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Mathematica Slovaca, Vol. 41 (1991), No. 1, 11--16

Persistent URL: http://dml.cz/dmlcz/132387

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TRANSMISSION IN GRAPHS : A BOUND AND VERTEX REMOVING

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ABSTRACT. The transmission of a graph G is the sum of all distances in G. Strict upper bound on the transmission of a connected graph with a given number of vertices and edges is provided. Changes of the transmission caused by removing a vertex are studied.

1. Introduction

All graphs considered in this paper are undirected without loops and multiple edges. For all terminology on graphs not explained here we refer to [1].

If S is set, then $|S|$ denotes the *cardinality of* S. Given a graph G, $V(G)$ and $E(G)$ denote its *vertex-set* and *edge-set*, respectively. The cardinalities $|v(G)|$ and $|E(G)|$ are often denoted n and m, respectively. If v and w are the vertices of G, then $d_G(v, w)$ or, briefly, $d(v, w)$ denotes *the distance from* v *to* w *in* G, $ec_G(v)$ or $ec(v)$ denotes *the eccentricity of* v.

The transmission of a vertex v *of a graph* G is defined by

$$\sigma_G(v) = \sum_{w \in V(G)} d_G(v, w).$$

The transmission $\sigma(G)$ *of a graph* G is the sum of the transmissions of all its vertices.

The main subject of this paper is the transmission. Several results on this notion are surveyed in [5]. The strict upper bound on the transmission of a connected graph with a given number of vertices and edges is provided in this paper. Changes of the transmission caused by removing a vertex are studied.

2. An upper bound for transmission

Entringer, Jackson and Snyder [1] have given some upper bounds for transmission of a connected graph with n vertices and m edges. But they are

AMS Subject Classification (1985): Primary 05C99.
Key words: Connected graph, Transmission of a graph, Diameter of a graph

not sharp for each m. Now we are going to establish the sharp upper bound.

Let u be an isolated vertex or one endvertex of a path. Let us join u with at least one vertex of a complete graph. This new graph is called a *path-complete graph* and denoted by $PK_{n,m}$, where n and m are the cardinalities of its vertex-set and edge-set, respectively (see. Fig. 1.). One can verify that there is exactly one path-complete graph $Pk_{n,m}$ for all $1 \leq n-1 \leq m \leq \binom{n}{2}$.

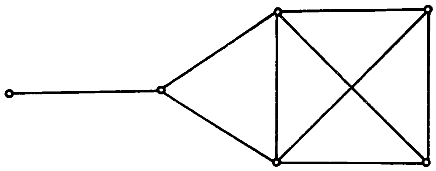

Fig. 1

The maximal distance in G is the *diameter* of G, *diam* (G). The following upper bound on the diameter, depending on the number of vertices and edges, was given by Harary [4].

Lemma 1 ([4]). *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then we have diam $(G) \leq$ diam $(PK_{n,m})$.*

If $R \subseteq V(G)$, then $G(R)$ is the induced subgraph of G with the vertex-set R. For a graph G and integer $k \geq 1$ let $S_k(G)$ be the set of all unordered pairs of such not adjacent vertices in G that their distance does not exceed k. Hence $S_1(G) = \emptyset$ holds. The following lemma gives the sharp lower bound for the cardinality of the set $S_k(G)$ with respect to the order and the diameter of a graph G.

Lemma 2. *Let G be a connected graph with $n \geq 2$ vertices and diameter $d \geq 3$. Then for any integer k, $2 \leq k \leq d-1$, we have*

$$|S_k(G)| \geq \sum_{i=2}^{k} (n-i) = (k-1)n - k(k+1)/2 + 1\,. \tag{1}$$

Moreover, the equality occurs if G is a path-complete graph.

Proof. Let G_0 be a shortest path in G joining two vertices with distance d. Then we can denote the vertices not lying in G_0 by the symbols $v_1, v_2, \ldots$ $\ldots, v_{n-d-1}$ in such a way that the graphs $G_j := G(V(G_0) \cup \{v_1, v_2, \ldots v_j\})$ are connected for all $j \leq n-d-1$. Let k be a fixed integer, $2 \leq k \leq d-1$. Obviously the equality occurs in (1) for $G = G_0$.Clearly, $S_k(G_j)$ contains $S_k(G_{j-1})$ for all $1 \leq j \leq n-d-1$. Thereby Lemma 2 will be established if we show that the set $S_k(G_j) - S_k(G_{j-1})$ has at least $k - 1$ elements. Now we distinguish two cases.

Case 1. Let $ec_{G_j}(v_j) \geq k$. Obviously, for at least $k - 1$ vertices z in G_{j-1} we have $2 \leq d_{G_j}(v_j, z) \leq k$.

Case 2. Let $ec_{G_j}(v_j) < k$. Note that the vertex v_j is adjacent to at most 3 vertices from G_0. That is why there are at least $d - 2$ vertices z such that $(v_j, z) \in S_k(G_j)$. Clearly, $d - 2 \geq k - 1$ holds.

One can directly verify that the equality occurs in (1) if G is a path-complete graph with diameter at least 3. ∎

Theorem 1. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then $\sigma(G) \leq \sigma(PK_{n,m})$ holds.*

Proof. Let D and d be the diameters of the graphs $PK_{n,m}$ and G, respectively. If $d \leq 2$ holds, then we have $\sigma(G) = 2n(n-1) - 2m \leq \sigma(PK_{n,m})$. Next we shall suppose that $d \geq 3$ holds. Let s_i be the number of unordered pairs of vertices in G with distance i, for integer $i \geq 0$. Note that

$$s_1 = m \text{ and } s_1 + s_2 + \ldots + s_d = m + |S_d(G)| = \binom{n}{2}$$

holds. A little calculation gives

$$\sigma(G)/2 = \sum_{i=1}^{d} is_i = s_1 + |S_d(G)| + \sum_{i=1}^{d-1} (|S_d(G)| - |S_i(G)|) =$$

$$= \binom{n}{2} + \sum_{i=1}^{d-1} \left(\binom{n}{2} - m - |S_i(G)|\right).$$

Now Lemma 1 gives $D \geq d$ and from Lemma 2 the inequality

$$\sigma(G)/2 \leq \binom{n}{2} + \sum_{i=1}^{D-1} \left(\binom{n}{2} - m - |S_i(PK_{n,m})|\right) = \sigma(PK_{n,m})/2$$

follows. ∎

3. The removal of a vertex

Now we shall study how the transmission will change if we remove a vertex from a graph. We shall obtain the graphs $G - e$, $G - v$ if we remove from G the edge e or the vertex v, respectively. Favaron, Kouider and Maheo in [3] solved a certain problem suggested by Plesnik in [5]. They have found the maximum value of $\sigma(G - e) - \sigma(G)$, as a function of n, where e is an edge of the graph G and $G - e$ is connected. Next we shall study a similar problem for removing a vertex.

Let f be a real function of two integer variables. Then we define the real function F_f such that for a connected graph G and such its vertex v that $G - v$ is connected we have $F_f(G, v) := f(\sigma(G - v), \sigma(G))$.

Next we shall consider the following three properties of a function f:

(Dj): the function $f(i, j)$ is decreasing with respect to j

(Ii): the function $f(i, j)$ is increasing with respect to i

(Ih): the function $g(h) := f(h, h + t)$ is increasing for any fixed integer $t > 0$.

Finally, by $w + PK_{n,m}$ we mean the graph obtained from $PK_{n,m}$ in such a way that we join the new vertex w to every vertex of $PK_{n,m}$ by an edge. Next we shall study the extremal values of a function F_f.

Theorem 2. *Let v be a vertex of a graph G with $n \geq 2$ vertices and $m \geq 2n - 3$ edges and both G and $G - v$ be connected. If a function $f(i, j)$ fulfils* (Dj) *and* (Ii) *then we have*

$$F_f(G, v) \leq F_f(w + Pk_{n-1, m-n+1}, w).$$

Proof. Note that $\sigma(G) \geq 2n(n-1) - 2m = \sigma(w + PK_{n-1, m-n+1}, w)$ holds. Further, for the graph $G - v$ with $n - 1$ vertices and m' edges, $m - (n - 1) \leq m' \leq m$, we get $\sigma(G - v) \leq \sigma(PK_{n-1, m-n-1})$ from Theorem 1. Using the properties (Dj) and (Ii) we complete this proof. ■

Theorem 3. *Let G be a connected graph of order $n \geq 2$, $v \in V(G)$ and the graph $G - v$ be connected. If the function $f(i, j)$ fulfils* (Dj) *and* (Ii) *then we have*

$$F_f(G, v) \leq \max_{2n-3 \leq m \leq n(n-1)/2} F_f(w + PK_{n-1, m-n+1}, w).$$

Proof. If we add to G an edge incident to v, then the value of F_f increases. That is why we can restrict ourselves to graphs with at least $2n - 3$ edges. The rest follows from Theorem 2. ■

Let $T_{n,t}$ be the set of all connected graphs of the order n which contain a vertex having the transmission t. The following lemma shows that the path-complete graph has the maximal transmission of all the graphs from $T_{n,t}$.

Lemma 3. *Let two integers $n \geq 2$ and t, $n - 1 \leq t \leq \binom{n}{2}$ be given. Then for any graph $G' \in T_{n,t}$ we have $\sigma(G') \geq \sigma(PK_{n,m})$, where $m = (n+2)(n-1)/2 - t$. Moreover, the equality occurs if and only if $G \cong PK_{n,m}$.*

Proof. Let G be the graph from $T_{n,t}$ having the minimal transmission, v be its vertex with the transmision t, r be the eccentricity of v and N_i be the set of such vertices u that $d(v, u) = i$, for any integer i.

The minimality of the transmission gives that

$$G(N_i \cup N_{i+1}) \text{ are the complete graphs for all } i \leq r - 1. \qquad (2)$$

If $r = 1$, then $G = G(N_0 \cup N_1)$ is the complete graph, hence it is the path-complete graph on n vertices and $\binom{n}{2}$ edges.

Now we can suppose that $r \geq 2$ holds. Here it is sufficient to prove that the set N_i contains just one element for each $0 \leq i \leq r - 2$. This together with (2) gives that G is a path-complete graph.

We prove it indirectly. Suppose that i is the smallest number such that N_i has at least two elements and $i \leq r - 2$. Clearly $N_0 = \{v\}$, hence $i \geq 1$ holds. Let $v_i \in N_i$, $v_r \in N_r$. Now we shall construct a graph H such that we "move v_i from N_i to N_{i+1} and move v_r from N_r to N_{r-1}". More formally, we omit the edge $v_{i-1}v_i$ where $N_{i-1} = \{v_{i-1}\}$, we add the edges v_iv_{i+2}, v_rv_{r-2} for all $v_{i+2} \in N_{i+2}$, $v_{r-2} \in N_{r-2}$. Finally we shall add the edge v_iv_r if $r - i \leq 3$ holds.

Note that the distance of any vertices u, z from $V(G) - \{v_i, v_r\}$ unchanged. Further the sum $d(z, v_i) + d(z, v_r)$ did not change or decreased. The last term unchanged for $z = v$, hence $\sigma_H(v) = t$ holds and so $H \in T_{n,t}$. Finally $d(v_i, v_r)$ decreased, which gives $\sigma(H) < \sigma(G)$, a contradiction. Thus G is a path-complete graph.

Note that for the vertex u of $PK_{n,m}$ with the smallest degree we have $\sigma(u) + m = \binom{n}{2} + n - 1 = (n + 2)(n - 1)/2$. For $m = n - 1$ this equality holds and if we alter $PK_{n,m}$ to $PK_{n,m-1}$, then we omit one edge and $\sigma(u)$ increases by one. This completes the proof. ∎

Theorem 4. *Let v be a vertex of a graph G on $n \geq 2$ vertices and both G and $G - v$ be connected. If a function $f(i, j)$ fulfils* (Dj) *and* (Ih), *then we have*

$$\min_{n-1 \leq m \leq \binom{n}{2} - (n-2)} F_f(Pk_{n,m}, u_{n,m}) \leq F_f(G, v),$$

where $u_{n,m}$ is the endvertex of the graph $PK_{n,m}$.

Proof. The property (Dj) means that if we omit from G an edge incident to v, then the value of F_f decreases. So we can restrict it to the case when v is an endvertex of G. Therefore

$$2n - 3 \leq \sigma(v) \leq \binom{n}{2} \tag{3}$$

holds. Moreover, we have

$$\sigma(G) = 2\sigma(v) + \sigma(G - v) \tag{4}$$

and so

$$F_f(G, v) = f(\sigma(G - v), 2\sigma(v) + \sigma(G - v)) \tag{5}$$

holds. Let us put $t' = \sigma(v)$. Then Lemma 3, the equality (5) and the property (Ih) together give $F_f(PK_{n,m}, u_{n,m}) \leq F_f(G, v)$ where $\sigma(u_{n,m}) = t'$ and so $m = (n+2)(n-1)/2 - t'$. Further, the inequalities (3) give $n - 1 \leq m \leq \binom{n}{2} - (n-2)$. This establishes the theorem. ■

Remark. Now we shall consider two special choices of the function f. Note that the function $f(i, j) = i/j$ fulfils (Dj), (Ii) and (Ih). So we can apply Theorems 2, 3, 4 to the ratio $\sigma(G - v)/\sigma(G)$.

Next we shall study the extremes of the function $a\sigma(G - v) + b\sigma(G)$ where a, b are real. The case $ab \geq 0$ is trivial. The other cases can be reduced to the form $\sigma(G - v) - q\sigma(G)$ with $q > 0$. The function $i - qj$ fulfils (Dj), (Ii) and also (Ih) if $0 < q < 1$. But if we want to find the minimal value of f as a function of n for $q \geq 1$, then we can restrict ourselves to the case when v is an endvertex (it follows from (Dj)). Hence (4) holds and we immediately get

$$F_f(G, v) = -(2q\sigma(v) + (q-1)\,\sigma(G - v)),$$

which is minimal if and only if G is the path on n vertices. We wil not deal here with further technical details.

Eventually the following unsolved problem is presented.

Problem. Find all such graphs G that the equality $\sigma(G) = \sigma(G - v)$ holds for all their vertices v. We know just one such graph — the cycle on 11 vertices.

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Received April 17,1989

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