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THE LINEAR ARBORICITY OF 10-REGULAR GRAPHS

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The concept of linear arboricity was introduced by Harary [8], [2] as one of the covering invariants of graphs.

A linear forest is a graph in which each component is a path. The linear arboricity $\Xi(G)$ of a graph G is the minimum number of linear forests whose union is G.

The value of linear arboricity so far has been determined only for a few special classes of graphs, e.g. for trees, complete graphs and complete bipartite graphs. (See [1] and [2].) The following conjecture expressed in [2] has a fundamental importance in the research on linear arboricity.

Conjecture. *The linear arboricity of an r-regular graph is* $\left\{\frac{r+1}{2}\right\}$.

The general proof of this conjecture does not seem to be simple and so we are meanwhile satisfied with partial results. The conjecture was proved for $r=3$ and $r=4$ by Akiyama, Exoo and Harary in [2] and [3], the cases of $r=5, 6$ were solved independently by Enomoto [4], Peroche [10] and Tomasta [12] (only for $r=6$) and the case of $r=8$ was proved by Enomoto and Peroche [5].

The aim of this paper is to prove the validity of the conjecture for the case of $r=10$.

Let us at first introduce some necessary notions and notations. In the paper by a graph we mean an undirected finite simple graph, by a spanning linear forest we mean a linear forest which is a factor with minimum degree one. Let us denote by $V(G)$ the set of vertices of a graph G and by $E(G)$ the set of edges of G. Further let us denote by $V_r(G)$ the set of vertices of degree r of G, let $N_G(v)$ denote the set of vertices adjacent to a vertex v in G, let $\langle M\rangle$ denote the subgraph induced by a subset M of vertices, let (u, v) denote the undirected edge joining vertices u and v and let $\Delta(G)$ denote the maximum degree of G. Any terminology not defined in the paper can be found in [9].

The basic fact which implies our new results about the conjecture is the following theorem.

Theorem 1. *Let G be a graph with the degree sequence* $(6, 5, 5, \ldots, 5)$. *Then* $\Xi(G)=3$.

As the complete proof of this result is too long for publication in a journal, we shall only outline the method of the proof. The detailed proof of Theorem 1 and of Lemmas 1—3 can be found in [6].

The proof of Theorem 1 is based on the strong Theorem 2 proved by Enomoto [4] and on Lemmas 1—3.

Theorem 2. *Let G be a graph with $\Delta(G)=4$. Let $\Delta(\langle V_4(G)\rangle)\leqslant 1$. Then $\Xi(G)=2$.*

Lemma 1. *Let G be a graph with one vertex v of degree 6 and all other vertices of degree 5. Let $v_1, v_2, \ldots, v_6 \in V(G)$ be adjacent to v. Let there exist a spanning linear forest P in G such that*

(i) *$(v_1, v), (v_2, v)\in E(P)$, i.e. v is not an endvertex of P,*
(ii) *v_3, v_4, v_5 are not endvertices of P,*
(iii) *either v_6 is not an endvertex of P or no vertex $y\in(N_G(v_6)-\{v_{61}\})$ is an endvertex of P, where $(v_6, v_{61})\in E(P)$.*

Then $\Xi(G)=3$.

Lemma 2. *Let G be a graph with one vertex v of degree 6 and all other vertices of degree 5. Let $v_1, v_2, v_3, v_4, v_5, v_6$ be adjacent to v. Let there exist a spanning linear forest P' in G such that*

(i) *$(v, v_1)\in E(P')$ and $(v, v_2), (v, v_3), \ldots, (v, v_6)\notin E(P')$, i.e. v is an endvertex of P',*
(ii) *$v_2, v_3, \ldots, v_6$ are not endvertices of P'.*

Then $\Xi(G)=3$.

Lemma 3. *Let G be a graph, let $V(G)=M_1\cup M_2$, $M_1\cap M_2=\emptyset$. Let $N(M_1)=\{y\in M_2;\ \exists x\in M_1,\ (y, x)\in E(G)\}$, let $N(M_1)\neq\emptyset$. Let there exist a spanning linear forest P in $\langle M_2\rangle$ such that $\deg_P(x)=2$ for all $x\in N(M_1)$. Let there exist an integer $\delta>1$ such that $\delta\leqslant\deg_G(u)\leqslant 2\delta$ for all $u\in M_2$. Then there exists a spanning linear forest P_1 in $\langle M_2\rangle$ with the property that there exists a vertex $v_0\in N(M_1)$ such that $\deg_{P_1}(v_0)=1$ and for all $y\in N(M_1)$, $y\neq v_0$ we have $\deg_{P_1}(y)=2$.*

The main idea of the proof of Theorem 1 is the following: Let v be the vertex of degree 6 and let $v_1, v_2, v_3, v_4, v_5, v_6$ be adjacent to v. Consider the graph $G_1=G-(v, v_1)$. From [4] or [10] it follows that G_1 can be decomposed into three linear forests F_1', F_2', F_3'. Then after adding the edge (v, v_1) back to G_1 we can either find a decomposition of G into three linear forests F_1, F_2, F_3 by modifying F_1, F_2, F_3 or determine a linear forest P (resp. P') in G which fulfils the conditions of Lemma 1 (resp. Lemma 2). There are considered 4 main cases in this proof, according to the values of the numbers

$$p_i=\sum_{j=1}^{6}\deg_{F_i-v}(v_j)\qquad\text{for}\quad i=1, 2, 3.$$

The main cases are then analysed and divided into more detailed subcases, the whole proof consists of verifying 19 subcases altogether.

Theorem 3. *Let r be an odd integer, $r \geqslant 5$. Let the linear arboricity of every r-regular graph be $\left\{\frac{r+1}{2}\right\}$, i.e. let the Akiyama—Exoo—Harary conjecture hold for r. Then the linear arboricity of every $(r+5)$-regular graph is $\left\{\frac{(r+5)+1}{2}\right\}$, i.e. the conjecture holds also for the case of $r+5$.*

Proof. Let G be an arbitrary $(r+5)$-regular graph. The inequality $\Xi(G) \geqslant \frac{(r+5)+1}{2}$ is obvious. By Petersen [11] we can decompose G into a 10-regular factor H and an $(r-5)$-regular factor $(G-H)$.

I. Let $|V(G)|$ be even. Consider a Eulerian trail in H. Colour the edges of this trail alternately with two colours. We obtain two 5-regular factors H_1 and H_2. By [4] or [10] H_1 can be decomposed into three linear forests and $\Xi(G-H_1) = \left\{\frac{r+1}{2}\right\}$ by assumption. Hence $\Xi(G) = \left\{\frac{(r+5)+1}{2}\right\}$.

II. Let $|V(G)|$ be odd. Once again consider a Eulerian trail in H. Colour the edges of this trail alternately with two colours. We obtain a decomposition of H into two factors H_1 and H_2. The factor H_1 has the degree sequence $(6, 5, 5, \ldots, 5)$ and H_2 has $(5, 5, \ldots, 5, 4)$. By Theorem 2 H_1 can be decomposed into 3 linear forests and $\Xi(G-H_1) = \left\{\frac{r+1}{2}\right\}$ follows from the assumption. Hence $\Xi(G) = \left\{\frac{(r+5)+1}{2}\right\}$.

As the conjecture has already been proved for the case of $r=5$, Theorem 3 implies the following important corollary.

Corollary. *The linear arboricity of every* 10*-regular graph is* 6.

A similar implication as in Theorem 3 from an add r to $r+3$ can be easily proved by a generalization of Tomasta's method of the proof of conjecture for $r=6$. The implication from an odd r to $r+1$ follows easily from the fact that every $(r+1)$-regular graph (for an odd r) contains a spanning linear forest (because every 2-factor contains a spanning linear forest) and so we can formulate the following theorem in the conclusion.

Theorem 4. *If the Akiyama—Exoo—Harary conjecture holds for some odd r, then it holds for $r+1$, $r+3$, $r+5$, too.*

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ЛИНЕЙНАЯ ДРЕВЕСНОСТЬ 10-ПРАВИЛЬНЫХ ГРАФОВ

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Резюме

Линейная древесность $\Xi(G)$графа G — это минимальное число линейных лесов, объединение которых равно G. В работе [2] была высказана гипотеза, что линейная древесность r-правильного графа равна

$$\left\{\frac{r+1}{2}\right\}.$$

До сих пор была доказана для $r=2, 3, 4, 5, 6, 8$. В этой статье показывается, что гипотеза правильна тоже для $r=10$.