P_1 : x(\cdot) \mapsto X(\cdot)X^{-1}(a)x(a)$$

$$P_2 : D \to \operatorname{Ker} L \qquad P_2 : y(\cdot) = \sum_{i=1}^{n} \lambda_i \varphi_i(\cdot) \mapsto \sum_{i=1}^{k} \lambda_i \varphi_i(\cdot)$$

be linear topological projections.

Then $P = P_2 \circ P_1 \colon C \to \operatorname{Ker} L \subset C$ is a topological projection in the space C onto $\operatorname{Ker} L$. Then $C = \operatorname{Ker} L \oplus C_{I-P}$.

Let $K_P \colon \operatorname{Im} L \to (\operatorname{dom} L) \cap C_{I-P}$ be the inverse operator of $L\big|_{(\operatorname{dom} L)\cap C_{I-P}}$.

It has been proved in [1] that K_P has the form:

$$K_P\big(b(t), r\big) = X(t)JT_0^{-1}\big(r - Tz(t,a,0)\big) + z(t,a,0) =$$

$$= X(t)JT_0^{-1}\left(r - T\int_a^t X(t)X^{-1}(s)b(s)\,\mathrm{d}s\right) + \int_a^t X(t)X^{-1}(s)b(s)\,\mathrm{d}s, \quad (2.5)$$

where $I \colon C \to C$ is the identity mapping, $C_{I-P} = \operatorname{Ker} P$ and $z(t,a,0)$ is a solution of the system:

$$\dot z(t) - A(t)z(t) = b(t), \quad (2.6)$$

which satisfies the elementary condition $z(a) = 0$. $T_0 = (T\varphi_{k+1}, \dots, T\varphi_n)$ is an $m \times m$ matrix and J is an immersion of $\mathbb{R}^m$ into $\mathbb{R}^n$. For demonstration see [1].

The equation (1.1), or the system (2.1)–(2.2), are equivalent, as it is stated in theorem 1.1, to equality (1.6).

Further, let $p(t)$, $q(t) \in C\big(\langle a;\infty), \mathbb{R}\big)$ be non-negative locally integrable functions on $\langle a;\infty)$ such that:

(i) $\int_a^{a+m} p(t)\,\mathrm{d}t = \Gamma_m < \infty$, $\int_a^{a+m} q(t)\,\mathrm{d}t = \Lambda_m < \infty$ for each natural number m,

(ii) $\|X^{-1}(t)f(t,u)\| \le p(t)\|u\| + q(t)$, $u \in \mathbb{R}^n$.

R e m a r k 2.1. Each maximal solution $x(t)$ of the system (2.1) is defined on $\langle a;\infty)$.

R e m a r k 2.2. With respect to (2.5), the operator M is defined on the set: $A = \left\{ g \in C \colon \int_a^t X(t)X^{-1}(s)f\big(s, g(s)\big)\,\mathrm{d}s \in \operatorname{dom} T \right\}$. For demonstration see [1].

LEMMA 2.1. *If* $\operatorname{dom} T = C$, *then the operator* M *is defined on* C *and it is continuous.*

P r o o f. By the definition of the operators L and N we have: If $g \in C$, then $Ng = \big(f(\cdot, g(\cdot)), r\big) \in \operatorname{Im} L$ if and only if there exists a solution $x(t)$ of the system:

$$\dot x(t) - A(t)x(t) = f\big(t, g(t)\big)$$

with $x(t) \in \operatorname{dom} T$ and $Tx = r$. This solution is of the form:

$$x(t) = y(t) + \int_a^t X(t)X^{-1}(s)f\big(s, g(s)\big)\,\mathrm{d}s \qquad a \le t \le \infty,$$

where $y(t)$ is a solution of (2.3) which satisfies the condition $y(a) = x(a)$. According to (2.4) $y(t) \in \operatorname{dom} T$. Therefore $x(t) \in \operatorname{dom} T$ if and only if

$$\int_a^t X(t)X^{-1}(s)f\big(s, g(s)\big)\,\mathrm{d}s \in \operatorname{dom} T = C. \tag{2.8}$$

But this condition is satisfied. In fact, if

$$T\Big(\int_a^t X(t)X^{-1}(s)f\big(s, g(s)\big)\,\mathrm{d}s\Big) = r_0,$$

then by (2.4) there exists such $y(t) \in D$ that $Ty = r - r_0$ and so $Tx = r$. Therefore $Ng \in \operatorname{Im} L$ for each $g \in C$ and $A = \operatorname{dom} M = C$.

Further, we shall prove continuity of the operator M. Since P is a continuous projection, it suffices to prove continuity of the operator $K_P N$. Let $\{x_j\}_{j=1}^{\infty}$ be a sequence of functions from C such that it converges to x in C. Now let us prove that:

$$X(t)\int_a^t X^{-1}(s)\big[f\big(s, x_j(s)\big) - f\big(s, x(s)\big)\big]\,\mathrm{d}s \tag{2.9}$$

uniformly converges to 0 in $\langle a; a+m\rangle$ for each natural number m, which means that it converges to 0 in C. The sequence $\{x_j\}_{j=1}^{\infty}$ converges uniformly to x on $\langle a; a+m\rangle$ for each natural number m. The function $X^{-1}(t)f(t,u)$ is continuous on the compact set $\langle a; a+m\rangle \times \overline{U(0,R)}$, where $R > \|x(t)\|$ for each $t \in \langle a; a+m\rangle$. So it is uniformly continuous. This means that: For each $\varepsilon > 0$ there exists $\delta > 0$ such that for any two numbers $s_1, s_2 \in \langle a; a+m\rangle$ and for any two points $u_1, u_2 \in U(0,R)$ there holds:

If $|s_1 - s_2| < \delta$ and $| u_1 - u_2\| < \delta$, then

$$\|X^{-1}(s_1)f(s_1,u_1) - X^{-1}(s_2)f(s_2,u_2)\| <$$

and to $\;> 0$ there exi ts a natural numb r j_0 such th t for each j an l for each $s \in \langle a; a+m\rangle$ there h lds: $|x_j(s \quad x(\;)\| <$. Ther fore w have:

For each $\varepsilon > 0$ there exists a natural number j_0 such that for each $j \geq j_0$ and for each $s \in \langle a; a+m\rangle$ there holds:

$$\big\|X^{-1}(s)\big[f\big(s, x_j(s)\big) - f\big(s, x(s)\big)\big]\big\| < \varepsilon,$$

which means that $X^{-1}(s)f\big(s, x_j(s)\big)$ uniformly converges to $X^{-1}(s)f\big(s, x(s)\big)$ in $\langle a; a+m\rangle$ for an arbitrary natural number m and $j \to \infty$. Further, there holds:

$$\int_a^t \big\|X^{-1}(s)\big[f\big(s, x_j(s)\big) - f\big(s, x(s)\big)\big]\big\| \,\mathrm{d}s \leq \varepsilon(t-a) \leq \varepsilon m$$

for each $t \in \langle a; a+m\rangle$, $j \geq j_0$. Since $X(t)$ is bounded on each $\langle a; a+m\rangle$, the sequence $X(t)\int_a^t X^{-1}(s)f\big(s, x_j(s)\big)\,\mathrm{d}s$ converges uniformly to

$$X(t)\int_a^t X^{-1}(s)f\big(s, x(s)\big)\,\mathrm{d}s$$

in $\langle a; a+m\rangle$ for each natural number m and so it converges in C for $j \to \infty$. If we take into account the continuity of the operators T and T_0, we obtain the continuity of $K_P N$ from (2.5).

LEMMA 2.2. *The operator M: $\operatorname{dom} M = A \subset C \to C$ maps an arbitrary bounded subset of A to relatively compact set in C.*

Proof. Since P is a linear continuous operator and $\dim(\operatorname{Im} P) < \infty$ (and hence P is compact), it is sufficient to prove the assertion for the mapping $K_P N$.

Let Ω be a bounded set, $\Omega \subset A$, this means that:
If $x \in \Omega$, then $p_m(x) \leq \mu_m$. By (2.5) we have:

$$\begin{aligned}\big\|(K_P N x)(t)\big\| &\leq \big\|X(t)JT_0^{-1}\big(r - Tx(t,a,0)\big)\big\| + \big\|x(t,a,0)\big\| \leq \\ &\leq \big\|X(t)\big\| \cdot \big\|JT_0^{-1}\big(r - Tx(t,a,0)\big)\big\| + \big\|X(t)\big\| \cdot \Big\|\int_a^t X^{-1}(s)f\big(s, x(s)\big)\,\mathrm{d}s\Big\| \leq \\ &\leq \big\|X(t)\big\| \cdot \big\|JT_0^{-1}\big\|\big(\|r\| + \|Tx(t,a,0)\|\big) + \big\|X(t)\big\| \cdot \Big\|\int_a^t X^{-1}(s)f\big(s, x(s)\big)\,\mathrm{d}s\Big\|,\end{aligned}$$

$$
\begin{aligned}
p_m(K_P N x) \le& \sup\{\|X(t)\| \cdot \|JT_0^{-1}\|(\|r\| + \|Tx(t,a,0)\|);\ t \in \langle a; a+m\rangle\} + \\
&+ \sup\Big\{\|X(t)\| \cdot \Big\|\int_a^t X^{-1}(s) f\big(s, x(s)\big)\,\mathrm{d}s\Big\|;\ t \in \langle a; a+m\rangle\Big\} \le \\
\le& H_m \|JT_0^{-1}\|\big[\|r\| + \gamma H_{m_0}(\Gamma_{m_0}\mu_{m_0} + \Lambda_{m_0})\big] + H_m(\Gamma_m \mu_m + \Lambda_m) = \nu_m
\end{aligned}
$$

for each $x \in \Omega$ and for each natural number m. So $M(\Omega)$ is uniformly bounded on each $\langle a; a+m\rangle$. Further we shall prove that $M(\Omega)$ is equicontinuous on these intervals. Let $t_1 < t_2$ be two points of $\langle a; a+m\rangle$, $m \in \mathbb{N}$. Let

$$
\begin{aligned}
\delta(t,x) &= \int_a^t X^{-1}(s) f\big(s, x(s)\big)\,\mathrm{d}s \qquad a \le t < \infty \\
V &= JT_0^{-1}\big(r - TX(t)\delta(t,x)\big).
\end{aligned}
$$

Then there holds:

$$
\begin{aligned}
&\|(K_P N x)(t_2) - (K_P N x)(t_1)\| = \\
&= \|X(t_2)V + X(t_2)\delta(t_2,x) - X(t_1)V - X(t_1)\delta(t_1,x)\| \le \\
&\le \|X(t_2) - X(t_1)\| \cdot \|V\| + \|X(t_2) - X(t_1)\|\big\{\|JT_0^{-1}\|\big[\|r\| + \gamma H_{m_0}(\Gamma_{m_0}\mu_{m_0} + \\
&\qquad + \Lambda_{m_0})\big] + \Gamma_m \mu_m + \Lambda_m\big\} + H_m\Big(\mu_m \int_{t_1}^{t_2} p(s)\,\mathrm{d}s + \int_{t_1}^{t_2} q(s)\,\mathrm{d}s\Big).
\end{aligned}
$$

Therefore the set $M(\Omega)$ is equicontinuous on each $\langle a; a+m\rangle$, for each $m \in \mathbb{N}$ and so by the Ascoli-Arzelà lemma there is a relatively compact set on each $\langle a; a+m\rangle$. This means that if $\{x_i\}_{i=1}^{\infty}$ is a sequence of the functions from $M(\Omega)$ and $I_m = \langle a; a+m\rangle$, $m \in \mathbb{N}$, then it is possible to choose a subsequence $\{x_i^1(t)\}_{i=1}^{\infty}$ of the sequence $\{x_i(t)\}_{i=1}^{\infty}$ which uniformly converges on I_1. Analogously there exists a subsequence $\{x_i^2(t)\}_{i=1}^{\infty}$ of $\{x_i^1(t)\}_{i=1}^{\infty}$ such that it is uniformly convergent on I_2. We can repeat this procedure for each $m \in \mathbb{N}$. In this way we obtain a family of subsequences of $\{x_i\}_{i=1}^{\infty}$. By Cantor's diagonal process we have that there exists a sequence $\{x_i^i(t)\}_{i=1}^{\infty}$ of $\{x_i(t)\}_{i=1}^{\infty}$ which uniformly converges on each interval I_m, and so $M(\Omega)$ is a relatively compact set in C.

The following lemma follows from Lemma 2.1 and Lemma 2.2.

LEMMA 2.3. *If* $\operatorname{dom} T = A$, *then the operator* $M\colon C \to C$ *is completely continuous.*

Using Tichonov's theorem we shall prove the following theorem which states the existence of a solution to (2.1)–(2.2).

THEOREM 2.1. *Let the system* (2.1)–(2.2) *satisfy the following conditions:*

(2.10) $A(t)$ *is a real* $n \times n$ *matrix, defined and continuous on* $\langle a;\infty)$, $X(t)$ *is a fundamental matrix of* (2.3) *with*

$$H_m = \sup\{\|X(t)\|;\ t \in \langle a; a+m\rangle\}, \qquad m = 1,2,\dots;$$

(2.11) $f \in C\big(\langle a;\infty) \times \mathbb{R}^n, \mathbb{R}^n\big)$ *and it satisfies:* $\|X^1(t)f(t,u)\| \le p(t)\|u\| + q(t)$, $u \in \mathbb{R}^n$, *where* $p(t), q(t)$ *are non-negative locally integrable functions such that*

$$\int_a^{a+m} p(t)\,\mathrm{d}t = \Gamma_m < \infty, \qquad \int_a^{a+m} q(t)\,\mathrm{d}t = \Lambda_m < \infty$$

for each natural number m;

(2.12) T *is a linear bounded operator from* $\operatorname{dom} T = C$ *onto* $\mathbb{R}^m$ *and the rank of the matrix* $TX(t)$ *is* m;

(2.13) $\gamma H_{m_0} H_m \|JT_0^{-1}\|\Gamma_{m_0} + H_m\Gamma_m < 1$ *for* $m \ge m_0$.

Then the system (2.1)–(2.2) *has at least one solution in* C.

P r o o f. The operator $M_0 = K_P N$ is completely continuous by Lemma 2.3 and so it is sufficient to find a bounded set K which satisfies the hypotheses of Tichonov's theorem. Let $K = \{x \in C\colon p_m(x) \le \alpha_m,\ m \ge m_0\}$, whereby we shall determine α_m by the following consideration. There holds:

$$\begin{aligned}
\|(M_0x)(t)\| &= \|(K_PNx)(t)\| \le \\
&\le \Big\|X(t)JT_0^{-1}\Big[r - T\int_a^t X(t)X^{-1}(s)f\big(s,x(s)\big)\,\mathrm{d}s\Big]\Big\| + \\
&\qquad + \Big\|\int_a^t X(t)X^{-1}(s)f\big(s,x(s)\big)\,\mathrm{d}s\Big\| \le \\
&\le \|X(t)\|\cdot\|JT_0^{-1}\|\Big[\|r\| + \Big\|T\int_a^t X(t)X^{-1}(s)f\big(s,x(s)\big)\mathrm{d}s\Big\|\Big] + \\
&\qquad + \|X(t)\|\cdot\Big\|\int_a^t X^{-1}(s)f\big(s,x(s)\big)\,\mathrm{d}s\Big\|
\end{aligned}$$

for $a \le t \le a+m$ and

$$\begin{aligned}
&p_m(K_P N x) \le \\
&\le H_m \|JT_0^{-1}\| \big[\|r\| + \gamma H_{m_0}\big(\Gamma_{m_0} p_{m_0}(x) + \Lambda_{m_0}\big)\big] + H_m\big(\Gamma_m p_m(x) + \Lambda_m\big) \le \\
&\le H_m \big[\|JT_0^{-1}\|\big(\|r\| + \gamma H_{m_0}\Lambda_{m_0}\big) + \Lambda_m\big] + \\
&\qquad\qquad + \big(H_m \|JT_0^{-1}\| \gamma H_{m_0}\Gamma_{m_0} + H_m \Gamma_m\big) p_m(x).
\end{aligned}$$

If we choose

$$\alpha_m \ge \frac{\big\{H_m\big[\|JT_0^{-1}\|\big(\|r\| + \gamma H_{m_0}\Lambda_{m_0}\big) + \Lambda_m\big]\big\}}{\big(1 - H_m\|JT_0^{-1}\|\gamma H_{m_0}\Gamma_{m_0} - H_m\Gamma_m\big)},$$

then $M_0 \colon K \to K$. Further, K is a non-empty, closed, convex and bounded set in C, $M_0(K)$ is a compact set by Lemma 2.2, M_0 is a continuous mapping, therefore by Tichonov's theorem there exists a fixed point of M_0 in K. By Corollary 1.1 it follows that this fixed point of M_0 is a solution of (1.1) which satisfies $Px = 0$. So it is a solution of the boundary problem (2.1) (2.2) in C_{I-P}.

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