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ON EXTREME FIRST RETURN PATH DERIVATIVES

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ABSTRACT. It is shown that the right (left) first return systems of paths are right (left) continuous and the extreme first return path derivatives of functions in B_α (Borel measurable functions, Lebesgue measurable functions) are elements of $B_{\alpha+2}$ (Borel measurable, Lebesgue measurable). It is also shown that even though the return path systems are the thinnest possible in a bilateral sense, the extreme first return path derivatives of continuous functions have some similarities with their Dini derivatives. We also provide an example illustrating that the extreme first return derivatives are not identical with their corresponding Dini derivatives.

1. Introduction

In [6] the concept of a path system is introduced and it is shown that many theorems about the differentiability of functions could be obtained from conditions on the thickness of paths and how they intersect each other. In [1] we introduce the concept of a continuous system of paths and show that this concept is another factor in the differentiability structure of functions. This concept was generalized by Milan Matejdes for the study of extreme path derivatives (see [11], [12], [13]).

Motivated by the Poincare first return map of differentiable dynamics, R. J. O'Malley [14] introduced a new type of path system which he calls first return systems. He shows that, though these are extremely thin paths, the systems possess an interesting intersection property that makes their differentiation theory as rich as those of much thicker path systems. For example, every first return path differentiable function is in DB_1^* and every first return path derivative is in DB_1. First return systems have been extensively investigated

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in a series of papers by U. B. Darji, M. J. Evans, P. D. Humke and R. J. O'Malley (see [7], [8], [9] and some of their references).

In this paper the extreme first return path derivatives of Baire class α functions are investigated. We show that these are elements of $B_{\alpha+2}$. It is also shown that, even though the first return path systems are the thinnest possible in a bilateral sense, the extreme first return path derivatives of continuous functions have some similarities with the Dini derivatives. Some early thoughts and questions that resulted in this paper were announced in the summer symposia in Real Analysis (see [2], [3]).

2. Notation and definitions

We take our definitions from [6] and [7].

Definition 2.1. A *trajectory* is a sequence P_n, $n = 0, 1, \dots$, with the following properties.

(i) $P_0 = 0$, $P_1 = 1$,
(ii) $P_n \neq P_m$, $n \neq m$,
(iii) $0 \leq P_n \leq 1$ for all n,
(iv) $\{P_n : n = 0, 1, \dots\}$ is dense in $[0, 1]$.

Our notation for a trajectory will be $\{P_n\}$. For a given $k \geq 1$, Π_k will represent the partition of the interval $[0, 1]$ generated by the initial segment $\{P_0, P_1, \dots, P_k\}$. The ith interval of that partition will be denoted as $\Pi_{k,i}$.

Definition 2.2. Let x belong to $[0, 1]$. A *path leading to* x is a set $R_x \subseteq [0, 1]$ such that $x \in R_x$ and x is a point of accumulation of R_x. A *system of paths* R is a collection $\{R_x : x \in [0, 1]\}$, where each R_x is a path at x.

Definition 2.3. Let $\{P_n\}$ be a fixed trajectory. For a given interval $(a, b) \subset [0, 1]$, $r(a, b)$ will be the first element of the trajectory in (a, b).

For $0 \leq y < 1$, the *right first return path to* y, R_y^+, is defined recursively via $y_1^+ = y$, $y_2^+ = 1$ and $y_{k+1}^+ = r(y, y_k^+)$ for $k \geq 2$. For $0 < y \leq 1$, the *left first return path to* y, R_y^-, is defined similarly. For $0 < y < 1$, we set $R_y = R_y^- \cup R_y^+$, and $R_0 = R_0^+$, $R_1 = R_1^-$.

The path systems $R^+ = \{R_x^+ : x \in [0, 1)\}$, $R^- = \{R_x^- : x \in (0, 1]\}$ and $R = \{R_x : x \in (0, 1)\} \cup \{R_0^+, R_1^-\}$ are called the *right first return system*, the *left first return system* and the *first return system of paths* generated by $\{P_n\}$, respectively.

DEFINITION 2.4. Let $F\colon [0,1] \to \mathcal{R}$ and let R be any path system. If the

$$\lim_{\substack{y \to x \\ y \in R_x \setminus \{x\}}} \frac{F(y) - F(x)}{y - x} = f(x)$$

exists and is finite, then we say F is *R-differentiable at x*. If F is R-differentiable at every point x, then we say that F is *path differentiable* and f is the *path derivative* of F and is denoted by $F'_R = f$.

If the system of paths is a first return system, then f is called the *first return path derivative of F*.

The *extreme path derivatives* $\overline{F}'_R$ and $\underline{F}'_R$ are defined in the usual way.

Note that when path derivatives exist, they have finite values while the extreme path derivatives could accept infinite values.

DEFINITION 2.5. A path system R is said to have the *external intersection condition*, denoted by *E.I.C.* (*intersection condition* denoted by *I.C.*, *internal intersection condition* denoted by *I.I.C.*) if there is a positive function $\delta(x)$ on $[0,1]$ such that $R_x \cap R_y \cap (y, 2y-x) \neq \emptyset$ and $R_x \cap R_y \cap (2x-y, x) \neq \emptyset$ ($R_x \cap R_y \cap [x,y] \neq \emptyset$, $R_x \cap R_y \cap (x,y) \neq \emptyset$, respectively), whenever $0 < y - x < \min\{\delta(x), \delta(y)\}$.

DEFINITION 2.6. Let δ be a positive function and let X be a set of real numbers. By a *δ-decomposition of X* we shall mean a sequence of sets $\{X_n\}$, which is a relabeling of the countable collection

$$Y_{m,j} = \{x \in X : \delta(x) > \tfrac{1}{m}\} \cap [\tfrac{j}{m}, \tfrac{j+1}{m}],$$
$$m = 1,2,3,\ldots \quad \text{and} \quad j = 0, \pm 1, \pm 2, \pm 3, \ldots.$$

The key features of such a decomposition of a set X are:

(i) $\bigcup_{n=1}^{\infty} X_n = X$;

(ii) if x and y belong to the same set X_n, then $|x - y| < \min\{\delta(x), \delta(y)\}$,

(iii) if $x \in \overline{X}_n$, then there are points y in X_n with $|x-y| < \min\{\delta(x), \delta(y)\}$.

We take the following definition from [1].

DEFINITION 2.7. Let $R = \{R_x : x \in [0,1]\}$ be a system of paths with each R_x compact. We endow R with the Hausdorff metric d_H to form a metric space. If the function $P\colon x \mapsto R_x$ is a continuous function, we say R is a *continuous system of paths*. The *left continuous* and *right continuous* systems of paths are defined similarly.

3. Results

We first show that the intersection conditions are not sufficient to guarantee the measurability of the extreme path derivatives of a Borel measurable function. We then show that the right (left) first return system of paths are right (left) continuous. We use this fact to obtain some results about the Borel measurability of extreme first return path derivatives. We conclude with some monotonicity results.

THEOREM 3.1. *There exists a function $F \in DB_1$ and a bilateral system of paths $R = \{R_x : x \in [0,1]\}$ satisfying both the E.I.C. and the I.I.C. for which $\overline{F'}_R$ is nonmeasurable.*

P r o o f . Let $P \subseteq [0,1]$ be a Cantor like set of positive measure. If $[0,1] \setminus P = \bigcup\limits_{n=1}^{\infty} (c_n, d_n)$, define

$$F(x) = \begin{cases} \frac{1}{(x-c_n)(d_n-x)} \sin\left[\frac{1}{(x-c_n)(d_n-x)}\right] & \text{for } x \in (c_n, d_n)\,, \\ 0 & \text{for } x \in P\,. \end{cases}$$

Clearly $F \in DB_1$. Choose $A \subseteq P$ nonmeasurable. Define the system of paths $R = \{R_x : x \in [0,1]\}$ as follows:

$$R_x = \begin{cases} \{t:\ t > x,\ F(t) \le t - x\} \cup \{t : t < x,\ F(t) \ge t - x\} \cup \{x\} & \forall (x \in P \setminus A), \\ \{t:\ t > x,\ F(t) \le \frac{1}{2}(t-x)\} \cup \{t:\ t < x,\ F(t) \ge \frac{1}{2}(t-x)\} \cup \{x\} & \forall (x \in A), \\ \mathcal{R} & \forall (x \in [0,1] \setminus P). \end{cases}$$

Then R satisfies both of E.I.C. and I.I.C., but

$$\overline{F'}_R(x) = \begin{cases} 1 & \forall (x \in P \setminus A)\,, \\ \frac{1}{2} & \forall (x \in A)\,. \end{cases}$$

It follows that $\overline{F'}_R$ is not measurable. □

THEOREM 3.2. *Let $\{P_n\}$ be any trajectory, and take R^+ and R^- to be the right and left first return path system generated by $\{P_n\}$. Then R^+ is a right continuous system of paths and R^- is a left continuous system of paths.*

P r o o f . We first note that for every $x \in [0,1]$, R_x, the path leading to x, is a closed subset of $[0,1]$. Fix $0 \le x \le 1$, and let ε be an arbitrary positive number. We consider two cases.

(i) $x \in [0,1] \setminus \{P_n\}_{n=1}^{\infty}$,

(ii) $x = P_k$ for some $k = 1, 2, \dots$.

Suppose δ_k is the length of the longest subinterval of partition Π_k. Obviously, $\lim\limits_{k\to\infty} \delta_k = 0$. In case (i), choose k_0 large enough so that $\delta_{k_0} < \frac{\varepsilon}{2}$. Let $\Pi_{k_0,x}$

$[c,d]$ be the subinterval of Π_{k_0} containing x, and set $\delta = \frac{1}{2}\min\{|d-x|,|c-x|\}$. Now if $|y-x| < \delta$, then y and x are both contained in the interval (c,d). From [14] we know that $R_y^- \cap [0,c] = R_x^- \cap [0,c]$ and $R_y^+ \cap [d,1] = R_x^+ \cap [d,1]$. Thus the points of R_x and R_y which are not identical lie in the interval $[c,d]$, which has a length less than $\varepsilon/2$. In this case, then $d_H(R_x, R_y) < \frac{\varepsilon}{2}$.

In case (ii), let $x = P_{k_1}$. Since R_x^- and R_x^+ are monotone subsequences of $\{P_n\}$ converging to x, there exists a positive integer N_1 such that $|x_k^- - x| < \frac{\varepsilon}{2}$ and $|x_k^+ - x| < \frac{\varepsilon}{2}$ for all $k \geq N_1$. Let k_2 be a positive integer so that Π_{k_2} contains the points $x_1^-, x_1^+, \ldots, x_{N_1}^-, x_{N_1}^+, x$ as end points and $\delta_{k_2} < \frac{\varepsilon}{2}$. Suppose δ'_{k_2} is the length of the smallest subinterval of Π_{k_2}, and $\delta = \min\{\delta_{k_2}, \delta'_{k_2}\}$. If $x \leq y < y + \delta$, then y and x both lie in the same subinterval of Π_{k_2}, namely $\Pi_{k_2,i} = [x,d]$. Since $y \in [x,d)$, we have $R_x^+ \cap [d,1] = R_y^+ \cap [d,1]$, so that the points of R_x^+ and R_y^+ which are not identical lie in the interval $[x,d]$ of length less than $\varepsilon/2$, implying that $d_H(R_x^+, R_y^+) < \varepsilon/2$. Similarly for $x - \delta < y \leq x$, we have $d_H(R_x^-, R_y^-) < \varepsilon/2$. We conclude that the path systems R^+ and R^- are right and left continuous, respectively. □

First return systems of paths satisfy the internal intersection condition (see [14]). We now give an example of a bilateral continuous system of paths which does not satisfy any of the intersection conditions, so that it cannot be a first return system with respect to any trajectory in $[0,1]$.

EXAMPLE 3.1. There exists a bilateral continuous system of paths $R = \{R_x : x \in [0,1]\}$ that does not satisfy any of the intersection conditions.

Let $\{h_n\}_{n=1}^{\infty}$ be a sequence of positive rational numbers with $\lim_{n\to\infty} h_n = 0$. Let $A = \{0\} \cup \{h_n, -h_n\}_{n=1}^{\infty}$. For each $x \in [0,1]$, choose $R_x = (A + x) \cap [0,1]$. It is easy to see that the system of paths $R = \{R_x : x \in [0,1]\}$ is bilateral and continuous. We show that R does not satisfy any of the intersection conditions. Suppose, to the contrary, that R satisfies one of the intersection conditions, and δ is the function associated with R for that intersection condition. Let $\{F_j\}_{j=1}^{\infty}$ be a δ-decomposition of $[0,1]$. Since the unit interval is not countable there exists j, $1 \leq j < \infty$, and x_j, y_j both in F_j such that $x_j - y_j$ is irrational. We show that $R_{x_j} \cap R_{y_j} = \emptyset$. If there is a $z \in R_{x_j} \cap R_{y_j}$, then $z = x_j + h_j = y_j + k_j$ for some h_j and k_j elements of A. This implies that $x_j - y_j = k_j - h_j$, so that $x_j - y_j$ is a rational number. We conclude that R does not satisfy any of the intersection conditions.

Remark 3.1. A first return derivative is a function in DB_1 (see [14]). One may ask if the same is true for R-derivatives when R is a bilateral continuous system of paths. In this regard it is easy to see that for a continuous function, any R-derivative tailored with a bilateral system of paths has the Darboux property. From [1; Theorem 5], we know that for a continuous function, its path derivative is in B_1 when the system of paths is also continuous. Thus, the path derivative of any continuous function tailored with a bilateral and continuous system of paths is in DB_1. However, this is not true in general. In fact, one can easily construct a bilateral continuous system of paths R, and a function in Baire class two which is R-differentiable and the R-derivative is not an element of B_1. In [1] we provide an example of a function in B_2 and a continuous system of paths whose path derivative is not Borel measurable. We also show that the extreme path derivatives of a Borel measurable functions are measurable when the system of paths is continuous (see [1; Theorem 16]). As we saw in the proof of Theorem 2, the first return systems of paths are continuous at all points except the points of the trajectory. We have also shown that the right (left) first return systems are right (left) continuous. One might ask if a result similar to [1; Theorem 5, Theorem 16] is possible. In fact, by mimicking the proofs of these theorems, we could obtain some weaker results. In what follows, we use a different technique to show that the extreme first return derivatives of a function are better behaved than the corresponding extreme path derivatives whenever the path system is continuous.

THEOREM 3.3. *Let $\{P_n\}$ be any trajectory with R the first return path system generated by $\{P_n\}$.*

(i) *If $F \in B_\alpha$, then $\overline{F}'_R \in B_{\alpha+2}$, $\overline{F}'_{R+} \in B_{\alpha+2}$ and $\overline{F}'_{R-} \in B_{\alpha+2}$.*

(ii) *If F is a Borel measurable function, then $\overline{F}'_R$, $\overline{F}'_{R+}$ and $\overline{F}'_{R-}$ are also Borel measurable functions.*

(iii) *If F is a Lebesgue measurable function, then $\overline{F}'_R$, $\overline{F}'_{R+}$ and $\overline{F}'_{R-}$ are also Lebesgue measurable functions.*

P r o o f . Let R^+ and R^- be the right and left first return systems generated by $\{P_n\}$. For each natural number n, let $E^+_{P_n} = \{x : P_n \in R^+_x\}$. We first show that each $E^+_{P_n}$ is a closed set. To see this, let $\{x_m\}_{m\geq 1} \subseteq E^+_{P_n}$ with $\lim_{m\to\infty} x_m = x_0$. If for some $m \geq 1$, $x_m \leq x_0$, then $R^+_{x_0} \cap [P_n, 1] = R^+_{x_m} \cap [P_n, 1]$ which implies that $x_0 \in E^+_{P_n}$. Otherwise, for each $m \geq 1$, we have that $x_m \geq x_0$. From the right continuity of R^+ and the fact that for each $m \geq 1$, $P_n \in R^+_{x_m}$, it follows that $x_0 \in E^+_{P_n}$.

Suppose $r \in \mathcal{R}$ and $\overline{F}'_{R^+}(z) \geq r$. Then for every natural number m, $\overline{F}'_{R^+}(z) > r - 1/m$, so that there exists a sequence $\{y_{m,k}\}_{k\geq 1} \subseteq R^+_z$ such that $y_{m,k} \neq z$, $\lim_{k\to\infty} y_{m,k} = z$ and $\big(F(y_{m,k}) - F(z)\big)/(y_{m,k} - z) > r - 1/m$. It follows that $z \in E^+_{y_{m,k}} \cap \{x : \big(F(x) - F(y_{m,k})\big)/(x - y_{m,k}) > r - 1/m\}$ for $k = 1, 2, 3, \ldots$. On the other hand, if there is a sequence of real numbers $\{n_m\}_{m\geq 1}$ for which $z \in E^+_{P_{n_m}}$, and $z \neq P_{n_m}$ and $\big(F(P_{n_m}) - F(z)\big)/(P_{n_m} - z) >$

$r - 1/m$, then from the fact that R_z^+ is a sequence converging to z from the right and $\{P_{n_m}\}_{m\geq 1}$ is a subsequence of R_z^+, one sees that $\lim_{m\to\infty} P_{n_m} = z$ and $\liminf_{m\to\infty}\big(F(P_{n_m}) - F(z)\big)/(P_{n_m} - z) \geq r$, implying $\overline{F}'_{R^+}(z) \geq r$. Thus,

$$\{x:\ \overline{F}'_{R^+}(x) \geq r\} = \bigcap_{m\geq 1}\bigcup_{n\geq m}\{E_{P_n}^+ \cap A_{m,n}\}, \tag{1}$$

where $A_{m,n} = \{x : x \neq P_n \text{ and } \big(F(x) - F(P_n)\big)/(x - P_n) > r - 1/m\}$.

In a similar way we may show that

$$\{x:\ \overline{F}'_{R^-}(x) \geq r\} = \bigcap_{m\geq 1}\bigcup_{n\geq m}\{E_{P_n}^- \cap A_{m,n}\}, \tag{2}$$

where $E_{P_n}^- = \{x :\ P_n \in R_x^-\}$ and $A_{m,n} = \{x :\ x \neq P_n \text{ and } \big(F(x) - F(P_n)\big)/(x - P_n) > r - 1/m\}$.

Our conclusion now follows from equalities (1), (2), and the fact that $\overline{F}'_R(x) = \sup\{\overline{F}'_{R^+}(x), \overline{F}'_{R^-}(x)\}$. □

Remark 3.2. For any ordinal number $\alpha > 1$ and any trajectory $P = \{P_n\}_{n\geq 0}$, let $E \subset [0,1]\setminus P$ be a set which is not of additive class α. Suppose R is the first return path generated by $\{P_n\}$ and $F(x) = \chi_E$. It is easy to see that $\overline{F}'_R(x) = \begin{cases} 0 & x \notin E, \\ +\infty & x \in E, \end{cases}$ so that $\overline{F}'_R$ is not a function in Baire class α. Similarly, we can construct functions whose extreme first return derivatives are not Borel or Lebesgue measurable functions. In fact, this also illustrates that we cannot expect to have a result similar to Hajek's Theorem (see [10]) for extreme first return derivatives.

THEOREM 3.4. *Let $\{P_n\}$ be a trajectory and R be the path system generated by $\{P_n\}$. If $F\colon \mathcal{R} \to \mathcal{R}$ is continuous, then*

(i) $\sup\{\overline{F}'_{R^+}(x) :\ x \in [a,b)\} = \sup\left\{\frac{F(t)-F(s)}{t-s} :\ t,s \in \{P_n\}\cap[a,b),\ t \neq s\right\}$,

(ii) $\sup\{\overline{F}'_{R^-}(x) :\ x \in (a,b]\} = \sup\left\{\frac{F(t)-F(s)}{t-s} :\ t,s \in \{P_n\}\cap(a,b],\ t \neq s\right\}$,

(iii) $\sup\{\overline{F}'_{R}(x) :\ x \in (a,b)\} = \sup\left\{\frac{F(t)-F(s)}{t-s} :\ t,s \in \{P_n\}\cap(a,b),\ t \neq s\right\}$,

(iv) $\inf\{\overline{F}'_{R^+}(x) :\ x \in [a,b)\} = \inf\left\{\frac{F(t)-F(s)}{t-s} :\ t,s \in \{P_n\}\cap[a,b),\ t \neq s\right\}$,

(v) $\inf\{\overline{F}'_{R^-}(x) :\ x \in (a,b]\} = \inf\left\{\frac{F(t)-F(s)}{t-s} :\ t,s \in \{P_n\}\cap(a,b],\ t \neq s\right\}$,

(vi) $\inf\{\overline{F}'_{R}(x) :\ x \in (a,b)\} = \inf\left\{\frac{F(t)-F(s)}{t-s} :\ t,s \in \{P_n\}\cap(a,b),\ t \neq s\right\}$.

Results analogous to (i), (ii), (iii), (iv), (v) *and* (vi) *hold for* $\underline{F}'_{R^+}$, $\underline{F}'_{R^-}$ *and* $\underline{F}'_R(x)$.

Proof. Since the verifications of parts (i) through (vi) are very similar, we will prove only the first part. Our method of proof is much like that of

[4; Theorem 1.2]. Suppose that for some $x_0 \in [a,b)$, $\overline{F}'_{R^+}(x_0) > M$. Then there exists $y \in R^+_{x_0} \cap [x_0,b)$ such that $\frac{F(y)-F(x_0)}{y-x_0} > M$. Let $\{P'_m\}_{m=1}^{\infty} \subseteq R^+_{x_0} \cap \{P_n\} \cap [x_0,b)$ such that $\lim\limits_{m\to\infty} P'_m = x_0$. Since F is continuous, we have $\lim\limits_{m\to\infty} \frac{F(y)-F(P'_m)}{y-P'_m} = \frac{F(y)-F(x_0)}{y-x_0} > M$, thus we can choose $t, s \in \{P_n\} \cap [a,b)$ such that $t \neq s$ and $\frac{F(t)-F(s)}{t-s} > M$. It follows that the right hand side of (i) is bigger than or equal to the left side of (i). Now, suppose $\frac{F(t)-F(s)}{t-s} = M$ for some $t > s$, both of which are contained in $\{P_n\} \cap [a,b)$, and define G by $G(x) = F(x) - Mx$ for all $x \in [a,b]$. Then $G(t) = G(s)$. The function G is continuous on $[s,t]$, and therefore it attains a minimum m on $[s,t]$ at some point $x_0 \in [s,t]$. If $G(t) = m$, then also $G(s) = m$. Thus, we may suppose $x_0 \in [s,t)$, and conclude that $\underline{F}'^{+}_{R}(x_0) \geq M$. This gives left hand side of (i) bigger than or equal to the right hand side of (i), so that the equality in (i) follows. □

THEOREM 3.5. *Let $F\colon \mathcal{R} \to \mathcal{R}$ be a continuous function. If $\overline{F}'_{R^+}$ is finite and continuous at x_0, then it is differentiable there.*

P r o o f . Let $\varepsilon > 0$ be given, and take δ to be the corresponding positive number due to continuity of $\overline{F}'_{R^+}$. Suppose $I_1 = (x_0 - \delta, x_0 + \delta)$. Using the above theorem, we have

$$\sup\Big\{ \tfrac{F(t)-F(s)}{t-s} \ :\ s,t \in \{P_n\} \cap I_1\,,\ \ t \neq s \Big\} - \inf\Big\{ \tfrac{F(t)-F(s)}{t-s} \ :\ s,t \in \{P_n\} \cap I_1\,,\ \ t \neq s \Big\} < 2\varepsilon\,.$$

Choose $\{s'_k\}_{k=1}^{\infty} \subseteq \{P_n\} \cap R^+_{x_0} \cap I_1$ such that $\lim\limits_{k\to\infty} s'_k = x_0$. For every $y \in I_1$, $y \neq x_0$, choose $\{s_k\}_{k=1}^{\infty} \subseteq \{P_n\} \cap R^+_y \cap I_1$ such that $\lim\limits_{k\to\infty} s_k = y$ and $s_k \neq s'_k$ for large k. Since F is continuous, $\lim\limits_{k\to\infty} \frac{F(s_k)-F(s'_k)}{s_k-s'_k} = \frac{F(y)-F(x_0)}{y-x_0}$. Thus, for every $y \in I_1$, $y \neq x_0$, we have

$$\begin{aligned} &\liminf\Big\{ \tfrac{F(s)-F(t)}{s-t} \ :\ s,t \in \{P_n\} \cap I_1\,,\ \ s \neq t \Big\} \\ &\leq \frac{F(y) - F(x_0)}{y - x_0} \\ &\leq \limsup\Big\{ \tfrac{F(s)-F(t)}{s-t} \ :\ s,t \in \{P_n\} \cap I_1\,,\ \ s \neq t \Big\}, \end{aligned}$$

implying the differentiability of F at x_0. □

Remark 3.3. At the points x_0 where $\overline{F}'_{R^+}(x_0)$ is infinite and continuous, we may use continuity and infinite limit definition, that is $\lim\limits_{x\to x_0} \overline{F}'_{R^+}(x) =$

$\overline{F}'_{R+}(x_0) = \infty$. Thus for every $M > 0$ there exists $\delta > 0$ so that $\overline{F}'_{R+}(x) > M$ for $x \in (x_0 - \delta, x_0 + \delta)$. Using our Theorem 3.4 and [4; p. 63, Theorem 1.2] we have:

$$\begin{aligned}
&\inf\{\overline{F}'_{R+}(x) : \ x \in (x_0 - \delta, x_0 + \delta)\} \\
&= \inf\Big\{\tfrac{F(t)-F(s)}{t-s} : \ t, s \in \{P_n\} \cap (x_0 - \delta, x_0 + \delta), \ t \neq s\Big\} \\
&= \inf\{D_+F(x) : \ x \in (x_0 - \delta, x_0 + \delta)\} \\
&= \inf\{D_F(x) : \ x \in (x_0 - \delta, x_0 + \delta)\} \geq M .
\end{aligned}$$

This implies that F has an infinite derivative at x_0, so Theorem 3.5 is also true at points where $\overline{F}'_{R+}$ is continuous and has infinite value.

Remark 3.4. The conclusion that the extreme first return derivatives (extreme right first return derivatives, extreme left first return derivatives) of a continuous function belong to B_2 is the most we can say. Suppose C is the Banach space of continuous real-valued functions on $[0, 1]$, and let $\mathcal{H}$ be a residual set of functions in C which do not have a finite or infinite derivative at any point (see [5; Corollary 2.3]). Then for each $F \in \mathcal{H}$ and each first return system of paths R generated by a given trajectory $\{P_n\}$ on $[0, 1]$, we have $\overline{F}'_{R+} \in B_2 \setminus B_1$. To see this, let $\overline{F}'_{R+} \in B_1$. Then it is continuous on a residual set Y. If $\overline{F}'_{R+}$ is finite at some point of the residual set Y, then the above theorem implies that F should be differentiable at that point which is a contradiction. If $\overline{F}'_{R+}(x) = \infty$ for all $x \in Y$, then by Remark 3.3, F should have infinite derivative at each point of Y, again a contradiction.

THEOREM 3.6. *Let F and G be two real valued continuous functions defined on $[0, 1]$, and R be the first return path system generated by a given trajectory $\{P_n\}$. If $F'_R(x) = G'_R(x)$ for every $x \in [a, b]$, then F and G differ by a constant on $[a, b]$.*

P r o o f. This follows immediately from Theorem 3.4. □

THEOREM 3.7. *Let F be a real valued continuous function defined on $[0, 1]$ and R be the first return path system generated by a given trajectory $\{P_n\}$. If one of the extreme derivatives $\overline{F}'_{R+}$, $\overline{F}'_{R-}$, $\underline{F}'_{R+}$, $\underline{F}'_{R-}$ is nonnegative on $[a, b]$, then F must be nondecreasing on $[a, b]$.*

P r o o f. We will prove our result for the case of $\overline{F}'_{R+}$. The other cases follow in a similar manner. Let $\overline{F}'_{R+}(x) \geq 0$ on $[a, b]$. Then, by Theorem 3.4, for every $s, t \in \{P_n\} \cap [a, b]$, with $s \neq t$, we have that $\frac{F(s)-F(t)}{s-t} \geq 0$. Now suppose x and y are two points in $[a, b]$ with $x < y$, and $\{s_k\}_{k=1}^{\infty}$, $\{t_k\}_{k=1}^{\infty}$ are two subsequences of $\{P_n\}$ such that for each k, $x < s_k < t_k < y$ and $\lim\limits_{k\to\infty} s_k = x$, $\lim\limits_{k\to\infty} t_k = y$.

Since $\frac{F(s_k)-F(t_k)}{s_k-t_k} \geq 0$, we have $F(t_k) \geq F(s_k)$, which implies $F(y) \geq F(x)$. □

Even though the return path systems are the thinnest possible in a bilateral sense, the extreme return path derivatives of continuous functions have some similarities with their corresponding Dini derivatives. We now show via an example that these need not be identical. In fact, similar to the following example, one can construct a continuous function with different corresponding extreme first return and Dini derivatives at a countably infinite number of points.

EXAMPLE 3.2. There exists a continuous function F defined on $[0,1]$ and a trajectory Q in $[0,1]$ such that D^+F and $\overline{F}'_{R+}$ are two different functions.

Proof. Let $Q = \{0,1,1/2,1/4,3/4,1/8,3/8,5/8,7/8,1/16,\ldots\}$, so that Q is a trajectory in $[0,1]$. Let E be the first return path system generated by Q. For every $x \in Q \cap (0,1)$, suppose m_1 and m_2 are the least positive integers such that $1/2^{m_1} < x$ and $1/2^{m_2} < 1-x$, respectively. Let $R_x^- = \{x\} \cup \{x - 1/2^n\}_{n \geq m_1}$, $R_x^+ = \{x\} \cup \{x + 1/2^n\}_{n \geq m_2}$ and $R_x = R_x^+ \cup R_x^-$. It is easy to see that for each $x \in Q \cap (0,1)$, E_x is eventually the same as R_x, $E_0 = \{0\} \cup \{1/2^n\}_{n \geq 0}$, and $E_1 = \{1\} \cup \{(2^n-1)/2^n\}_{n \geq 0}$. Let F be a continuous function defined on $[0,1]$ so that $F(x) = \begin{cases} 0 & x \in \{0\} \cup \{1/2^n\}_{n\geq 0}, \\ x & x \in \{3/2^{(n+2)}\}_{n \geq 0} \end{cases}$ and F is linear between these points. Then $\overline{F}'_{E+}(0) = \overline{F}'_{R+}(0) = 0$, while $D^+F(0) \geq 1$. □

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