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Peter Mihók

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ON THE STRUCTURE OF THE POINT ARBORICITY CRITICAL GRAPHS

PETER MIHÓK

1. Introduction

A colouring of the points of a graph is called "acyclic" if no cycle has all its points the same colour. The "point arboricity" $\varrho(G)$ of a graph G is defined (see [3]) as the minimum number of colours in an acyclic colouring of the points of G. In this paper we investigate graphs which are critical with respect to point arboricity. A graph G is k-critical if G is connected, $\varrho(G)=k$ and for each line e of G, $\varrho(G-e)<\varrho(G)$. It is easy to see that the only 2-critical graphs are cycles, therefore we take $k\geqslant 3$. The properties of k-critical graphs with respect to $\varrho(G)$ have been investigated in [4], [5]. In [4] it was shown that every k-critical graph G has a minimum degree $\delta(G)\geqslant 2k-2$. The structure of the subgraph of k-critical graph G induced by the set of points of degree $2k-2$ is presented in §2. All k-critical graphs having at most one point of degree greater than $2k-2$ are described in §3.

2. The point arboritic analogues to Gallai's and Brooks' Theorems

In general, we follow the notation and terminology of book [2]. For any set S of points of a graph G the subgraph (S) induced by S is the maximal subgraph of G with a point set S. By a colouring of a graph G we always mean an acyclic colouring of the points of G. The set of all points with any one colour is called a colour-class. Let v be any point of a graph G; then a k-colouring of G is called a $\{v\}$-colouring of G if $\varrho(G)=k$ and one of the colour-classes consists of only v. The colour of the point v in the colouring f of G is denoted by $f(v)$. If the path P: $v_0v_1\dots v_n$ of a graph G has all its points the same colour c, then it is called a c-path. Let f be a colouring of the points of a path P: $v_0v_1\dots v_n$; then by recolouring the points of P we mean such a colouring f' that $f'(v_i)=f(v_{i+1})$ for $i=0, 1, \dots, n-1$; and $f'(v_n)=f(v_0)$. The point v of a k-critical graph G is called "secondary" if $\deg v=2k-2$ or "primary" if $\deg v>2k-2$. The diagonal of a cycle C in a graph G is a line of G joining two points of C, but not belonging to C.

The following lemmas are used in the proofs of our results. As Lemmas 1—4 ollow immediately from the definition of k-critical graphs, we omit the simple proofs.

Lemma 1. *Let v be a secondary point of a k-critical graph G. Then in any $\{v\}$-colouring f of G there exists for every colour i, $i \neq f(v)$, an i-path joining a pair of points adjacent to v.*

Lemma 2. *Let v be a secondary point of a k-critical graph G and let u be a point of G adjacent to v. If we change in a $\{v\}$-colouring of G the colours of v and u, then we obtain a $\{u\}$-colouring of G.*

Lemma 3. *If P is a $u-v$ path in a k-critical graph G and each point of P is secondary, then after recolouring the points of P we obtain from the $\{u\}$-colouring of G a $\{v\}$-colouring of G.*

Lemma 4. *Let C: $v_0v_1 \dots v_n$ be a cycle in a k-critical graph G and let any point of C be secondary. If we change in a $\{v_0\}$-colouring of C cyclically the colours of the points $v_1, v_2, \dots, v_{n-1}$, then we obtain again a $\{v_0\}$-colouring of G.*

Lemma 5. *Let C: $v_0v_1 \dots v_n$ be an even cycle in a k-critical graph G and let each point of C be secondary. If there is a point v_j of C which is incident with no diagonal of C in G, then C contains no diagonal in G.*

Proof. If the point v_j is incident with no diagonal of C in G, then among the points of C only v_{j-1} and v_{j+1} are adjacent to v_j. By Lemma 1, in any $\{v_j\}$-colouring of G the points adjacent to v_j are coloured pairwise with the same colour and this property, according to Lemma 4, is preserved after cyclical change of the colours of points $v_{j+1} \dots v_nv_1 \dots v_{j-1}$.
Since C is even, it follows that in any $\{v_j\}$-colouring of G all points but the point v_j of C have the same colour and thus C has no diagonal in G.

Lemma 6. If each even cycle in the block B of a graph G has at least two diagonals in G, then the block B is a complete subgraph of G.

Proof. This lemma follows immediately from Theorem 1.9 of [1, p. 170].

The following theorem is a point-arboritic analogue to Theorem 1 of T. Gallai [1].

Theorem 1. *Let G be a k-critical graph, $k \geq 3$, and let M be the set of all secondary points of G. Then the blocks of the subgraph (M) of G induced by the set M are complete graphs K_j, $0 \leq j \leq 2k-1$ or cycles.*

Proof. We consider the following three cases:

(1) The block B of (M) contains no even cycle. Then either $B = K_2$ or B contains an odd cycle C_{2n+1}. In case $B \neq C_{2n+1}$, then either C_{2n+1} has a diagonal in

M or there is a point of B not belonging to C_{2n+1}. In both cases there is an even cycle in B which contradicts our assumption.

(2) There is an even cycle C_{2n}: $v_0v_1\ldots v_{2n}$ in the block B of (M) which contains a point v_j lying in no diagonal of C_{2n} in (M).

We shall show, that C_{2n} is a block of (M). Let us suppose that C_{2n} is a proper subgraph of the block B. Since, by Lemma 5, C_{2n} contains no diagonal in (M), there exists a point u of B not belonging to C_{2n}. Let C': $v_0v_1\ldots u\ldots v_{2n}$ be a cycle in B containing the point u and the line v_0v_1 and let us denote by u_1 (resp. u_2) the first (respectively last) point of C' not belonging to C_{2n}. Let us take any $\{v_0\}$-colouring f of G. By Lemma 5, all points of C but v_0 have the same colour $c \neq (u_1)$. However, after recolouring of the points of C' we obtain a contradiction.

(3) The block B of (M) contains an even cycle C_{2n} different from B. Then by (2) each point of any even cycle in B is incident with one diagonal in (M) at least. According to Lemma 6, B is a complete subgraph of G.

The proof of Theorem 1 is completed.

Theorem 2 is a point-arboritic analogue to the well-known Brooks'Theorem. We are presenting another proof of this Theorem, first proved by Kronk—Mitchem in [5].

Theorem 2. *If G is connected, not complete and $\varrho(G)=k$, $k\geqq 3$, then $\Delta(G)\geqq 2k-1$.*

Proof. Let us assume that G is connected, not complete, $\varrho(G)=k$ and $\Delta(G)\leqq 2k-2$. We can assume that G is the smallest graph with the above mentioned properties. Then G is k-critical, $\delta(G)\geqq 2k-2$, so that all the points of G are secondary. According to Theorem 1, G is complete and this contradiction proves Theorem 2.

Corollary. *The only k-critical graph G without principal points is K_{2k-1}.*

3. Critical graphs having exactly one principal point

In [4, 5] it was shown that if G is a 2-connected graph with $\varrho(G)=k$ having at most one point of degree exceeding $2k-2$, then G is k-critical. The following two theorems describe the structure of all k-critical graphs having exactly one point of degree exceeding $2k-2$. The structure of all k-critical graphs having two (or more) principal points is much more complicated and it cannot be characterized in a similar way.

A block B of a graph G is called a K_j-block if it is a complete graph K_j and a C_n-block if it is a cycle C_n. An end-block of a graph G is a block containing exactly one cut point of G.

Theorem 3. *A graph G is a k-critical graph, $k\geqq 4$, having exactly one principal point, denoted z, if and only if all of the following conditions* (1)—(6) *hold:*

(1) *$G-z$ is connected.*
(2) *The degree of any point v in $G-z$ is $2k-3 \leqslant \deg v \leqslant 2k-2$.*
(3) *$G-z$ consists of K_2, K_{2k-3}, K_{2k-2} and C_n-blocks.*
(4) *Each cutpoint v of $G-z$ lies in a K_{2k-3}-block or in a K_{2k-2}-block of $G-z$.*
(5) *$G-z$ consists of at least three blocks.*
(6) *vz is a line of G if and only if $\deg v$ in $G-z$ is equal to $2k-3$.*

Proof. Let G be a k-critical graph having exactly one principal point z. Then G is 2-connected, $\delta(G) \geqslant 2k-2$ so that (1), (2) and (6) hold. According to Theorem 1, all the blocks of $G-z$ are complete graphs or cycles. Let v be a cutpoint of $G-z$. Let us assume that the greatest natural number j, for which v is a point of a K_j-block of $G-z$, is smaller than $2k-3$ or that all the blocks of $G-z$ containing the point v are cycles. For $A=\{v; z\}$, denote by L_i, $i=1, 2, \ldots, r$ the connected components of $G-A$ and by G_i the subgraphs of G induced by $V(L_i) \cup A$. Then, since $k \geqslant 4$, in each $\{v\}$-colouring f of G there is a colour $c_i \neq f(z)$ in any block B_i of G_i, which contains the point v, so that at most one point of B_i is coloured with c_i. Let us recolour the point v in G_i, $i=1, 2, \ldots, r$ by the colour c_i; then we obtain to each $i=1, 2, \ldots, r$ a $(k-1)$-colouring f_i of G_i, in which $f_i(v) \neq f_i(z)$. However, this implies that there exists a $(k-1)$-colouring of G, which is in contradiction with the fact that $\varrho(G)=k$. Hence each cutpoint v of $G-z$ lies in a K_{2k-3}-block or a K_{2k-2}-*block of* $G-z$ and (3), (4) and (5) hold.

Conversely, let G be a graph satisfying the conditions (1)—(6). In verifying that G is k-critical, it suffices to show that $\varrho(G) \geqslant k$. We use the induction on the number m of the K_{2k-3}-blocks of $G-z$.

I. If $m=0$, the desired result is implied by Theorem 3.3 of [1] (p 184) which states that if $m=0$ and G satisfies (1)—(6), then G has the chromatic number $\chi(G)=2k-1$. Since $\chi(G) \leqslant 2\varrho(G)$ (see [3]), we have $\varrho(G) \geqslant k$.

II. Let us assume for any graph G satisfying conditions (1)—(6) and having less than m K_{2k-3}-blocks that $\varrho(G) \geqslant k$. We shall assume that G^* is a graph satisfying (1)—(6) having exactly m K_{2k-3}-blocks, $\varrho(G^*) \leqslant k-1$ and we shall show that this leads to a contradiction. Let f be a $(k-1)$-colouring of G^* and let v be a cutpoint of G^*-z. Further, let us for $A=\{v, z\}$ denote by L_i, $i=1, 2, \ldots, r$ the connected components of G^*-A and by G_i^* the subgraphs of G^* induced by $A \cup V(L_i)$, $i=1, 2, \ldots, r$.

Let us consider the following cases:

(1) v is a point of a C_n-block B of G^*-z;

1.1. If $f(z) \neq f(v)$, then let us denote by U the points of this connected component L_j of G^*-A which contains the points of B. We define a graph H as follows: Let $G_1=K_{2k-3}$, $G_2=K_{2k-2}$ and $G_3=G^*-U$ are mutually disjoint graphs, v_1 and v_2 arbitrary points of G_1 and G_2, respectively. Let us join the point v of G_3 with v_1 and v_2 and the point z of G_3 with all the remaining points of G_1 and G_2.

Then it is easy to see that the obtained graph H has a $(k-1)$-colouring and it satisfies (1)—(6), which contradicts our inductive assumption.

1.2. If $f(z)=f(v)$, then one of the points of B has the colour different from $f(z)$ and thus we can continue as in case 1.1.

(2) G^*-z contains no C_n-block; Let us select an arbitrary K_{2k-3}-block B of G^*-z and a point v_0 of B for which $f(v_0)=f(z)$. Since $\varrho(K_{2k-3}) = k-1$, such a point v_0 exists.

2.1. If v_0z is a line of G^*, then there is no $f(v_0)$-path joining the points v_0 and z which passes a K_2-block of G^*-z. We define a graph H as follows: For $i=0, 1, 2, \ldots, 2k-4$ let v_i be the points of B, let $A_i = \{z, v_i\}$; now we denote by L^i an arbitrary connected component of $G-A_i$ containing no points of B. Let us remove from G^* the line v_0z and all points and lines of the subgraph $G^i=(V(L^i))$ of G^* induced by the points of L^i, $i=1, 2, \ldots, 2k-4$. Now we take a new point u and we join it with all the points of B. The obtained graph H obviously satisfies (1)—(6) and it has a $(k-1)$-colouring, however, by the inductive assumption this is impossible.

2.2. If v_0z is not a line of G^*, then we proceed similarly as in the case 2.1.

This completes the proof.

Since K_3 is a cycle, the structure of the 3-critical graphs is somewhat complicated. We shall describe it in the following theorem.

Theorem 4. *A graph G is a 3-critical graph having exactly one principal point, denoted z, if and only if all of the following conditions* (1)—(6) *hold:*

(1) *$G-z$ is connected.*
(2) *The degree of any point v in $G-z$ fulfils the inequalities* $3\leq \deg v\leq 4$.
(3) *$G-z$ consists of K_2, K_4 and C_n-blocks.*
(4) *The set M of all C_n-blocks of $G-z$ is divided into two disjoint classes M_1, M_2, so that no two blocks of the same class have a common point and each cutpoint v of $G-z$ lies in a K_{4k-2}-block of $G-z$ or in a C_n-block of M_1.*
(5) *$G-z$ consists of at least three blocks.*
(6) *vz is a line of G if and only if* $\deg v$ *in $G-z$ is* 3.

The proof of Theorem 4 is omitted, it proceeds similarly to the proof of Theorem 3.

Remark. Similarly as in [1, p. 186—189], using Theorem 1, the following theorem can be proved.

Theorem 5. *Let G be a k-critical graph, $k\geq 3$, with $n>2k-1$ points and m lines, then*

$$m>n(k-1)+\frac{n}{4k+10}.$$

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Katedra geometrie a algebry
Prírodovedeckej fakulty UPJŠ
Komenského 14
041 54 Košice

О СТРУКТУРЕ ВЕРШИННО-ДРЕВЕСНОСТНЫХ КРИТИЧЕСКИХ ГРАФОВ

Петер Мигок

Резюме

Граф $G=(X, \Gamma)$ называется ациклически раскрашенным k красками, если каждая его вершина раскрашена одной из k красок и вершины ни одного цикла в G не получают одинакого цвета.

Вершинная древесность $\varrho(G)$ графа G определяется как наименьшее k, для которого существует ациклическая раскраска графа G k красками.

Граф $G=(X, \Gamma)$ называется k-критическим, если G связен, $\varrho(G)=k$ и для любого ребра $e\in\Gamma$, $\varrho(G-e)<\varrho(G)$. В [4] показано, что если G k-критический граф, то $\delta(G)\geq 2(k-1)$.

В статье доказывается, что в k-критическом графе G блоки подграфа, порожденного вершинами степени $2k-2$, являются полными графами и циклами. Далее изучаются графы которых степень любой вершины, кроме одной, равна $2k-2$.