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HILBERT-SYMBOL EQUIVALENCE OF GLOBAL FUNCTION FIELDS

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ABSTRACT. Hilbert-symbol equivalence of degree ℓ between two global fields containing a primitive ℓth root of unity is an isomorphism between the groups of ℓth power classes of these fields preserving Hilbert symbols of degree ℓ. The Hilbert-symbol equivalence of degree ℓ is said to be tame if it preserves the $\mathfrak{p}$-orders modulo ℓ. In the paper we prove that if ℓ is an odd prime number, then any two global function fields are Hilbert equivalent. We find also necessary and sufficient conditions for tame Hilbert-symbol equivalence of global function fields for all prime numbers $\ell \geq 2$.

1. Introduction

Let ℓ be a prime number and let K and L be global fields of characteristic prime to ℓ containing primitive ℓth roots of unity. *Degree ℓ Hilbert-symbol equivalence* (or ℓ-Hilbert-symbol equivalence) between K and L is defined to be a triple of maps

$$f\colon \mu_\ell(K) \to \mu_\ell(L), \qquad t\colon \dot{K}/\dot{K}^\ell \to \dot{L}/\dot{L}^\ell, \qquad T\colon \Omega(K) \to \Omega(L),$$

where f is an isomorphism between the groups of ℓth roots of unity, t is an isomorphism between the groups of ℓth power classes of the two fields and T is a bijective map between the sets of all primes of K and L, with (f,t,T) preserving Hilbert symbols of ℓth degree in the sense that

$$(a,b)_{\mathfrak{p}}^f = (ta,tb)_{T\mathfrak{p}} \qquad \text{for all} \quad a,b \in \dot{K}/\dot{K}^\ell, \ \ \mathfrak{p} \in \Omega(K).$$

We say that K and L are *degree ℓ Hilbert-symbol equivalent* when there exists a degree ℓ Hilbert-symbol equivalence between K and L.

The 2-Hilbert-symbol equivalence was introduced in [PSCL] in order to classify the global fields with respect to isomorphism of Witt rings. For $\ell > 2$ the

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ℓ-Hilbert-symbol equivalence was first discussed in [CS1]. In [S] it was shown that:

K and L are degree ℓ Hilbert-symbol equivalent if and only if there is an isomorphism of graded rings $\mathsf{K}(K)/\ell\mathsf{K}(K) \cong \mathsf{K}(L)/\ell\mathsf{K}(L)$, sending $\{-1\}_\ell$ onto $\{-1\}_\ell$ (here $\mathsf{K}(F)$ stands for the Milnor ring of the field F).

Hilbert-symbol equivalences come in two types: tame and wild. The equivalence (f,t,T) of degree ℓ is said to be *tame at non-archimedean place* $\mathfrak{p}$ of K, when

$$\operatorname{ord}_{\mathfrak{p}}(a) \equiv \operatorname{ord}_{T\mathfrak{p}}(ta) \pmod{\ell} \qquad \text{for all} \quad a \in \dot{K}/\dot{K}^{\ell}. \tag{1}$$

Otherwise the equivalence is *wild* at $\mathfrak{p}$. The equivalence (f,t,T) is said to be *tame* when it is tame at all finite non-archimedean places $\mathfrak{p}$ of K.

So far ℓ-Hilbert-symbol equivalence has been investigated in details for all global fields only for $\ell = 2$ (see [PSCL] and [C]). On the other hand for $\ell > 2$ the ℓ-Hilbert-symbol equivalence, and the tame Hilbert-symbol equivalence has been studied only for the algebraic number fields (see [CS2], [CZ1], [CZ2]).

This paper completes the picture by finding necessary and sufficient conditions for degree ℓ Hilbert-symbol equivalence of global function fields in case $\ell > 2$ (in Section 2), and for tame ℓ-Hilbert-symbol equivalence of global function fields for all $\ell \geq 2$ (in Section 3).

2. Hilbert-symbol equivalence

The 2-Hilbert-symbol equivalence can be described in terms of field invariants. Carpenter in [C] has shown:

2.1. *The algebraic number fields K and L are degree 2 Hilbert-symbol equivalent if and only if they have the same level, the same number of real primes and there exists a bijection from dyadic primes of K to those of L which preserves the local degrees and local levels.*

2.2. *The global function fields K and L of odd characteristic are degree 2 Hilbert-symbol equivalent if and only if they have the same level.*

The counterpart of Carpenter's theorem 2.1 for degree $\ell > 2$ was proved in [CS1]:

2.3. *The algebraic number fields K and L containing primitive ℓth roots of unity are degree ℓ Hilbert-symbol equivalent if and only if there exists a bijection from ℓ-adic primes of K to those of L which preserves the local degrees.*

Here we will show the following result which is the counterpart of Carpenter's theorem 2.2 for degree $\ell > 2$:

THEOREM 2.4. *Let $\ell > 2$ be a prime number. Then any two global function fields K and L containing primitive ℓth roots of unity are degree ℓ Hilbert-symbol equivalent.*

We will obtain the proof of Theorem 2.4 by adapting the approaches presented in [CS1] to the global function fields. First we recall the notions and facts necessary in the proof.

Suppose ℓ is a prime number and K is a global function field containing a primitive ℓth root of unity. In case $\ell = 2$ we assume additionally that the characteristic of K is different from 2. Those assumptions guarantee that the characteristic of K is prime to ℓ.

Let ζ_K be a fixed primitive ℓth root of unity in K. For any prime $\mathfrak{p} \in \Omega(K)$ the group $\dot{K}_\mathfrak{p}/\dot{K}_\mathfrak{p}^\ell$ of the ℓth power classes of the local field $K_\mathfrak{p}$ can be viewed as 2-dimensional inner product space over the ℓ-element field $\mathbf{F}_\ell$, with the bilinear form $\beta_\mathfrak{p}$ defined by the $\mathfrak{p}$-adic Hilbert symbol of degree ℓ in the following way

$$(x, y)_\mathfrak{p} = \zeta_K^{\beta_\mathfrak{p}(x,y)} .$$

If u is arbitrary $\mathfrak{p}$-adic unit which is not a local ℓth power and $\pi_\mathfrak{p}$ is a local uniformizer at $\mathfrak{p}$, then the set $\{u, \pi_\mathfrak{p}\}$ forms a basis of this space. The form $\beta_\mathfrak{p}$ is symmetric when $\ell = 2$ and antisymmetric when $\ell > 2$.

The characteristic of the residue class field of $K_\mathfrak{p}$ is prime to ℓ, thus the Hilbert symbol $(,)_\mathfrak{p}$ of degree ℓ is tame. From the explicit formula for the value of tame Hilbert symbol (c.f. [CF; Example 2]) one can deduce that there exists a local $\mathfrak{p}$-adic unit $u \in U_\mathfrak{p}$ with the property

$$\beta_\mathfrak{p}(u, x) = \operatorname{ord}_\mathfrak{p} x \bmod \ell \qquad \text{for every} \quad x \in \dot{K}_\mathfrak{p} .$$

Assume S is a finite nonempty subset of $\Omega(K)$. We call such a set S *sufficiently large* if ℓ does not divide the class number $h^S(K)$ of the ring $\mathcal{O}_K(S)$ of S-integers of K. We consider the inner product space $\big(G(S), \beta_S\big)$ over $\mathbf{F}_\ell$ which is the orthogonal product of $(\dot{K}_\mathfrak{p}/\dot{K}_\mathfrak{p}^\ell, \beta_\mathfrak{p})$, $\mathfrak{p} \in S$, that is,

$$G(S) = \prod_{\mathfrak{p}\in S} \dot{K}_\mathfrak{p}/\dot{K}_\mathfrak{p}^\ell \qquad \text{and} \qquad \beta_S\big((x_\mathfrak{p})_{\mathfrak{p}\in S}, (y_\mathfrak{p})_{\mathfrak{p}\in S}\big) = \sum_{\mathfrak{p}\in S} \beta_\mathfrak{p}(x_\mathfrak{p}, y_\mathfrak{p}) .$$

The dimension (over $\mathbf{F}_\ell$) of the space $G(S)$ is equal to $2\#S$.

We write $U_K(S)$ for the group of S-units of K. According to Dirichlet's unit theorem, we have $\operatorname{rk}_\ell U_K(S)/U_K(S)^\ell = \#S$. We have a homomorphism $i\colon U_K(S)/U_K(S)^\ell \to G(S)$, which is the composite map

$$U_K(S)/U_K(S)^\ell \longrightarrow \dot{K}/\dot{K}^\ell \xrightarrow{\text{diag}} G(S) .$$

Using the same arguments as in the proofs of [CS2; Lemma 3.1, Lemma 3.2], and [CZ2; Lemma 3.6] we obtain the following facts:

2.5. *If S is a sufficiently large set of primes of K, then*

(i) *An ℓth power class $a\dot{K}^\ell$ lies in $U_K(S)/U_K(S)^\ell$ if and only if* $\mathrm{ord}_\mathfrak{p}(a) \equiv 0 \pmod{\ell}$ *for every $\mathfrak{p}$ outside S.*

(ii) *The map $i\colon U_K(S)/U_K(S)^\ell \to G(S)$ is injective.*

(iii) *The image of the group $U_K(S)/U_K(S)^\ell$ under the monomorphism i is a self-orthogonal subspace of $G(S)$*
(i.e. $i(U_K(S)/U_K(S)^\ell) = i(U_K(S)/U_K(S)^\ell)^\perp$).

LEMMA 2.6. *If S is a finite non-empty set of primes of K and $(\alpha_\mathfrak{p})_{\mathfrak{p}\in S} \in G(S)$, then there exists $\mathfrak{q} \in \Omega(K) \setminus S$ and $q \in \dot{K}$ such that*

(i) $q\dot{K}_\mathfrak{p}^\ell = \alpha_\mathfrak{p}\dot{K}_\mathfrak{p}^\ell$ *for all $\mathfrak{p} \in S$,*

(ii) $\mathrm{ord}_\mathfrak{q}\, q = 1$,

(iii) $\mathrm{ord}_\mathfrak{p}\, q \equiv 0 \pmod{\ell}$ *for all $\mathfrak{p} \in \Omega(K) \setminus (S \cup \{\mathfrak{q}\})$.*

P r o o f. This is immediate from [LW; Lemma 2.1]. □

It has become a standard that the Hilbert symbol equivalence is achieved by using the concept of an S-equivalence. Let S be a sufficiently large set of primes of K. By *S-equivalence* between K and L we mean a quadruple $(f, T, t_S, (t_\mathfrak{p})_{\mathfrak{p}\in S})$ where $f\colon \mu_\ell(K) \to \mu_\ell(L)$ is a group isomorphism, T is a bijection of S onto a sufficiently large set $S' = TS$ of primes of L, $t_S\colon U_K(S)/U_K(S)^\ell \to U_L(S')/U_L(S')^\ell$ is a group isomorphism, $(t_\mathfrak{p})_{\mathfrak{p}\in S}$ is a family of isomorphisms $t_\mathfrak{p}\colon \dot{K}_\mathfrak{p}/\dot{K}_\mathfrak{p}^\ell \to \dot{L}_{T\mathfrak{p}}/\dot{L}_{T\mathfrak{p}}^\ell$ preserving Hilbert symbols in the sense that $(x,y)_\mathfrak{p}^f \doteq (t_\mathfrak{p}x, t_\mathfrak{p}y)_{T\mathfrak{p}}$ for all $x, y \in \dot{K}_\mathfrak{p}$ and the following diagram commutes

$$\begin{array}{ccccc} 1 & \longrightarrow & U_K(S)/U_K(S)^\ell & \xrightarrow{\ i\ } & G(S) \\ \downarrow & & \downarrow{\scriptstyle t_S} & & \downarrow{\scriptstyle \prod t_\mathfrak{p}} \\ 1 & \longrightarrow & U_L(S')/U_L(S')^\ell & \xrightarrow{\ i\ } & G(S') \end{array}$$

An S-equivalence is said to be *tame* if each isomorphism $t_\mathfrak{p}$ is tame.

THEOREM 2.7. *A S-equivalence of degree ℓ can be extended to a degree ℓ Hilbert-symbol equivalence which is tame outside S.*

P r o o f. Arguments are the same as those used in the proof of [CS2; Theorem 3.4]. □

Now we turn to:

P r o o f o f T h e o r e m 2.4. Assume $\ell > 2$ is a prime number. Let K, L be number fields and let ζ_K, ζ_L be fixed primitive ℓth roots of unity in K and L, respectively. From [OM; 33:13a] it follows that there exist sufficiently large sets of primes S, S' of the fields K and L, respectively. Adding, if necessary, some primes to one of these sets, we can assume they consist of the same number of elements. Let $T\colon S \to S'$ be a bijection between the sets S and S'. For every $\mathfrak{p} \in S$ the inner product spaces $(\dot{K}_\mathfrak{p}/\dot{K}_\mathfrak{p}^\ell, \beta_\mathfrak{p})$ and $(\dot{L}_{T\mathfrak{p}}/\dot{L}_{T\mathfrak{p}}^\ell, \beta_{T\mathfrak{p}})$ are isometric. Using the same arguments as in the proof of [CS2; Theorem 4.1] we conclude that there exists a small equivalence of degree ℓ between K and L. Now the statement follows immediately from Theorem 2.7. □

3. Tame Hilbert-symbol equivalence

Again let ℓ be a prime number, K be a global function field of characteristic prime to ℓ and let $\zeta_K \in K$ be a fixed primitive ℓth root of unity. By E_K we shall denote the constant field of K. Let us observe that the level $s(K)$ of K is equal to 1, when -1 is a square in E_K and is equal to 2, otherwise. We write C_K for the zero-degree divisor class group of K, that is C_K is the factor group $\mathcal{D}_0/\mathcal{P}$, where $\mathcal{D}_0$ is the group of divisors of degree 0, and $\mathcal{P}$ is the group of principal divisors.

We define the group of ℓ-singular (or briefly singular) elements of K,

$$K_{\mathrm{si}} = \{x \in \dot{K} : \operatorname{ord}_\mathfrak{p} x \equiv 0 \pmod{\ell} \text{ for all primes } \mathfrak{p} \text{ of } K\}.$$

It is obvious that K_{si} is a subgroup of $\dot{K}$ and it contains the group $\dot{K}^\ell$.

Lemma 3.1. *We have*

$$\operatorname{rk}_\ell K_{\mathrm{si}}/\dot{K}^\ell = 1 + \operatorname{rk}_\ell C_K .$$

P r o o f. Let ${}_\ell C_K$ be the subgroup of C_K consisting of elements of order $\le \ell$. The map

$$K_{\mathrm{si}} \to {}_\ell C_K , \qquad x \mapsto \operatorname{cl} \prod_\mathfrak{p} \mathfrak{p}^{(\operatorname{ord}_\mathfrak{p} x)/\ell}$$

is a surjective homomorphism with the kernel $\dot{E}_K\dot{K}^\ell$. Thus $\operatorname{rk}_\ell K_{\mathrm{si}}/E_K\dot{K}^\ell = \operatorname{rk}_\ell C_K$. The groups $\dot{E}_K/\dot{K}^\ell$ and $\dot{E}_K/\dot{E}_K^\ell$ are isomorphic and the ℓ-rank of $\dot{E}_K/\dot{E}_K^\ell$ is equal to 1. This proves the lemma. □

Assume S is a finite nonempty set of primes of K. We consider the group of ℓ-singular elements with respect to S.

$$K_{\mathrm{si}}(S) = \{x \in \dot{K} : \operatorname{ord}_\mathfrak{p} \equiv 0 \pmod{\ell} \text{ for all } \mathfrak{p} \in \Omega(K) \setminus S\}.$$

Similarly as in the proof of Lemma 3.1 we see that the map $x \mapsto \operatorname{cl} \prod_{\mathfrak{p}\notin S} \mathfrak{p}^{(\operatorname{ord}_\mathfrak{p} x)/\ell}$ is a surjective homomorphism of $K_{\mathrm{si}}(S)$ onto ${}_\ell C_K(S)$ with the kernel $U_K(S)\dot{K}^\ell$. Since $U_K(S)\dot{K}^\ell/\dot{K}^\ell \cong U_K(S)/U_K(S)^\ell$ and $\operatorname{rk}_\ell U_K(S)/U_K(S)^\ell = \#S$, we have

$$\operatorname{rk}_\ell K_{\mathrm{si}}(S)/\dot{K}^\ell = \#S + \operatorname{rk}_\ell C_K(S). \tag{2}$$

Lemma 3.2. *If $\mathfrak{p}_0$ is a prime of K of degree prime to ℓ and $S_0 = \{\mathfrak{p}_0\}$, then*

(i) $K_{\mathrm{si}}(S_0) = K_{\mathrm{si}}$,
(ii) $\mathrm{rk}_\ell C_K(S_0) = \mathrm{rk}_\ell C_K$.

Proof.

(i) Let x be an arbitrary element of $K_{\mathrm{si}}(S_0)$. The degree of principal divisor (x) is equal to 0. Thus we have

$$\mathfrak{f}_{\mathfrak{p}_0} \operatorname{ord}_{\mathfrak{p}_0} x + \sum_{\mathfrak{p} \neq \mathfrak{p}_0} \mathfrak{f}_{\mathfrak{p}} \operatorname{ord}_{\mathfrak{p}} x = 0,$$

where $\mathfrak{f}_{\mathfrak{p}}$ denotes the degree of the prime $\mathfrak{p}$. Since ℓ divides $\operatorname{ord}_{\mathfrak{p}} x$ for every prime $\mathfrak{p} \neq \mathfrak{p}_0$ and ℓ does not divide $\mathfrak{f}_{\mathfrak{p}_0}$, hence ℓ divides $\operatorname{ord}_{\mathfrak{p}_0} x$, so $x \in K_{\mathrm{si}}$.

(ii) It follows immediately from (i), Lemma 3.1 and (2). □

Lemma 3.3. *Assume that the elements $a_1, \dots, a_n \in \dot{K}$ are ℓ-independent and $\varepsilon_1, \dots, \varepsilon_n \in \{0, \dots, \ell - 1\}$. Then there are infinitely many primes $\mathfrak{p}$ of K for which*

$$\left(\frac{a_i}{\mathfrak{p}}\right)_\ell = \zeta_K^{\varepsilon_i}$$

holds for $i = 1, \dots, n$.

Proof. Let $L_i = K(\sqrt[\ell]{a_i})$ for $i = 1, \dots, n$. The extension L_i/K is normal with the cyclic Galois group $G(L_i/K)$. Let σ_i be its generator acting on $\sqrt[\ell]{a_i}$ by $(\sqrt[\ell]{a_i})^{\sigma_i} = \zeta_K \sqrt[\ell]{a_i}$. Consider the field $L = L_1 \cdots L_n = K(\sqrt[\ell]{a_1}, \dots, \sqrt[\ell]{a_n})$.

From the Kummer theory it follows that L/K is a Galois extension of degree ℓ^n with the Abelian Galois group $G(L/K) = \prod_{i=1}^{n} G(L_i/K)$. Let $\sigma = (\sigma_1^{\varepsilon_1}, \dots, \sigma_n^{\varepsilon_n}) \in G(L/K)$. According to Chebotarev density theorem (see [W; Chap. XII, Theorem 12]) there exist infinitely many primes $\mathfrak{p}$ of K for which the Frobenius automorphism $F_{L/K}(\mathfrak{p})$ is equal to σ. It follows that $F_{L/K}(\mathfrak{p}) = (F_{L_1/K}(\mathfrak{p}), \dots, F_{L_i/K}(\mathfrak{p}))$, so $F_{L_i/K}(\mathfrak{p}) = \sigma_i^{\varepsilon_i}$ for $i = 1, \dots, n$.

On the other hand, we have $(\sqrt[\ell]{a_i})^{F_{L_i/K}(\mathfrak{p})} = \left(\frac{a_i}{\mathfrak{p}}\right)_\ell \sqrt[\ell]{a_i}$ (see [CF; Example 1]), hence $\left(\frac{a_i}{\mathfrak{p}}\right)_\ell = \zeta_K^{\varepsilon_i}$. □

Now we prove the second main theorem of the paper.

Theorem 3.4.

(i) *The global function fields K and L of odd characteristic are degree 2 tamely Hilbert symbol equivalent if and only if they have the same level and the zero-degree divisor class groups of K and L have the same 2-rank.*

(ii) *Let ℓ be an odd prime number and K, L be global function fields containing a primitive ℓth root of unity. The fields K and L are degree ℓ tamely*

Hilbert symbol equivalent if and only if the zero-degree divisor class groups of K and L have the same ℓ-rank.

Proof. We will prove simultaneously (i) and (ii).

Suppose (f, t, T) is the degree ℓ tame Hilbert-symbol equivalence of K and L. The isomorphism t induces a group isomorphism $t\colon K_{\mathrm{si}}/\dot{K}^\ell \to L_{\mathrm{si}}/\dot{L}^\ell$. By the Lemma 3.1 we have $\mathrm{rk}_\ell C_K = \mathrm{rk}_\ell C_L$. When $\ell = 2$, $t(-1) = -1$ holds (see [PSCL]), thus we get additionally $s(K) = s(L)$.

Now we prove the sufficiency part of (i) and (ii).

Let us fix the ℓth roots of unity $\zeta_K \in K$ and $\zeta_L \in L$. From the assumptions we have $\mathrm{rk}_\ell K_{\mathrm{si}}/\dot{K}^\ell = \mathrm{rk}_\ell L_{\mathrm{si}}/\dot{L}^\ell = 1 + n$, where $n = \mathrm{rk}_\ell C_K = \mathrm{rk}_\ell C_L$. There exist elements $a_0 \in E_K$, $b_0 \in E_L$ which are not global ℓth powers. When $\ell = 2$ and -1 is not a square in both K and L, we choose $a_0 = b_0 = -1$.

From [AT; Chap. 5, Theorem 5] it follows that the least positive divisor degree of K is equal to 1. Hence there exists a prime divisor $\mathfrak{p}_0$ of K of degree $\mathfrak{f}_{\mathfrak{p}_0}$ prime to ℓ. The corresponding completion $K_{\mathfrak{p}_0}$ is a field of power series with coefficients in a finite field $\widetilde{E}_K$, where $\widetilde{E}_K$ is finite extension of the field E_K of degree $\mathfrak{f}_{\mathfrak{p}_0}$. Since ℓ does not divide $[\widetilde{E}_K : E_K]$, the element a_0 is not an ℓth power in the field $\widetilde{E}_K$. This implies that $a_0 \notin \dot{K}_{\mathfrak{p}_0}/\dot{K}^\ell_{\mathfrak{p}_0}$, and so $\left(\frac{a_0}{\mathfrak{p}_0}\right)_\ell \neq 1$. Replacing, if necessary, the element a_0 with its power we can assume that $\left(\frac{a_0}{\mathfrak{p}_0}\right)_\ell = \zeta_K$. In the same way, there exists a prime divisor $\mathfrak{q}_0$ of L of degree $\mathfrak{f}_{\mathfrak{q}_0}$ prime to ℓ and we can assume that $\left(\frac{b_0}{\mathfrak{q}_0}\right)_\ell = \zeta_L$.

Let $\{a_0, a_1, \dots, a_n\}$ be a basis for $K_{\mathrm{si}}/\dot{K}^\ell$ and $\{b_0, b_1, \dots, b_n\}$ be a basis for $L_{\mathrm{si}}/\dot{L}^\ell$. Using Lemma 3.3 we pick up primes $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ of K and primes $\mathfrak{q}_1, \dots, \mathfrak{q}_n$ of L, such that

$$\left(\frac{a_i}{\mathfrak{p}_i}\right)_\ell = \zeta_K, \qquad \left(\frac{b_i}{\mathfrak{q}_i}\right)_\ell = \zeta_L, \qquad \left(\frac{a_j}{\mathfrak{p}_i}\right)_\ell = \left(\frac{b_j}{\mathfrak{q}_i}\right)_\ell = 1$$

for each $i \in \{1, \dots, n\}$, $j \in \{0, 1, \dots, n\}$, $i \neq j$.

Multiplying, if necessary, the elements a_i $(i = 1, \dots, n)$ by powers of a_0 we can assume that $\left(\frac{a_i}{\mathfrak{p}_0}\right)_\ell = 1$ for $i = 1, \dots, n$. Similarly, we can assume that $\left(\frac{b_i}{\mathfrak{q}_0}\right)_\ell = 1$ for $i = 1, \dots, n$. Let $S_0 = \{\mathfrak{p}_0\}$ and $S_0' = \{\mathfrak{q}_0\}$.

Claim. *The set of classes of primes $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ is linearly independent (over $\mathbf{F}_\ell$) in $C_K(S_0)/C_K(S_0)^\ell$.*

For otherwise there exists $x \in K$ and a fractional S_0-ideal $\mathfrak{a}$ such that

$$x\mathcal{O}_K(S_0) = \mathfrak{p}_1^{e_1} \dots \mathfrak{p}_n^{e_n} \mathfrak{a}^\ell$$

for some $e_1, \dots, e_n \in \{0, \dots, \ell - 1\}$ and $e_i > 0$ for certain i. The element a_i is $\mathfrak{r}$-adic unit (modulo local ℓth power) for each prime $\mathfrak{r}$ of K and the element

x is $\mathfrak{r}$-adic unit (modulo local ℓth power) for every prime $\mathfrak{r} \notin \{\mathfrak{p}_0, \dots, \mathfrak{p}_n\}$. Taking into account the connection between the residue norm symbol and Hilbert symbol (c.f. [CF; Example 2]) we have

$$(a_i, x)_{\mathfrak{p}_i} = \left(\frac{a_i}{\mathfrak{p}_i}\right)_\ell^{\mathrm{ord}_{\mathfrak{p}_i}(x)} = \zeta_K^{e_i} \neq 1, \qquad (a_i, x)_{\mathfrak{p}_j} = \left(\frac{a_i}{\mathfrak{p}_j}\right)_\ell^{\mathrm{ord}_{\mathfrak{p}_j}(x)} = 1$$

for all $j \in \{1, \dots, n\}$, $j \neq i$, and $(a_i, x)_{\mathfrak{r}} = 1$ for every prime $\mathfrak{r} \notin \{\mathfrak{p}_0, \dots, \mathfrak{p}_n\}$. This contradicts Hilbert's reciprocity and establishes the claim.

Analogously we claim that the set of classes of primes $\mathfrak{q}_1, \dots, \mathfrak{q}_n$ is linearly independent in $C_L(S_0')/C_L(S_0')^\ell$.

We put $S = \{\mathfrak{p}_0, \mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ and $S' = \{\mathfrak{q}_0, \mathfrak{q}_1, \dots, \mathfrak{q}_n\}$.

The claims imply that the ℓ-rank of the groups $C_K(S)$ and $C_L(S')$ are equal to 0, thus the sets S and S' are sufficiently large.

From 2.5(i) we infer that the group $K_{\mathrm{si}}/\dot{K}^\ell$ is a subgroup of $U_K(S)/U_K(S)^\ell$. Because these groups have the same ℓ-rank we get $K_{\mathrm{si}}/\dot{K}^\ell = U_K(S)/U_K(S)^\ell$ and similarly $L_{\mathrm{si}}/\dot{L}^\ell = U_L(S')/U_L(S')^\ell$.

Now we construct an S-equivalence of K and L.

We define an isomorphism $f\colon \mu_\ell(K) \to \mu_\ell(L)$, a bijection $T\colon S \to S'$ and an isomorphism $t_S\colon K_{\mathrm{si}}/\dot{K}^\ell \to L_{\mathrm{si}}/\dot{L}^\ell$ by putting $f(\zeta_K) = \zeta_L$, $T(\mathfrak{p}_i) = \mathfrak{q}_i$ and $t_S(a_i) = b_i$ for $i = 0, 1, \dots, n$.

For $i \in \{0, 1, \dots, n\}$ the element a_i is $\mathfrak{p}_i$-adic unit, which is not a local ℓth power at $\mathfrak{p}_i$, thus the set $\{a_i, \pi_{\mathfrak{p}_i}\}$ forms a basis of $\dot{K}_{\mathfrak{p}_i}/\dot{K}_{\mathfrak{p}_i}^\ell$. We define the isomorphism $t_{\mathfrak{p}_i}\colon \dot{K}_{\mathfrak{p}_i}/\dot{K}_{\mathfrak{p}_i}^\ell \to \dot{L}_{\mathfrak{q}_i}/\dot{L}_{\mathfrak{q}_i}^\ell$ by sending $a_i \mapsto b_i$, $\pi_{\mathfrak{p}_i} \mapsto \pi_{\mathfrak{q}_i}$. Of course the isomorphism $t_{\mathfrak{p}_i}$ is tame and preserves Hilbert symbols. For $j \neq i$ the element a_j is a local ℓth power at $\mathfrak{p}_i$ and b_j is a local ℓth power at $\mathfrak{q}_i$; hence the diagram

$$\begin{array}{ccccc}
1 & \longrightarrow & K_{\mathrm{si}}/\dot{K}^\ell & \xrightarrow{\mathrm{diag}} & G(S) \\
\big\downarrow & & \big\downarrow{\scriptstyle t_S} & & \big\downarrow{\scriptstyle \prod_i t_{\mathfrak{p}_i}} \\
1 & \longrightarrow & L_{\mathrm{si}}/\dot{L}^\ell & \xrightarrow{\mathrm{diag}} & G(S')
\end{array}$$

commutes. Therefore f, T, t_S and the family $(t_{\mathfrak{p}_i})$, $i = 0, \dots, n$, determine an S-equivalence of K and L. By Theorem 2.7 this S-equivalence can be extended to degree ℓ Hilbert-symbol equivalence (f, t, T) which is tame outside S. To finish the proof, it is sufficient to notice that the equivalence is in fact tame, because the isomorphism $t_{\mathfrak{p}_i}$ is tame for every $i \in \{0, 1, \dots, n\}$. □

Remark. A criterion for tame Hilbert-symbol equivalence, as simple as in Theorem 3.4, does not exist for the global number fields. But for algebraic number fields there is a necessary and sufficient condition for tame Hilbert-symbol equivalence that can be viewed as a finiteness condition, for details see [CZ1] and [CZ2].

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