Ján Mináč
Some counterexamples in p-adic analysis

Mathematica Slovaca, Vol. 30 (1980), No. 3, 305--311

Persistent URL: http://dml.cz/dmlcz/132119

Math. Slovaca 30, 1980, No. 3, 305—311

SOME COUNTEREXAMPLES IN p-ADIC ANALYSIS

JÁN MINÁČ

The class of differentiable functions on the p-adic integer numbers Z_p to the p-adic numbers Q_p is "rich". (One of the reasons is the total discontinuity of the topological space Z_p.) In this paper we shall show that this class is not contained in some classes defined later.

Throughout the paper, a similar notation as in [1] (especially chapter 2, § 2) will be used. Evidently in some theorems of [1], chapt. 2, § 2 some additional assumptions are missing (e.g. strict differentiability, see the end of the present paper), because in the cited version the theorems are false.

The theorems of [1], chapt. 2, § 2 are based on the following lemma.

Let f be a differentiable function on the set of p-adic integers Z_p with values in the set of p-adic numbers Q_p and let x_n, $y_n \in Z_p$, $x_n \neq y_n$ $(n = 1, 2, 3, \ldots)$, $\lim_{n\to\infty} (x_n - y_n) = 0$. Then there exists such an element x_0 from Z_p and such a monotone increasing sequence of natural numbers n_k, that

$$\lim_{k\to\infty} \frac{f(y_{n_k}) - f(x_{n_k})}{y_{n_k} - x_{n_k}} = f'(x_0).$$

The mistake in the proof of this lemma lies in the fact that some denominator $(\lim_{k\to\infty} (1 + u_{n_k}^{-1}))$ in this computation may be equal to zero.

First we give a counterexample to the lemma.

Example 1. Let f be a function defined as follows:

$$f(x) = \begin{cases} p^{2i} + p^{2i+1} & \text{if } x = p^i + p^{2i+1} + \ldots \\ & \text{(the first two non-zero coefficients are fixed and the others can be arbitrary)}, \\ p^{2i} & \text{if } x = p^i + p^{2i+2} + \ldots, \\ 0 & \text{otherwise,} \end{cases}$$

where we assume that every $x \in Z_p$ is written in the usual p-adic form $x = a_0 + a_1 p + \ldots + a_n p^n + \ldots$, where $0 \leqq a_i \leqq p - 1$.

This function f is locally constant on $Z_p - \{0\}$. Indeed, if we take the usual p-adic norm $|\cdot|$ $\left(|p|=\frac{1}{p}\right)$, then for every $x_0 \neq 0$ the value $f(x)$ in the neighborhood $\left\{x \,\|x-x_0| < \frac{1}{p^{2k+2}}\right\}$, where the first non-zero coefficient of x_0 is a_k, is equal to $f(x_0)$. This holds since then sufficiently many coefficients of such x and x_0 are equal. Thus the derivative of f on Z_p exists and is equal to zero. Indeed, it is clear, because if x is sufficiently small, then the quotient

$$\frac{f(x)-f(0)}{x} = \begin{cases} p^j \text{ multiplied by a unit in } Z_p, \text{ if } & x = p^j + p^{2j+1} + \dots \text{ or} \\ & x = p^j + p^{2j+2} + \dots, \\ & 0 \text{ otherwise } (x \neq 0) \end{cases}$$

(by a unit in Z_p we understand an element of Z_p which has an inverse element in Z_p) is arbitrarily small. (In the sense of p-adic topology.) Now we take the following sequences

$$x_n = p^n + p^{2n+2}, \quad y_n = p^n + p^{2n+1}, \quad \text{then} \quad \lim_{n\to\infty} (x_n - y_n) = 0, \quad x_n \neq y_n,$$

but

$$\frac{f(x_n)-f(y_n)}{x_n - y_n} = \frac{p^{2n} - p^{2n} - p^{2n+1}}{p^{2n+2} - p^{2n+1}} = (1-p)^{-1}.$$

Thus we cannot find such a sequence n_k of natural numbers as desired.

This example is also a counterexample to theorem 2.8 of [1], which asserts:

If f is a differentiable function on Z_p with values in Q_p and if $f'(x) \equiv 0$, then for every $\varepsilon > 0$ there exists such a $\delta(f, \varepsilon) > 0$ that

$$|f(x)-f(y)| \leqq \varepsilon |x-y| \quad \text{if} \quad |x-y| < \delta(f, \varepsilon).$$

Really, the function of example 1 has the derivative equal to zero on whole Z_p and

$$|f(p^n + p^{2n+1}) - f(p^n + p^{2n+2})| = |p^{2n+1}| = \frac{1}{p^{2n+1}},$$

but

$$|p^n + p^{2n+1} - (p^n + p^{2n+2})| = \frac{1}{p^{2n+1}},$$

hence, e.g. for $\varepsilon = \frac{1}{2}$ it does not hold that

$$|f(p^n + p^{2n+1}) - f(p^n + p^{2n+2})| \leqq \frac{1}{2} |p^n + p^{2n+1} - (p^n + p^{2n+2})|,$$

although

$$\lim_{n\to\infty} (p^n + p^{2n+1} - (p^n + p^{2n+2})) = 0.$$

This example is also an counterexample to theorem 2.10 of [1] (more precisely, to one half of this theorem) which asserts:

The derivative $f'(x)\equiv 0$ on Z_p if and only if

$$\lim_{n\to\infty}\frac{\delta_n}{|p^n|}=0, \quad \text{where} \quad \delta_n=\sup\left\{|f(x)-f(y)|\,\middle|\,|x-y|\leqq\frac{1}{p^n}\right\}.$$

It is clear that for the function f of Example 1 we have

$$\overline{\lim}\frac{\delta n}{|p^n|}\geqslant 1.$$

But it is clear that if a function f satisfies the above condition, then f has indeed the derivative identically equal to zero on Z_p.

Theorem 2.6 of [1] asserts that if f is differentiable on Z_p with values in Q_p, then f' is bounded on Z_p. A counterexample can be constructed by a little change of the preceding example.

Example 2. Let

$$f(x)=\begin{cases} p^{2n_1}+a_2p^{n_2-n_1}+a_3p^{n_3-n_1}+\dots \\ \text{if } x=p^{n_1}+a_2p^{n_2}+\dots+a_kp^{n_k}+ \\ \text{(this form respects only nonzero coefficients of } x, \text{ i.e. } a_2, a_3, \dots\neq 0) \\ n_1<3n_1<n_2<n_3<\dots<n_k<\dots \\ \text{(it is possible also } x=p^{n_1}), \\ 0 \text{ otherwise.}\end{cases}$$

Since the function f on the set

$$Z_p-\{x=p^{n_1}+a_2p^{n_2}+ \\ +\dots+a_kp^{n_k}+\dots n_1<3n_1<n_2<n_3<\dots,\ 0\leqq a_i\leqq p-1,\ i\geqq 2\}\cup\{0\}$$

is locally constant, the derivative f' is equal to zero on this set. Now compute the derivative in zero. Suppose $n_2>3n_1$. Since

$$\frac{f(p^{n_1}+a_2p^{n_2}+\dots)}{p^{n_1}+a_2p^{n_2}+\dots}=\frac{p^{n_1}(1+a_2p^{n_2-3n_1}+\dots)}{1+a_2p^{n_2-n_1}+\dots}$$
$$=p^{n_1}(\text{a unit in } Z_p)$$

is arbitrarily small if n_1 is sufficiently large, we have that $f'(0)=0$. Now compute the following quotient for $n_2>3n_1$ and a fixed natural number $k_0\geqq 2$, $x\neq y$. (The denotation we use here means that the first k_0 coefficients of the following terms x, y are equal.)

$$\frac{f(x)-f(y)}{x-y}=\frac{f(p^{n_1}+a_2p^{n_2}+\dots+a_{k_0}p^{n_{k_0}}+a_{k_0+1}p^{n_{k_0+1}}+\dots)}{(p^{n_1}+\dots+a_{k_0}p^{n_{k_0}}+\dots)-(p^{n_1}+\dots+a_{k_0}p^{n_{k_0}}+b_{k_0+1}p^{n_{k_0+1}}+\dots)}-$$

$$-\frac{f(p^{n_1}+a_2p^{n_2}+\ldots+a_{k_0}p^{n_{k_0}}+b_{k_0+1}p^{n_{k_0+1}}+\ldots)}{(p^{n_1}+\ldots a_{k_0}p^{n_{k_0}}+\ldots)-(p^{n_1}+\ldots+a_{k_0}p^{r_0}+b_{k_0+1}p^{n_{k_0+1}}+\ldots)}=$$

$$=\frac{\dfrac{(x-p^{n_1})}{p^r}-\dfrac{(y-p^{n_1})}{p^{n_1}}}{x-y}=\frac{1}{p^{n_1}}.$$

Thus the derivative in any $x=p^{n_1}+a_2p^{n_2}+\ldots$ (with $n_2>3n_1$) is equal to $\frac{1}{p^{n_1}}$. Since the norm $\left|\frac{1}{p^{n_1}}\right|$ is equal to p^n, the derivative is unbounded.

Example 2 is clearly also a counterexample to theorem 2.9 of [1], which asserts that if f is differentiable on Z_p, then $f\in L_1$. (L_1 is the first Lipschitz class, i.e.

$$|f(x)-f(y)|\leqq A\,|x-y|,$$

for every function $f\in L_1$, for each pair $x, y\in Z_p$ and a suitable constant A.)

Indeed, e.g. for the sequences

$$x_n=p^n+p^{4n},\quad y_n=p^n+p^{4n}+p^{5n}$$

we have

$$|f(x_n)-f(y_n)|=|p^{2n}+p^{3n}-(p^{2n}+p^{3n}+p^{4n})|=|p^{4n}|=\frac{1}{p^{4n}}$$

and

$$|x_n-y_n|=\frac{1}{p^{5n}}.$$

Thus the sequence

$$\left\{\frac{|f(x_n)-f(y_n)|}{|x_n-y_n|}=p^n\right\}_1^\infty$$

tends to infinity and we cannot find such a constant A as desired.

Now we give the counterexample to theorem 2.7, of [1], which asserts:

Let f be a differentiable function

$$Z_p\to Q_p,\quad f'(x)\not\equiv 0,\quad A=\sup_{x\in Z_p} f'(x),\quad a=\inf_{x\in Z_p} f'(x).$$

Then there exists such a positive constant $\delta(f)$ that

$$a\,|x-y|\leqq|f(x)-f(y)|\leqq A\,|x-y| \tag{1}$$

if $|x-y|<\delta(f)$.

Example 3. For the counterexample we can take the function

$$f(x)=\begin{cases}p^n+p^{2n}+p^{4n}+a_1p^{4n+1}+\ldots & \text{if } x=p^n+p^{4n}+a_1p^{4n+1}+\ldots\ n\geqq 1\\ x & \text{otherwise}\end{cases}$$

It is clear that $f'(x)=1$ for all $x \in Z_p$. Indeed, if x has the form $p^n+p^{4n}+a_1p^{4n+1}+\ldots, n \geqq 1$, then $f(x)-f(y)=x-y$ for every y such that $|x-y|<\frac{1}{p^{4n}}$. Now compute the derivative in zero. If $x=p^n+p^{4n}+a_1p^{4n+1}+\ldots$, $n \geqq 1$, then

$$\frac{f(x)}{x}=\frac{x+p^{2n}}{x}=1+p^n. \text{ (an element of the ring } Z_p).$$

Thus it is clear that $f'(0)=1$, because the other possibility for x is trivial. It is clear that the derivative in nonzero x, which has not the form $p^n+p^{4n}+a_1p^{4n+1}+\ldots$ is equal to 1, because on the neighborhood

$$\left\{y \,\middle|\, |y-x|<\frac{1}{p^{4n}}\right\}$$

is $f(x)$ equal x. But for $x_n=p^n+p^{2n}+p^{4n}$, $y_n=p^n+p^{4n}$ we have

$$|f(x_n)-f(y_n)|=0, \quad |x_n-y_n|=\frac{1}{p^{2n}},$$

hence it is clear that the left side inequality (1) does not hold for any $\delta(f)>0$. That the function f does not satisfy the right side inequality of (1) we can see from the following: For $x_n=p^n$, $y_n=p^n+p^{4n}$ we have

$$|f(x_n)-f(y_n)|=|p^n-p^n-p^{2n}-p^{4n}|=$$
$$=|p^{2n}||1+p^{2n}|=\frac{1}{p^{2n}}, \quad |x_n-y_n|=\frac{1}{p^{4n}},$$

hence

$$\frac{|f(x_n)-f(y_n)|}{|x_n-y_n|}=p^{2n}$$

is unbounded. This proves that f does not satisfy the right inequality of (1) for any x, y such that $|x-y|<\delta(f)$, whichever $\delta(f)$ was taken.

Example 3 is also a counterexample to the theorem 2.7a of [1]: If $f'(x_0)\neq 0$ and f' is a continuous function in a point x_0, then there exists such a neighbourhood of x_0, that there holds

$$|f(x)-f(y)|=|f'(x_0)||x-y|.$$

(Indeed, we can take $x_0=0$ and then because in every neighborhood of 0 there lie almost all the elements x_n and y_n, $n=1, 2, \ldots$, we get a contradiction, since

$$|f(x_n)-f(y_n)|=p^{2n}|x_n-y_n|,$$

but not

$$|f(x_n)-f(y_n)|=|x_n-y_n|.)$$

Example 3 is also a counterexample to theorem 2.11 of [1], which asserts: If f has the continuous derivative f' and if $f'(x_0) \neq 0$, then there exists such a neighborhood of x_0, that there exists the inverse (continuous) function f^{-1}, $(x = f^{-1}(y))$, to the function f restricted on this neighborhood, for which

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{f'(x)}.$$

This theorem is false, since for the function f of Example 3 we have

$$f(p^n + p^{4n}) = f(p^n + p^{2n} + p^{4n}) = p^n + p^{2n} + p^{4n}$$

and because almost all of these elements lie in every neighborhood of zero, we have that the function f restricted to an arbitrary small neighbourhood of zero is not one-to-one.

Why is it so easy to construct differentiable functions on Z_p without the above prescribed properties? The reason is based on the fact that for constructing of differentiable functions on Z_p it is sufficient to construct systems of differentiable functions on open subsets of Z_p, which form a disjoint cover of Z_p and that these functions are independent of each other. (We need not consider the situation on the borders of the sets.) On the other hand we can take such a cover that the points of some different open sets of this cover can lie arbitrarily near.

It is easy to see that the sufficient condition for the validity of the assertions of these theorems is that f is strictly differentiable, i.e. the function f is such that the function

$$g(x, y) = \frac{f(x) - f(y)}{x - y}$$

from $Z_p \times Z_p - \{(x, x) | x \in Z_p\}$ to Q_p has a continuous extension on the whole $Z_p \times Z_p$ (see, e.g. [2]).

Remark. The cited theorems from [1] are not exactly in original version (for reason of simplicity) in the sense that in [1] the function f is not assumed to be from Z_p to Q_p, but a more general situation is considered: The function f is $V \to k$, where k is a complete non-archimedean field with respect to the absolute value $|\cdot|$ and V is the set $\{x | |x \leqq 1\}$. In some theorems of [1] there is neither explicitly written that constants $(\delta(f), \delta(f, \varepsilon))$ are positive, but the theorems have something to tell us only in this case (theorems 2.7, 2.8).

REFERENCES

[1] ЛЕНСКОЙ, Д. К.: Функции в неархимедовски нормированных полях, Издательство Саратовского университета, Саратов 1962.
[2] WEISMAN, C. S.: On p-adic differentiability, J. Number Theory 9 (1977), 79—86.

Received September 1978

Matematický ústav SAV
ul. Obrancov mieru 49
886 25 Bratislava

НЕКОТОРЫЕ КОНТРПРИМЕРЫ В p-АДИЧЕСКОМ АНАЛИЗЕ

Ян Минач

Резюме

В этой работе даются некоторые контепримеры для утверждений к некоторым теоремам из p-адического анализа, касающихся дифференцируемых функций.