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MAXIMUMS OF DARBOUX QUASI-CONTINUOUS FUNCTIONS

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ABSTRACT. In this article the functions which can be expressed as the maximum of Darboux quasi-continuous functions are studied. In particular, it is shown that Natkaniec's conjecture concerning characterization of such functions is false.

1. Preliminaries

The letters $\mathbb{R}$ and $\mathbb{N}$ denote the real line and the set of positive integers, respectively. The word *function* denotes a mapping from $\mathbb{R}$ into $\mathbb{R}$ unless otherwise explicitly stated. The word *interval* denotes a nondegenerate compact interval. For each $A \subset \mathbb{R}$ we use the symbol $\operatorname{Int} A$ to denote the interior of A.

Let f be a function and $x \in \mathbb{R}$. Set $c = \varliminf_{t \to x^-} f(t)$ and $d = \varlimsup_{t \to x^-} f(t)$. We say that $x \in \mathbb{R}$ is a *Darboux point of f from the left* if $c \le f(x) \le d$ and $f\big[(x-\delta, x)\big] \supset (c, d)$ for each $\delta > 0$. Similarly we define the notion of a *Darboux point from the right.* We say that x is a *Darboux point of f* if x is a Darboux point of f both from the left and from the right. Recall that f is a Darboux function[1] if and only if each $x \in \mathbb{R}$ is a Darboux point of f. (See, e.g., [1; Theorem 5.1].)

We say that a function f is *quasi-continuous* ([2]) at a point $x \in \mathbb{R}$ if for every open sets $U \ni x$ and $V \ni f(x)$ we have $\operatorname{Int}\big(U \cap f^{-1}(V)\big) \ne \emptyset$. Similarly we define *bilateral quasi-continuity* of f at x. Recall that f is quasi-continuous at x if and only if there exists a sequence (x_n) of continuity points of f such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} f(x_n) = f(x)$. Similarly we can characterize points of bilateral quasi-continuity. The symbols $\mathscr{C}_f$ ($\mathscr{Q}_f$, $\mathscr{Q}^*_f$) will stand for the set of points of continuity of f (the set of points of quasi-continuity of f, the set of

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[1] We say that f is a *Darboux function* if it maps connected sets onto connected sets.

all $x \in \mathbb{R}$ such that x is both a Darboux point and a point of quasi-continuity of f), respectively. If $\mathscr{Q}_f = \mathbb{R}$, then we say that f is *quasi-continuous*. Thus f is a Darboux quasi-continuous function if and only if $\mathscr{Q}_f^* = \mathbb{R}$.

Let f be a function. If $A \subset \mathbb{R}$ and x is a limit point of A, then let

$$\overline{\lim}(f, A, x) = \overline{\lim_{t \to x,\, t \in A}} f(x).$$

Similarly we define the symbols $\overline{\lim}(f, A, x^-)$ and $\overline{\lim}(f, A, x^+)$.

2. Introduction

In 1992, T. Natkaniec proved the following result. (Cf. [4; Proposition 3].)

THEOREM 2.1. *For every function f the following are equivalent:*

(a) *there are quasi-continuous functions g_1 and g_2 with $f = \max\{g_1, g_2\}$;*

(b) *$\mathbb{R} \setminus \mathscr{Q}_f$ is nowhere dense, and $f(x) \le \overline{\lim}(f, \mathscr{C}_f, x)$ for each $x \in \mathbb{R}$.*

He remarked also that if a function f can be written as the maximum of Darboux quasi-continuous functions, then

$$f(x) \le \min\{\overline{\lim}(f, \mathscr{C}_f, x^-), \overline{\lim}(f, \mathscr{C}_f, x^+)\} \qquad \text{for each} \quad x \in \mathbb{R}, \tag{1}$$

and asked whether the following conjecture is true [4; Remark 3].

CONJECTURE 2.2. If f is a function such that $\mathbb{R} \setminus \mathscr{Q}_f$ is nowhere dense and condition (1) holds, then there are Darboux quasi-continuous functions g_1 and g_2 with $f = \max\{g_1, g_2\}$.

We will show that Conjecture 2.2 is false (Example 3.2). On the other hand, if f is a function such that $\mathbb{R} \setminus \mathscr{Q}_f^*$ is nowhere dense and condition (1) holds, then there are Darboux quasi-continuous functions g_1 and g_2 with $f = \max\{g_1, g_2\}$ (Theorem 3.3). Alas, the condition "$\mathbb{R} \setminus \mathscr{Q}_f^*$ is nowhere dense" is not necessary for a function f to be the maximum of two Darboux quasi-continuous functions (Example 3.5). Thus the problem of characterization of the maximums of Darboux quasi-continuous functions is still open.

3. Main results

First we will construct a counter-example for Conjecture 2.2. The easy proof of Lemma 3.1 is left to the reader.

LEMMA 3.1. *Let $x_1 < x_2$, $y_1 < y_2$, and $P = [[x_1, x_2] \times [y_1, y_2]]$. There is a set $Q \subset \mathbb{R}^2$ such that the following conditions hold:*

- *there are intervals $J_1, J_2, \dots \subset (y_1, y_2) \setminus \{(y_1 + y_2)/2\}$ and pairwise disjoint intervals $I_1, I_2, \dots \subset (x_1, x_2)$ such that $Q = \bigcup_{n\in\mathbb{N}} [I_n \times J_n]$ and the length of each I_n is less than $(x_2 - x_1)/2$;*
- *the set $K = [x_1, x_2] \setminus \bigcup_{n\in\mathbb{N}} \operatorname{Int} I_n$ is nowhere dense and perfect;*
- *for each $x \in K$ and each $\delta > 0$, if the set $N^-_{x,\delta} = \{n \in \mathbb{N} : I_n \subset (x - \delta, x)\}$ is infinite, then $\bigcup_{n\in N^-_{x,\delta}} J_n = (y_1, y_2) \setminus \{(y_1 + y_2)/2\}$;*
- *for each $x \in K$ and each $\delta > 0$, if the set $N^+_{x,\delta} = \{n \in \mathbb{N} : I_n \subset (x, x + \delta)\}$ is infinite, then $\bigcup_{n\in N^+_{x,\delta}} J_n = (y_1, y_2) \setminus \{(y_1 + y_2)/2\}$.*

EXAMPLE 3.2. There is a bilaterally quasi-continuous function $h\colon [0,1] \to \mathbb{R}$ which is the maximum of Darboux quasi-continuous functions on no interval.

Construction. Define $I_{1,1} = J_{1,1} = [0,1]$. Use Lemma 3.1 with $P = [I_{1,1} \times J_{1,1}]$ to construct a set L_2 with the properties listed there. Next we proceed by induction. Fix a $k > 1$ and suppose we have already defined the set L_k such that there are intervals $J_{k,1}, J_{k,2}, \dots$ and pairwise disjoint intervals $I_{k,1}, I_{k,2}, \dots$ such that $L_k = \bigcup_{n\in\mathbb{N}} [I_{k,n} \times J_{k,n}]$. For each $n \in \mathbb{N}$ apply Lemma 3.1 with $P = [I_{k,n} \times J_{k,n}]$ to construct a set $Q_{k,n}$ with the properties listed there. Define $L_{k+1} = \bigcup_{n\in\mathbb{N}} Q_{k,n}$.

Fix an $x \in [0,1]$. We consider two cases.

- If $x \in \bigcap_{k>1} \bigcup_{n\in\mathbb{N}} I_{k,n}$, then notice that there is only one $y \in [0,1]$ such that $\langle x, y\rangle \in \bigcap_{k>1} L_k$, and define $h(x) = y$.
- In the other case notice that there is only one pair $\langle k, n\rangle \in \mathbb{N}^2$ such that $x \in I_{k,n} \setminus \bigcup_{m\in\mathbb{N}} I_{k+1,m}$, and define $h(x) = \min J_{k,n}$.

One can easily show that $\bigcap_{k>1} \bigcup_{n\in\mathbb{N}} I_{k,n} \subset \mathscr{C}_h$. Moreover, the graph of $h \restriction \mathscr{C}_h$ is bilaterally dense in the graph of h, whence h is bilaterally quasi-continuous.

Suppose that there is an interval $I \subset [0,1]$ and Darboux quasi-continuous functions $g_1, \ldots, g_m$ such that $h = \max\{g_1, \ldots, g_m\}$ on I. Without loss we may assume that $I = I_{k_0,n_0}$ for some $k_0, n_0 \in \mathbb{N}$, and that

(2) whenever I' is a subinterval of I and N is a proper subset of $\{1, \ldots, m\}$, then $h(x) \neq \max\{g_i(x) : i \in N\}$ for some $x \in I'$.

Put $y_1 = \min J_{k_0,n_0}$, $y_2 = \max J_{k_0,n_0}$, and $y = (y_1 + y_2)/2$. There is a $j \in \{1, \ldots, m\}$ such that $\sup g_j[I] = \sup h[I] = y_2 > y$. Since $\inf g_j[I] \leq \inf h[I] = y_1 < y$ and g_j is Darboux, so $g_j(x) = y$ for some $x \in I$. Then $h(x) \geq y$, whence there is an $n \in \mathbb{N}$ such that $x \in I_{k_0+1,n}$ and $\max J_{k_0+1,n} > y$. Put $y_0 = \min J_{k_0+1,n}$ and recall that $y_0 > y$ and $h(t) > y_0$ for $t \in I_{k_0+1,n}$. But g_j is bilaterally quasi-continuous [3; Lemma 2(a)], so there is an interval $I' \subset I_{k_0+1,n} \subset I$ such that $g_j < y_0$ on I'. It follows that $h = \max\{g_i : i \neq j\}$ on I', which contradicts (2).

Our next goal is the following theorem.

THEOREM 3.3. *If f is a function such that the set $\mathbb{R} \setminus \mathcal{Q}_f^*$ is nowhere dense and condition* (1) *holds, then there are Darboux quasi-continuous functions g_1 and g_2 with $f = \max\{g_1, g_2\}$. Moreover we can conclude that g_1 and g_2 are Lebesgue measurable or belong to Baire class α provided that f is so.*

In the proof we will need a technical lemma.

LEMMA 3.4. *Let f be a function. For any intervals $I \subset \mathcal{Q}_f^*$ and $J \subset (-\infty, \sup f[I])$ there are Darboux quasi-continuous functions g_1 and g_2 such that $f = \max\{g_1, g_2\}$ on I and for $i \in \{1,2\}$: $g_i[I] \supset J$ and $f(x) = g_i(x)$ whenever x is an endpoint of I. Moreover we can conclude that g_1 and g_2 are Lebesgue measurable or belong to Baire class α provided that f is so.*

P r o o f. Choose an $x_1 \in \operatorname{Int} I$ with $f(x_1) > \max J$. Put $x_0 = \min I$ and $x_2 = \max I$, and construct a continuous function φ such that $\varphi(x) = f(x)$ for $x \in \{x_0, x_1, x_2\}$ and $\max\{\inf \varphi[(x_0, x_1)], \inf \varphi[(x_1, x_2)]\} < \min J$. For $i \in \{1,2\}$ define

$$g_i(x) = \begin{cases} \min\{f(x), \varphi(x)\} & \text{if } x \in [x_{i-1}, x_i], \\ f(x) & \text{if } x \in [x_{2-i}, x_{3-i}], \\ \text{constant} & \text{on } (-\infty, x_0] \text{ and } [x_2, \infty). \end{cases}$$

Then clearly $f = \max\{g_1, g_2\}$ on I. Fix an $i \in \{1,2\}$. By [3; Theorem 2(3)], g_i is both Darboux and quasi-continuous. Moreover

$$\inf g_i[I] \leq \inf \varphi\big[(x_{i-1}, x_i)\big] < \min J < \max J < f(x_1) = g_i(x_1) \leq \sup g_i[I],$$

whence $J \subset g_i[I]$. □

Proof of Theorem 3.3. Find a family of nonoverlapping intervals, $\{I_n : n \in \mathbb{N}\}$, such that $\operatorname{Int} \mathscr{D}_f^* = \bigcup_{n\in\mathbb{N}} I_n$ and each $x \in \operatorname{Int} \mathscr{D}_f^*$ belongs to $\operatorname{Int}(I_n \cup I_m)$ for some $n, m \in \mathbb{N}$. For each $n \in \mathbb{N}$ set $b_n = \min\{\sup f[I_n] - n^{-1}, n\}$ and $a_n = \min\{b_n - 1, -n\}$, and use Lemma 3.4 to construct Darboux quasi-continuous functions g_{1n} and g_{2n} such that $f = \max\{g_{1n}, g_{2n}\}$ on I_n and for $i \in \{1,2\}$: $g_{in}[I_n] \supset [a_n, b_n]$ and $f(x) = g_{in}(x)$ whenever x is an endpoint of I_n. For $i \in \{1,2\}$ define $g_i(x) = g_{in}(x)$ if $x \in I_n$ for some $n \in \mathbb{N}$, and $g_i(x) = f(x)$ otherwise. Then evidently $f = \max\{g_1, g_2\}$ on $\mathbb{R}$. To complete the proof we will show that g_1 and g_2 are both Darboux and quasi-continuous. Fix an $i \in \{1,2\}$ and an $x \in \mathbb{R}$.

One can easily see that $\operatorname{Int} \mathscr{D}_f^* \subset \mathscr{D}_{g_i}^*$. So let $x \notin \operatorname{Int} \mathscr{D}_f^*$. By construction, for each $\delta > 0$ we have

$$g_i\big[(x-\delta, x) \cap \operatorname{Int} \mathscr{D}_f^*\big] \supset \big(-\infty, \overline{\lim}(f, \operatorname{Int} \mathscr{D}_f^*, x^-)\big)\,. \tag{3}$$

Hence by (1), x is a Darboux point of g_i from the left. Similarly we can show that x is a Darboux point of g_i from the right. Now condition (3) easily implies that $x \in \mathscr{D}_{g_i}$, which completes the proof. □

Finally we will show that the condition "$\mathbb{R} \setminus \mathscr{D}_f^*$ is nowhere dense" is not necessary for a function f to be the maximum of two Darboux quasi-continuous functions.

EXAMPLE 3.5. There is a bilaterally quasi-continuous function $f\colon [0,1] \to \mathbb{R}$ which is the maximum of two Darboux quasi-continuous functions and which is Darboux on no interval.

Construction. Let h be the function defined in Example 3.2. Put $f = -h$. Evidently f is bilaterally quasi-continuous and f is Darboux on no interval.

Let the symbols $I_{k,n}$ and $J_{k,n}$ $(k,n \in \mathbb{N})$ be defined as in Example 3.2. For each k and n put $A_{k,n} = \operatorname{Int} I_{k,n} \setminus \bigcup_{m\in\mathbb{N}} I_{k+1,m}$, and let $\varphi_{k,n,1}, \varphi_{k,n,2}\colon A_{k,n} \to J_{k,n}$ be such that $\min\{\varphi_{k,n,1}, \varphi_{k,n,2}\} = \min J_{k,n}$ on $A_{k,n}$ and $\varphi_{k,n,1}[I] = \varphi_{k,n,2}[I] = J_{k,n}$ whenever I is an interval intersecting $A_{k,n}$. For $i \in \{1,2\}$ define

$$g_i(x) = \begin{cases} -\varphi_{k,n,i}(x) & \text{if } x \in A_{k,n}\,,\ \ k,n \in \mathbb{N}, \\ f(x) & \text{otherwise.} \end{cases}$$

Clearly $f = \max\{g_1, g_2\}$ on $[0,1]$. Fix an $i \in \{1,2\}$. One can easily show that $\bigcap_{k>1} \bigcup_{n\in\mathbb{N}} I_{k,n} \subset \mathscr{C}_{g_i}$, so g_i is quasi-continuous.

To prove that g_i is Darboux fix an interval $I \subset [0,1]$. Set

$$k_0 = \max\{k \in \mathbb{N} :\ I \subset I_{k,n} \text{ for some } n \in \mathbb{N}\}\,.$$

Let $n_0 \in \mathbb{N}$ be such that $I \subset I_{k_0,n_0}$. Then $I \cap A_{k_0,n_0} \neq \emptyset$, so $g_i[I] \supset J_{k_0,n_0}$. (Notice that $I \not\subset \bigcup_{m\in\mathbb{N}} I_{k_0+1,m}$.) But the opposite inclusion is evident, whence $g_i[I]$ is an interval.

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