Mathematica Slovaca

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Mathematica Slovaca, Vol. 29 (1979), No. 3, 245--255

Persistent URL: http://dml.cz/dmlcz/132398

Math. Slovaca 29, 1979, No. 3

ON A TENSOR PRODUCT IN INITIALLY STRUCTURED CATEGORIES

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0. Introduction

Closed categories (see e.g. [4, p. 180]) introduced by Eilenberg and Kelly in [2] (called there symmetric monoidal closed categories) have been studied intensively and many useful results have been obtained. Thus it is often helpful to know that a given category is closed.

It is well known that the category of topological spaces and continuous maps is closed with respect to the "inductive" tensor product (which is obtained by proving $X \times Y$ with the "topology of separate continuity"). The direct generalization of this closed structure to the initialstructure categories over the category Set (which coincide with Herrlich's topological categories over the category Set) is given in [11, p. 432].

In the first section of this paper, this closed structure is extended to the initially structured categories in the sense of [5] and shown to be in a certain sense the smallest possible. In the second section, it is proved that in the category of closure spaces [1, p. 237] and continuous maps there is (up to a natural isomorphism) exactly one tensor product (i.e. exactly one structure of closed category). Note that $\mathscr{A}(X, Y)$ denotes the set of all $\mathscr{A}$-morphisms $X \to Y$ and ob $\mathscr{A}$ denotes the class of all $\mathscr{A}$-objects.

1. The smallest tensor product in initially structured categories

First recall the definition and some properties of *initially structured category.*

1.1. Definition [5]. (1) *Let* $U: \mathscr{A} \to \mathscr{B}$ *be a functor and* $(A \xrightarrow{a_i} A_i)_{i \in I}$ *a source in* $\mathscr{A}$. *To say that* (a_i) *is* U-*initial means that for any source* $(B \xrightarrow{b_i} A_i)_{i \in I}$ *and any*

$\mathscr{B}$-morphism f such that $Ua_i \circ f = Ub_i$ for all i there is precisely one $\mathscr{A}$-morphism c such that $Uc = f$ and $a_i \circ c = b_i$ for all $i \in I$.

(2) *A category $\mathscr{A}$ is said to be initially structured with forgetful functor U provided that there exists a functor $U: \mathscr{A} \to Set$ such that the following hold:*

1S1. *Any source $(X \xrightarrow{f_i} UA_i)_{i \in I}$ in Set has a factorization $(X \xrightarrow{e} UB \xrightarrow{Ug_i} UA_i)_{i \in I}$ such that e is an epimorphism, (Ug_i) is a mono-source and (g_i) is U-initial.*

1S2. *U has small fibres, i.e. for every object X in Set there is at most a set of pairwise non-isomorphic $\mathscr{A}$-objects A with $UA = X$.*

1S3. *There is precisely one $\mathscr{A}$-object P (up to isomorphism) such that UP is a singleton in Set.*

1.2. Examples [5]. The category $\mathscr{T}$ of all topological spaces, the category $\mathscr{T}_2$ of all Hausdorff spaces, the category of all partially ordered sets, the category of all closure spaces, etc. (with the usual classes of morphisms).

1.3. Basic properties [5]. *Let $\mathscr{A}$ be an initially structured category with forgetful functor U. Then*

(1) *U is faithful.*

(2) *If A, B are $\mathscr{A}$-objects and $c: UA \to UB$ is a constant map, then there exists $k: A \to B$ with $Uk = c$.*

(3) *The B in 1S1 is determined uniquely up to isomorphism.*

(4) *If $(\Pi A_i, (p_i))_{i \in I}$ is an $\mathscr{A}$-product, then $(U(\Pi A_i), (Up_i))_{i \in I}$ is a Set-product of $(UA_i)_{i \in I}$, i.e. there exists a bijection $m: \Pi_{i \in I} UA_i \to U(\Pi_{i \in I} A_i)$ such that $Up_i \circ m = q_i$ for all $i \in I$ ($q_i: \Pi_i UA_i \to UA_i$ is the i-th natural projection).*

(5) *Any non-trivial coreflective and any U-epireflective subcategory of an initially structured category is an initially structured category (g is an U-epimorphism iff Ug is an epimorphism).*

1.4. Remarks. (1) For the remainder of this section $\mathscr{A}$ will be an initially structured category with forgetful functor U.

(2) Since U is faithful the statement "for $f: UA \to UB$ there exists $k: A \to B$ with $Uk = f$" includes automatically the uniqueness of k.

(3) The natural projections of cartesian products in *Set* will be always denoted by q, in all other cases by p.

(4) An $\mathscr{A}$-morphism f which forms an U-initial mono-source will be called *an embedding.*

We can now start constructing a tensor product in an initially structured category $\mathscr{A}$. We shall use the following notation: Let X, Y, Z be sets and $f: X \times Y \to Z$ a map. Then for each $a \in X$ $f_a: Y \to U$ is defined by $f_a(y) = f(a, y)$ and for each $b \in Y$ $f^b: X \to Z$ is defined by $f^b(x) = f(x, b)$.

1.5. Definition. *Let A, B be $\mathscr{A}$-objects. Define $A \otimes B \in ob\mathscr{A}$ as follows: Let $\mathscr{S}_{AB}$ be the class of all maps $f: UA \times UB \to UC$, $C \in ob\mathscr{A}$ such that for each $a \in UA$ and*

$b \in UB$ *there exist* $\mathscr{A}$*-morphisms* u_a: $B \to C$ *and* v^b: $A \to C$ *with* $Uu_a = f_a$ *and* $Uv^b = f^b$. *According to* 1S1 *there exists a factorization* $(UA \times UB \overset{e_{AB}}{\to} U(S_{AB}) \overset{Ut_f}{\to} UC)$ *of the source* $\mathscr{S}_{AB}$ *where* e_{AB} *is an epimorphism,* $(Ut_f)_{f \in \mathscr{S}_{AB}}$ *is a mono-source and* $(t_f)_{f \in \mathscr{S}_{AB}}$ *is U-initial. For each pair* $(A, B) \in ob\mathscr{A}$ *make a definite choice of* S_{AB} *and* e_{AB} *(and therefore also of* (t_f)*). Put* $A \otimes B = S_{AB}$.

In the following we shall always use the notation from 1.5., i.e. e_{AB}: $UA \times UB \to U(A \otimes B)$ and $(t_f)_{f \in \mathscr{S}_{AB}}$ for the U-initial source corresponding to $\mathscr{S}_{AB}$.

Now let f: $A \to A'$, $g: B \to B'$ be $\mathscr{A}$*-morphisms, let* h: $UA' \times UB' \to UC$ belong to $\mathscr{S}_{A'B'}$ and $a \in UA$. Then $(h \circ (Uf \times Ug))_a = h_{Uf(a)} \circ Ug$. Since h belongs to $\mathscr{S}_{A'B'}$ there exists k_a: $B \to C$ with $Uk_a = h_{Uf(a)}$. Then, clearly, $U(k_a \circ g) = (h \circ (Uf \times Ug))_a$. Similarly, we can show that for each $b \in UB$ there exists s^b: $A \to C$ with $Us^b = (h \circ (Uf \times Ug))^b$. Hence $h \circ (Uf \times Ug)$ belongs to $\mathscr{S}_{AB}$ and therefore there exists $v_h = t_{h \circ (Uf \times Ug)}$; $A \otimes B \to C$ for which $Uv_h \circ e_{AB} = h \circ (Uf \times Ug)$. Thus we have the source $(v_h)_{h \in \mathscr{S}_{A'B'}}$ with $Ut'_h \circ (e_{A'B'} \circ (Uf \times Ug) \circ e_{AB}^{-1} = Uv_h$ (e_{AB} is bijective — see 1.7.) for all $h \in \mathscr{S}_{A'B'}$. Since (t'_h) (which corresponds to $\mathscr{S}_{A'B'}$) is U-initial there exists precisely one $\mathscr{A}$-morphism denoted by $f \otimes g$ such that $U(f \otimes g) = e_{A'B'} \circ (Uf \times Ug) \circ e_{AB}^{-1}$ and $t'_h \circ (f \otimes g) = v_h$ for all $h \in \mathscr{S}_{A'B'}$.

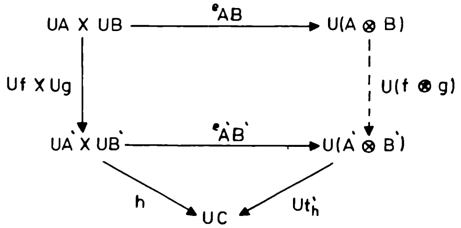

Fig. 1

1.6. Proposition. $\otimes$: $\mathscr{A} \times \mathscr{A} \to \mathscr{A}$; $(A, B) \mapsto A \otimes B$, $(f, g) \to f \otimes g$ *is a functor.*

1.7. Proposition: $e = (e_{AB})$: $UA \times UB \to U(A \otimes B)$ *is a natural isomorphism.*

Proof. Let A, B be $\mathscr{A}$-objects and m: $UA \times UB \to U(A \times B)$ the bijection mentioned in 1.3.(4). Using the fact that the natural projections q_1: $UA \times UB \to UA$, q_2: $UA \times UB \to UB$ belong to $\mathscr{S}_{AB}$ we can verify that m belongs to $\mathscr{S}_{AB}$. Therefore there exists t_m: $A \otimes B \to A \times B$ such that $Ut_m \circ e_{AB} = m$. Since m is a bijection, e_{AB} is bijective. The naturalness of e follows from the construction of $f \otimes g$. ∎

The last proposition implies that the properties of the "indukctive" tensor product in the initialstructure categories ([11]) like symmetry and associativity remain valid also for $\otimes$ in the initially structured categories. Clearly, if P is an $\mathscr{A}$-object for which $UP = \{*\}$, then P is a unit of $\otimes$.

Now we shall construct the internal hom functor $\mathscr{A}^{op} \times \mathscr{A} \to \mathscr{A}$ corresponding to $\otimes$.

Let B, C be $\mathscr{A}$-objects. Consider the $\mathscr{A}$-power $(C^{UB}, (p_b)_{b \in UB})$. Then by 1.3. (4) there exists a bijection m_{BC}: $UC^{UB} \to U(C^{UB})$ such that $Up_b \circ m_{BC} = q_b$ for all $b \in UB$ and, evidently, $m = (m_{BC})$ is a natural isomorphism. We know that the map U_{BC}: $\mathscr{A}(B, C) \to UC^{UB}$ is injective (1.3. (1)). By 1S1 there exists a factorization $\mathscr{A}(B, C) \overset{v_{BC}}{\to} U[B, C] \overset{Uk_{BC}}{\to} U(C^{UB})$ of the map $m_{BC} \circ U_{BC}$ where $[B, C]$ is an $\mathscr{A}$-object, v_{BC} is a surjection and k_{BC} is an embedding (see 1.4. (4)). For each $(B, C) \in ob\mathscr{A} \times ob\mathscr{A}$ choose $[B, C]$, v_{BC} and k_{BC}. Since $m_{BC} \circ U_{BC}$ is injective v_{BC} is bijective.

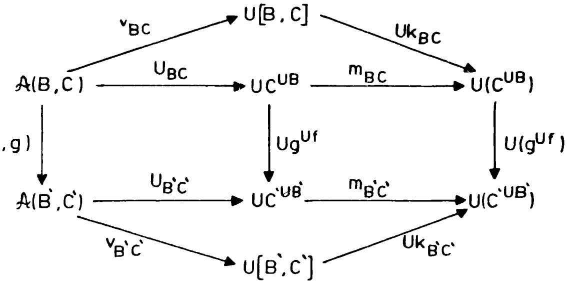

Fig. 2

Now consider the diagram where f: $B' \to B$, g: $C \to C'$ are arbitrary $\mathscr{A}$-morphisms (and as usual $\mathscr{A}(f, g)\,(u) \;=\; g \circ u \circ f$). The top region, the bottom region and the left region obviously commute. The right region also commutes by the naturalness of m. Since $k_{B'C'}$ is an embedding and $Uk_{B'C'} \circ (v_{B'C'} \circ \mathscr{A}\ (f,\ g) \circ v_{BC}^{-1}) = U(g^{Uf} \circ k_{BC})$ there exists an $\mathscr{A}$-morphism $[f,\ g]$: $[B,\ C] \to [B',\ C']$ for which $U[f,\ g] = v_{B'C'} \circ \mathscr{A}\ (f,\ g) \circ v_{BC}^{-1}$ and $k_{B'C'} \circ [f,\ g] = g^{Uf} \circ k_{BC}$. These two equations together with the next proposition show that $v = (v_{BC})$ and $k = (k_{BC})$ are natural transformations.

1.8. Proposition. $[-,\ -]$: $\mathscr{A}^{op} \times \mathscr{A} \to \mathscr{A}$; $(B, C) \mapsto [B, C]$, $(f, g) \mapsto [f, g]$ *is a functor.*

Proof. Straighthorward.

1.9. Theorem. $(\mathscr{A}, \otimes, [-,\ -])$ *is a closed category.*

Proof. By 1.7., [11] and [8; Theorem 4.4.] it suffices to prove that there exists a natural isomorphism $\gamma = (\gamma_{ABC})$: $\mathscr{A}(A \otimes B, C) \;\to\; \mathscr{A}(A, [B, C])$. Let u belong to $\mathscr{A}(A \otimes B, C)$. Then $Uu \circ e_{AB}$ belongs to $\mathscr{S}_{AB}$ so that there exists s_a: $B \to C$ with $Us_a = (Uu \circ e_{AB})_a$ for all $a \in UA$. Define $\tilde{u}$: $UA \to \mathscr{A}(B, C)$; $a \mapsto s_a$ and consider the diagram where for each $b \in UB$ w_b^u is the $\mathscr{A}$-morphism $A \to C$ for which $Uw_b^u = (Uu \circ e_{AB})^b$ and w^u is defined by the family $(w_b^u)_{b \in UB}$. For each $a \in UA$ $Up_b \circ Uk_{BC} \circ v_{BC} \circ \tilde{u}(a) \;=\; Up_b \circ m_{BC} \circ U_{BC}(s_a) \;=\; q_b \circ U(s_a) \;=\; q_b \circ (Uu \circ e_{AB})_a$

$= Uu\circ e_{AB}(a, b) = (Uu\circ e_{AB})^b(a) = Uw^u_b(a)$. Hence the exterior of the diagram commutes and therefore both triangles also commute. The morphism k_{BC} is an embedding, $Uk_{BC}\circ(v_{BC}\circ\tilde{u}) = Uw^u$ so that by 1S1 there exists (unique) $\hat{u}: A \to [B, C]$ with $U\hat{u} = v_{BC}\circ\tilde{u}$ and $k_{BC}\circ\hat{u} = w^u$. Define $\gamma_{ABC}: \mathscr{A}(A\otimes B, C) \to \mathscr{A}(A, [B, C])$ by $u\mapsto\hat{u}$. Clearly, $u_1\neq u_2$ implies $\tilde{u}_1\neq\tilde{u}_2$ and then $\hat{u}_1\neq\hat{u}_2$. Hence γ_{ABC} is injective. To show that γ_{ABC} is also surjective take $g\in\mathscr{A}, ([B, C])$. Then Ug is a map $UA\to U[B, C]$. Put $\varphi = v^{-1}_{BC}\circ Ug$ and $\psi = U_{BC}\circ\varphi$. Define $\bar{\psi}: UA\times UB\to UC$ by $\bar{\psi}(a, b) = \psi_a(b)$. Then $\bar{\psi}_a = (U_{BC}\circ\varphi)_a = U(\varphi_a)$ where φ_a belongs to $\mathscr{A}(B, C)$. $\bar{\psi}^b = q_b\circ U_{BC}\circ v^{-1}_{BC}\circ Ug = Up_b\circ Uk_{BC}\circ Ug = U(p_b\circ k_{BC}\circ g)$. Hence $\bar{\psi}\in\mathscr{S}_{AB}$ and therefore there exists $g^*: A\otimes B\to C$ with $Ug^*\circ e_{AB} = \bar{\psi}$. It is easy to check that $\gamma_{ABC}(g^*) = g$. Let $f: A'\to A$, $g: B'\to B$, $h: C\to C'$ be $\mathscr{A}$-morphisms, let u belong to $\mathscr{A}(A\otimes B, C)$ and $u' = \mathscr{A}(f\otimes g, h)(u) = h\circ u\circ(f\otimes g)$. Then u' belongs to $\mathscr{A}(A'\otimes B', C')$. Consider the diagram 5. The interior of this diagram

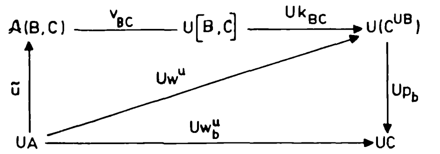

Fig. 3

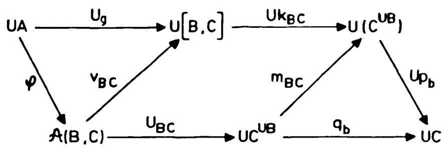

Fig. 4

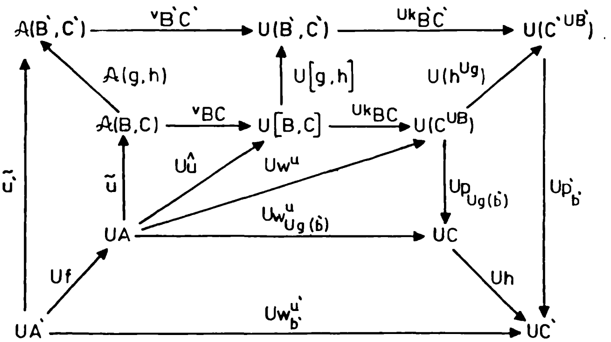

Fig. 5

commutes by the preceding part of this proof. The upper regions commute by the naturalness of $v = (v_{BC})$ and $k = (k_{BC})$, respectively, the right region evidently also commutes. By the first part of this proof $\tilde{u}: UA\to\mathscr{A}(B, C)$; $a\mapsto s_a$ such that $Us_a = (Uu\circ e_{AB})_a$ and $\tilde{u}': UA'\to\mathscr{A}(B', C')$; $a'\mapsto s'_{a'}$, such that $Us'_{a'} = (Uu'\circ e_{A'B'})_{a'}$. $\mathscr{A}(g, h)\circ\tilde{u}\circ Uf(a') = \mathscr{A}(g, h)(s_{Uf(a')}) = h\circ s_{Uf(a')}\circ g$. But $U(h\circ s_{Uf(a')}\circ g) = Uh\circ Us_{Uf(a')}\circ Ug = Uh\circ(Uu\circ e_{AB})_{Uf(a')}\circ Ug = (Uh\circ Uu\circ e_{AB}\circ(Uf\times Ug))_{a'} = (Uh\circ Uu\circ U(f\otimes g)\circ e_{A'B'})_{a'} = (U(h\circ u\circ(f\otimes g))\circ e_{A'B'})_{a'} = (Uu'\circ e_{A'B'})_{a'} = Us'_{a'}$. Hence $h\circ s_{Uf(a')}\circ g = s'_{a'}$ for each $a'\in UA'$ so that the left region of the diagram commutes. Now $w^{u'}_{b'} = Up'_{b'}\circ Uk_{B'C'}\circ v_{B'C'}\circ\tilde{u}'$

$= Uh \circ Up_{g(b')} \circ Uk_{BC} \circ v_{BC} \circ \tilde{u} \circ Uf = Uh \circ Uw^{u}_{b} \circ Uf$ so that the lower region also commutes. Therefore $Uw^{u'} = U(h^{Ug}) \circ UW^{u} \circ Uf = U(h^{Ug} \circ w^{u} \circ f)$ and then $U\hat{u}' = U(f \circ \hat{u} \circ [g, h])$. Thus we obtain $\hat{u}' = f \circ \hat{u} \circ [g, h]$. Finally $\gamma_{A'B'C'} \circ \mathscr{A}(f \otimes g, h)(u) = \gamma_{A'B'C'}(u') = \hat{u}' = f \circ \hat{u} \circ [g, h] = \mathscr{A}(f, [g, h])(\hat{u}) = \mathscr{A}(f, [g, h]) \circ \gamma_{ABC}(u)$. Thus $\gamma = (\gamma_{ABC})$ is natural. ■

The next theorem is very useful for our study. Recall (see [4, p. 26]) that *a concrete category* is a pair $(\mathscr{K}, V)$ where $\mathscr{K}$ is a category and $V\colon \mathscr{K} \to Set$ is a faithful functor.

1.11. Theorem [6]. *Let $(\mathscr{K}, V)$ be a concrete category with the following properties:*

(1) *For any constant map $c\colon VA \to VB$ there exists a $\mathscr{K}$-morphism $k\colon A \to B$ with $Vk = c$.*

(2) *For any bijection $f\colon VA \to X$ there exists a $\mathscr{K}$-isomorphism $s\colon A \to B$ with $Vs = f$.*

(3) *There is a $\mathscr{K}$-object A with card $VA \geqq 2$.*

If there is a closed structure $(\square, H)$ on $\mathscr{K}$, then there exists a closed structure $(\bigcirc, G)$ on $\mathscr{K}$ isomorphic with $(\square, H)$ with the following properties:

(a) *Card $VI = 1$, where I is the unit of $\bigcirc$.*

b) *$VA \times VB \subset V(A \bigcirc B)$ for all $A, B \in ob\mathscr{K}$.*

(c) *For $f, g\colon A \bigcirc B \to C$ $Vf \mid VA \times VB = Vg \mid VA \times VB$ implies $f = g$ ($\mid$ denotes a restriction of a map).*

d) *$V(f \bigcirc g) \mid VA \times VB = Vf \times Vg$.*

(e) *$VG(B, C) = \mathscr{K}(B, C)$.*

(f) *$V(\Theta f)(a) = (b \mapsto Vf(a, b))$, where $\Theta\colon \mathscr{K}(A \bigcirc B, C) \to \mathscr{K}(A, G(B, C))$ is the corresponding adjunction.*

g) *$V(\Theta^{-1} g)(a, b) = (Vg(a))(b)$.*

If, moreover, $\mathscr{K}$ satisfies

(4) *$X \subset VA$ implies that there exists a $\mathscr{K}$-morphism $j\colon B \to A$ such that $VB = X$ and $Vj\colon VB \to VA$; $x \mapsto x$ and*

(5) *for every $\mathscr{K}$-epimorphism g Vg is a surjection, then*

i) *$VA \times VB = V(A \bigcirc B)$ for all $A, B \in$ ob$\mathscr{K}$.*

It is obvious that any initially structured category together with its forgetful functor is a concrete category satisfying (1)—(3) of 1.18.

1.12. Theorem. *Let $\mathscr{A}$ be an initially structured category and $(\square, H)$ a closed structure on $\mathscr{A}$. Then there exist natural transformations $\eta\colon \otimes \to \square$ and $\Theta\colon H \to [-, -]$.*

Proof. By 1.11. we can suppose $(\square, H)$ to satisfy (a)—(g) of 1.11. and to have the same unit as $(\otimes, [-, -])$. Then it is easy to check that if $l^{\square} = (l^{\square}_{B})\colon P \square B \to B$ and $r^{\square} = (r^{\square}_{A})\colon A \square P \to A$ are natural isomorphisms corresponding to $\square$, then

$Ul_B^{\square} | UP \times UB = q_2^B$: $(*, y) \mapsto y$ and $Ur_A^{\square} | UA \times UP = q_1^A$: $(x, *) \mapsto x$ for all A, $B \in \mathrm{ob}\mathscr{A}$. Denote by j_{AB} the embedding $UA \times UB \subset U(A \square B)$ for all $A, B \in \mathrm{ob}\mathscr{A}$ and consider the following diagram

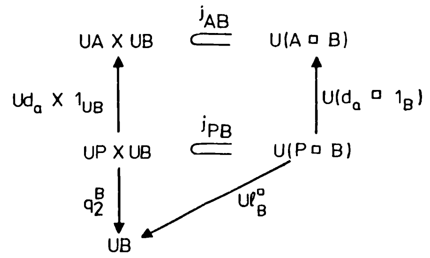

Fig. 6

Let $a \in UA$ and $b \in UB$. Denote by d_a the morphism $P \to A$ for which $Ud_a(*) = a$ and by d_b the morphism $P \to B$ with $Ud_b(*) = b$. Then it is easy to see that $(j_{AB})_a = U((d_a \square 1_B) \circ (l_B^{\square})^{-1})$ and $(j_{AB})^b = U((1_A \square d_b) \circ (r_A^{\square})^{-1})$. Thus j_{AB} belongs to $\mathscr{S}_{AB}$ so that there exists η_{AB}: $A \otimes B \to A \square B$ with $U\eta_{AB} \circ e_{AB} = j_{AB}$. The naturalness of $\eta = (\eta_{AB})$ follows from the naturalness of $j = (j_{AB})$ and $e = (e_{AB})$ and from the faithfulness of U. The existence of Θ immediately follows from the existence of η. ■

2. Tensor products in the category of closure spaces

Recall (see [1]) that *a closure space* is a pair (P, u) where P is a set and u: $2^P \to 2^P$ is a map satisfying (i) $u\emptyset = \emptyset$, (ii) $M \subset uM$ for all $M \subset P$ and (iii) $u(L \cup M) = uL \cup uM$ for all $L, M \in 2^P$. If (P, u), (Q, v) are closure spaces, then a map f: $P \to Q$ is said to be *continuous* provided that $f[uM] \subset vf[M]$ for each $M \subset P$. The category of all closure spaces and continuous maps will be denoted by $\mathscr{C}$. We shall often write only P instead of (P, u) and then $\bar{M}$ instead of uM for $M \subset P$. If (P, u), (Q, v) are closure spaces, then we shall write $(P, u) \leqq (Q, v)$ iff $P = Q$ and $uM \subset vM$ for each $M \subset P$. Note that $\mathscr{C}$ is an initially structured category.

Let (P, u), (Q, v) be closure spaces. Define $u \otimes v$ on $P \times Q$ by $(u \otimes v)M =$

$$= (\bigcup_{x \in P}(\{x\} \times vM_2^x)) \bigcup (\bigcup_{y \in Q}(uM_1^y \times \{y\})),$$

where $M_2^x = \{y \in Q;\ (x, y) \in M\}$ and $M_1^y = \{x \in P;\ (x, y) \in M\}$. Then $(P \times Q, u \otimes v)$ is a closure space called in [1] *the inductive product of* (P, u), (Q, v) and it will be denoted by $(P, u) \otimes (Q, v)$. It is easy to see that for arbitrary $P, Q, S \in \mathrm{ob}\mathscr{C}$ a map f: $P \otimes Q \to S$ is continuous iff f_a and f^b are continuous for each $a \in P$ and $b \in Q$. Hence the inductive product in $\mathscr{C}$ is a special case of the tensor product introduced in the first section (evidently $f \otimes g = f \times g$). Now let (Q, v), (S, w) be closure spaces and $(S, w)^Q$ a (object part of) $\mathscr{C}$-power. Denote by $[(Q, v), (S, w)]$ the subspace of $(S, w)^Q$ consisting of all continuous maps $(Q, v) \to (S, w)$. Then $[(Q, v), (S, w)]$ is the value of the internal hom functor corresponding to the inductive product at $((Q, v), (S, w))$ (obviously $[f, g](t) = g \circ t \circ f$). Evidently, $\mathscr{C}$ satisfies (1)—(5) of 1.11., so we can adopt the following restriction without loss of generality.

2.1. Convention. Throughout the remainder of this section all closed structures on $\mathscr{C}$ will be assumed to fulfil (a)—(i) of 1.11.

2.2. Proposition. *If $(\mathscr{C}, \Box, H)$ is a closed category, then $P \otimes Q \leqq P \Box Q \leqq P \times Q$ for all $P, Q \in ob\mathscr{C}$.*

Proof. The left part immediately follows from 1.12. Note that by 2.1. the natural isomorphism $l^\Box = (l^\Box_Q): \{*\} \Box Q \to Q$ fulfils $l^\Box_Q(*, y) = y$ and similarly $r^\Box = (r^\Box_P): P \Box \{*\} \to P$ fulfils $r^\Box_P(x, *) = x$. Then $r^\Box_P \circ (1_P \Box c) = p_1: P \Box Q \to P; (x, y) \mapsto x$ and $l^\Box_Q \circ (k \Box 1_Q) = p_2: P \Box Q \to Q; (x, y) \mapsto y$, where $c: Q \to \{*\}$, $k: P \to \{*\}$, are $\mathscr{C}$-morphisms so that $id_{P \times Q} = p_1 \times p_2: P \Box Q \to P \times Q; (x, y) \mapsto (x, y)$ is a $\mathscr{C}$-morphism. Thus $P \Box Q \leqq P \times Q$. ■

Now let A be an infinite set and $\mathscr{U}$ a non-principal ultrafilter on A. Let $a \notin A$ and let $G^{\mathscr{U}}$ denote the corresponding non-principal ultraspace defined on $A \cup \{a\}$ (for each $x \in A$ $\{x\}$ is open in $G^{\mathscr{U}}$ and $\{U \cup \{a\};\ U \in \mathscr{U}\}$ is the family of all neighbourhoods of a in $G^{\mathscr{U}}$). Then we shall always write $O_{\mathscr{U}}$ instead of a and $A_{\mathscr{U}} (= G^{\mathscr{U}} - \{O_{\mathscr{U}}\})$ instead of A. Denote by $\mathscr{L}$ the class of all non-principal ultraspaces. The coreflective hull $C(\mathscr{L})$ of $\mathscr{L}$ in $\mathscr{C}$ coincides with $\mathscr{C}$ (it can be easily verified that any closure space is an extremal quotient of a suitable coproduct of non-principal ultraspaces in $\mathscr{C}$). Hence every tensor product in $\mathscr{C}$ is uniquely determined by defining its values on $\mathscr{L} \times \mathscr{L}$ (because it preserves $\mathscr{C}$-coproducts and extremal $\mathscr{C}$-epimorphisms which coincide with regular ones in $\mathscr{C}$). In the following, by "ultraspace" we shall always mean "non-principal ultraspace" and B will always denote the Sierpinski doubleton defined on $\{0, 1\}$ where $\{\bar{0}\} = \{0\}$, $\{\bar{1}\} = \{0, 1\}$.

2.3. Remark. Let $G^{\mathscr{U}}$, $G^{\mathscr{V}}$ be ultraspaces and (x, y) belong to $(G^{\mathscr{U}} \times G^{\mathscr{V}}) - \{(O_{\mathscr{U}}, O_{\mathscr{V}})\}$. Then for each $M \subset G^{\mathscr{U}} \times G^{\mathscr{V}}$ $(x, y) \in \bar{M}$ in $G^{\mathscr{U}} \otimes G^{\mathscr{V}}$ iff $(x, y) \in \bar{M}$ in $G^{\mathscr{U}} \times G^{\mathscr{V}}$. Hence $G^{\mathscr{U}} \otimes G^{\mathscr{V}} < G^{\mathscr{U}} \Box G^{\mathscr{V}}$ $(\leqq G^{\mathscr{U}} \times G^{\mathscr{V}})$ for some tensor product $\Box$ in $\mathscr{C}$ iff there exists $M \leqq G^{\mathscr{U}} \Box G^{\mathscr{V}}$ with $(O_{\mathscr{U}}, O_{\mathscr{V}}) \in \bar{M}$ in $G^{\mathscr{U}} \Box G^{\mathscr{V}}$ and $(O_{\mathscr{U}}, O_{\mathscr{V}}) \notin \bar{M}$ in $G^{\mathscr{U}} \otimes G^{\mathscr{V}}$. Note also that a $\mathscr{C}$-morphism $f: (P, u) \to (Q, v)$ is an extremal $\mathscr{C}$-epimorphism iff $vX = f[uf^{-1}[X]]$ for each $X \subset Q$.

2.4. Proposition. *If $(\mathscr{C}, \Box, H)$ is a closed category and $B \Box B = B \otimes B$, then $\Box = \otimes$.*

Proof. Let $G^{\mathscr{U}}$, $G^{\mathscr{V}}$ be ultraspaces, $A_{\mathscr{U}} = G^{\mathscr{U}} - \{O_{\mathscr{U}}\}$, $A_{\mathscr{V}} = G^{\mathscr{V}} - \{O_{\mathscr{V}}\}$. To prove that $G^{\mathscr{U}} \Box G^{\mathscr{V}} = G^{\mathscr{U}} \otimes G^{\mathscr{V}}$ we need to show that $(O_{\mathscr{U}}, O_{\mathscr{V}}) \in \bar{M} - M$ in

$G^{\mathscr{U}} \Box G^{\mathscr{V}}$ implies that $O_{\mathscr{V}} \in \overline{M_2^{O_{\mathscr{U}}}}$ or $O_{\mathscr{U}} \in \overline{M_1^{O_{\mathscr{V}}}}$, i.e. $(O_{\mathscr{U}}, O_{\mathscr{V}}) \in \bar{M} - M$ in $G^{\mathscr{U}} \otimes G^{\mathscr{V}}$. Thus, on the contrary, let $(O_{\mathscr{U}}, O_{\mathscr{V}})$ belong to $\bar{M} - M$ in $G^{\mathscr{U}} \Box G^{\mathscr{V}}$ and $O_{\mathscr{U}} \notin \overline{M_1^{O_{\mathscr{V}}}}$, $O_{\mathscr{V}} \notin \overline{M_2^{O_{\mathscr{U}}}}$. Then, evidently, $(O_{\mathscr{U}}, O_{\mathscr{V}}) \notin \overline{\{O_{\mathscr{U}}\} \times M_2^{O_{\mathscr{U}}}}$ and $(O_{\mathscr{U}}, O_{\mathscr{V}}) \notin \overline{M_1^{O_{\mathscr{V}}} \times \{O_{\mathscr{V}}\}}$ in $G^{\mathscr{U}} \Box G^{\mathscr{V}}$ (because $G^{\mathscr{U}} \Box G^{\mathscr{V}} \leqq G^{\mathscr{U}} \times G^{\mathscr{V}}$). Clearly $M = (\{O_{\mathscr{U}}\} \times M_2^{O_{\mathscr{U}}}) \cup (M_1^{O_{\mathscr{V}}} \times \{O_{\mathscr{V}}\}) \cup M_3$, where $M_3 \subset A_{\mathscr{U}} \times A_{\mathscr{V}}$. Hence $(O_{\mathscr{U}}, O_{\mathscr{V}}) \in \bar{M}_3$ and therefore $(O_{\mathscr{U}}, O_{\mathscr{V}})$ belongs to $\overline{A_{\mathscr{U}} \times A_{\mathscr{V}}}$ in $G^{\mathscr{U}} \Box G^{\mathscr{V}}$. Define $f\colon G^{\mathscr{U}} \to B$; $O_{\mathscr{U}} \mapsto 0$, $f[A_{\mathscr{U}}] = \{1\}$ and $g\colon G^{\mathscr{V}} \to B$; $O_{\mathscr{V}} \mapsto 0$, $g[A_{\mathscr{V}}] = \{1\}$. Obviously f, g are $\mathscr{C}$-morphisms and therefore $f \Box g\colon G^{\mathscr{U}} \Box G^{\mathscr{V}} \to B \Box B$ is a $\mathscr{C}$-morphism. But then $f(O_{\mathscr{U}}, O_{\mathscr{V}}) = (0, 0) \in \overline{(f \Box g)[A_{\mathscr{U}} \times A_{\mathscr{V}}]} = \overline{\{(1, 1)\}}$ in $B \Box B = B \otimes B$ and this yields a contradiction. ■

Since $B \otimes B \leqq B \Box B \leqq B \times B$ implies $B \Box B = B \otimes B$ or $B \Box B = B \times B$, we have:

2.5. Proposition. *If $(\mathscr{C}, \Box, H)$ is a closed category and $\Box \neq \otimes$, then $B \Box B = B \times B$.*

2.6. Lemma. *If $(\mathscr{C}, \Box, H)$ is a closed category and $\Box \neq \otimes$, then for any ultraspaces $G^{\mathscr{U}}, G^{\mathscr{V}}$ $(O_{\mathscr{U}}, O_{\mathscr{V}}) \in \overline{A_{\mathscr{U}} \times A_{\mathscr{V}}}$ in $G^{\mathscr{U}} \Box G^{\mathscr{V}}$.*

Proof. The maps $f\colon G^{\mathscr{U}} \to B$; $O_{\mathscr{U}} \mapsto 0$, $f[A_{\mathscr{U}}] = \{1\}$, $g\colon G^{\mathscr{V}} \to B$; $O_{\mathscr{V}} \mapsto 0$, $g[A_{\mathscr{V}}] = \{1\}$ are evidently extremal $\mathscr{C}$-epimorphisms so that $f \Box g\colon G^{\mathscr{U}} \Box G^{\mathscr{V}} \to B \Box B$ is an extremal $\mathscr{C}$-epimorphism. Since $B \Box B = B \times B$ $(0, 0) \in \overline{\{(1, 1)\}}$ in $B \Box B$ so that $(O_{\mathscr{U}}, O_{\mathscr{V}}) \in \overline{A_{\mathscr{U}} \times A_{\mathscr{V}}} = \overline{(f \Box g)^{-1}[(1, 1)]}$ in $G^{\mathscr{U}} \Box G^{\mathscr{V}}$ (because $\{(O_{\mathscr{U}}, O_{\mathscr{V}})\} = (f \Box g)^{-1}[(0, 0)]$). ■

2.7. Theorem. *If $(\mathscr{C}, \Box, H)$ is a closed category, then $\Box = \otimes$.*

Proof. Suppose $\Box \neq \otimes$. Let $\{N_k^*; k \in N\}$ be a sequence of pairwise disjoint copies of N^* where N is the discrete space of all non-negative integers and N^* is the Alexandroff compactification of N (the new point will be denoted by ω, the elements of N_k^* will be indexed by k). Put $\mathscr{N} = \sqcup_{k \in N} N_k^*$ (the $\mathscr{C}$-coproduct). Denote by $\mathscr{N}'$ the extremal quotient of $\mathscr{N}$ in $\mathscr{C}$ obtained by identifying all ω_k into one point denoted by Ω. Then the map $e\colon \mathscr{N} \to \mathscr{N}'$; $\omega_k \mapsto \Omega$ for all $k \in N$ and $x \mapsto x$ otherwise is an extremal $\mathscr{C}$-epimorphism. Since for each $K \subset \mathscr{N}'$ $\bar{K} = K$ or $\bar{K} = K \cup \{\Omega\}$ in $\mathscr{N}'$ it follows that $\mathscr{N}' \in \operatorname{ob}\mathscr{T}$ so that e is also an extremal $\mathscr{T}$-epimorphism ($\mathscr{N}$ is obviously a $\mathscr{T}$-object). Consider the space $\mathscr{N}' \Box N^*$. Put $A_k = \bigcup_{j=k}^{\infty} N_j$ ($N_j = N_j^* - \{\omega_j\} \subset \mathscr{N}'$) and $M = \bigcup_{k \in N}(A_k \times \{k\})$. We claim that $(\Omega, \omega) \in \bar{M}$ in $\mathscr{N}' \Box N^*$. *Put* $\mathscr{F}_\Omega = \{U - \{\Omega\};$ U is a neighbourhood of Ω in $\mathscr{N}'\}$. Clearly, $\mathscr{S} = \{A_k; k \in N\} \cup \mathscr{F}_\Omega$ has a finite intersection property and $\cap \mathscr{S} = \emptyset$ in $\mathscr{N}' - \{\Omega\}$ so that there exists a non-principal ultrafilter $\mathcal{U}$ on the set $X = \bigcup_{k \in N} N_k$ $(= \mathcal{N}' - \{\Omega\})$ containing $\mathcal{S}$. Denote by $X^{\mathcal{U}}$ the corresponding (non-principal) ultraspace defined on $X \cup \{\Omega\}$. Obviously, the map $j: X^{\mathcal{U}} \to \mathcal{N}'$; $x \mapsto x$ is a $\mathcal{C}$-morphism. Now define the map $f: X \to N$ by $n_k \mapsto k$ for all $k \in N$ and $n_k \in N_k$. Put $\mathcal{V} = \{f[W]; W \in \mathcal{U}\}$. $\mathcal{V}$ is a non-principal ultrafilter on N (f is surjective) and denote by $N^{\mathcal{V}}$ the corresponding ultraspace on $N \cup \{\omega\}$. Then $g: X^{\mathcal{U}} \to N^{\mathcal{V}}$ defined by $g(\Omega) = \omega$ and $g(x) = f(x)$ otherwise is evidently an extremal $\mathcal{C}$-epimorphism. Obviously $M \subset X^{\mathcal{U}} \square N^{\mathcal{V}}$. Put $L = (g \square 1_{N^{\mathcal{V}}})[M] = \bigcup_{k \in N}([k, \omega) \times \{k\})$, where $[k, \omega) = \{n \in N; n \geqq k\}$. Let $(\omega, \omega) \notin \bar{L}$ in $N^{\mathcal{V}} \square N^{\mathcal{V}}$. Since by 2.6. $(\omega, \omega) \in \overline{N \times N}$ in $N^{\mathcal{V}} \square N^{\mathcal{V}}$, we have $(\omega, \omega) \in \overline{(N \times N) - L}$.. $\square$ is symmetric so that $c: N^{\mathcal{V}} \square N^{\mathcal{V}} \to N^{\mathcal{V}} \square N^{\mathcal{V}}$; $(x, y) \mapsto (y, x)$ is a $\mathcal{C}$-isomorphism. Consider $L' = c[L]$. Then $L' = \bigcup_{k \in N}(\{k\} \times [k, \omega))$ and it is easy to see that $(N \times N) - L \subset L'$. Therefore $(\omega, \omega) \in \bar{L}'$ in $N^{\mathcal{V}} \square N^{\mathcal{V}}$ but this is impossible. Hence $(\omega, \omega) \in \bar{L}$ in $N^{\mathcal{V}} \square N^{\mathcal{V}}$. Since $(g \square 1_{N^{\mathcal{V}}})^{-1}[L] = M$ and $(g \square 1_{N^{\mathcal{V}}})^{-1}[(\omega, \omega)] = \{(\Omega, \omega)\}$ it follows that $(\Omega, \omega) \in \bar{M}$ in $X^{\mathcal{U}} \square N^{\mathcal{V}}$. Evidently, $h: N^{\mathcal{V}} \to N^*$; $x \mapsto x$ is a $\mathcal{C}$-morphism so that $j \square h$: $X^{\mathcal{U}} \square N^{\mathcal{V}} \to \mathcal{N}' \square N^*$ is a $\mathcal{C}$-morphism and then $(\Omega, \omega) \in \bar{M}$ in $X^{\mathcal{U}} \square N^{\mathcal{V}}$ implies that $(\Omega, \omega) \in \bar{M}$ in $\mathcal{N}' \square N^*$. Finally, consider $e \square 1_{N^*}$: $\mathcal{N} \square N^* \to \mathcal{N}' \square N^*$. Clearly, $(e \square 1_{N^*})^{-1}[M] = M$, $\mathcal{N} \square N^* = (\sqcup_{k \in N} N_k^*) \square N^* = \sqcup_{k \in N}(N_k^* \square N^*)$. Obviously, $\bar{M} = \bigcup_{k \in N}\overline{(M \cap (N_k^* \square N^*))}$ in $\mathcal{N} \square N^*$. Since $M \cap (N_k^* \square N^*) = N_k \times \{0, 1, \ldots, k\}$, we obtain that $\overline{M \cap (N_k^* \square N^*)} = N_k^* \times \{0, 1, \ldots, k\}$ (because $N_k^* \otimes N^* \leqq N_k^* \square N^* \leqq N_k^* \times N^*$) so that $(\omega_k, \omega) \notin \bar{M}$ in $\mathcal{N} \square N^*$ for all $k \in N$. Hence $(e \square 1_{N^*})^{-1}[(\Omega, \omega)] \cap \overline{(e \square 1_{N^*})^{-1}[M]} = \emptyset$ in $\mathcal{N} \square N^*$. Therefore $e \square 1_{N^*}$ is not an extremal $\mathcal{C}$-epimorphism and this is impossible. ∎

2.8. Remark. Note that throughout this section we have not used the associativity of $\square$.

REFERENCES

[1] ČECH, E.: Topological spaces. Prague 1966.

[2] EILENBERG, S.—KELLY, G. M.: Closed categories. In: Proc. Conf. on Categorical Algebra, La Jolla, 1965. New Yor 1966, 421—562.

[3] HERRLICH, H.—STECKER, G. E.: Category theory. Boston 1973.

[4] Mac LANE, S.: Categories for the working mathematician. New York 1971.

[5] NEL, L. D.: Initially structured categories and cartesian closedness. Canad. J. Math. 27, 1975, 1361—1377.

[6] NIEDERLE, J.: A remark on tensor products in concrete categories with constant mappings (Czech). To appear.

[7] PAVELKA, J.: Tensor products in the category of convergence spaces. Comment. math. Univ. carol. 13, 1972, 693—709.

[8] PULTR, A.: Extending tensor products to structures of closed categories. Comment. math. Univ. carol. 13, 1972, 599—616.

[9] WILKER, P.: Adjoint product and hom functors in general topology. Pacific J. Math. 34, 1970, 269—283.
[10] WYLER, O.: Convenient categories for topology. Gen. Topology Appl. 3, 1973, 225—242.
[11] WISCHNEWSKY, M. B.: Aspects of categorical algebra in initialstructure categories. Cahiers Topo. Geo. diff. XV, 1974, 419—444.

Received July 11, 1977

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ОБ ОДНОМ ТЕНЗОРНОМ ПРОИЗВЕДЕНИИ В ИНИЦИАЛЬНО СТРУКТУРОВАННЫХ КАТЕГОРИЯХ

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Резюме

В работе показывается, что в каждой инициально структурованной категории в смысле Нела (определение 1.1.) можно определить структуру замкнутой категории так, что соответствующее тензорное произведение является в определенном смысле наименьшим. Доказывается также, что в категории пространств с замыканием (пространством с замыканием называется двойка (P, u), где P — множество и u: $2^P \to 2^P$ — отображение, исполняющее $u\emptyset$, $M \subset uM$ и $u(M \cup K) = uM \cup uK$ для всех $M, K \subset P$ и непрерывных отображений (f: $(P, u) \to (Q, v)$ непрерывно, если $f[uM] \subset vf[M]$ для каждого $K \subset P$) можно определить только одну структуру замкнутой категории.