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TOPOLOGIES ON QUANTUM LOGICS INDUCED BY MEASURES

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In the classical probability theory the collection of all experimentally verifiable propositions on a physical system is assumed to be a Boolean σ-algebra. The well-known Heisenberg Uncertainty Principle is the usual argument that this assumption is inadequately strong, in general. Namely, according to the Heisenberg Principle, the position and momentum of physical particle are quantities which cannot be measured simultaneously with arbitrary accuracy. Therefore, in quantum physics the family of propositions on a quantum mechanical system is mathematically described as a lattice of all closed subspaces of a separable Hilbert space. A quantum logic is a model for the collection of propositions on a general physical system. It includes both the Boolean σ-algebra and the lattice of subspaces of a Hilbert space.

In the quantum logic context, the notion of compatibility is of principal importance. The compatibility of two or more elements of a logic was discussed in many papers. However, the relationship between arbitrary elements (including incompatible ones) was apparently neglected by mathematicians. One feels intuitively that these relations deserve a mathematical description. In this paper we shall discuss this problem from the topological point of view. Given a logic $\mathscr{L}$, we shall first construct a topological space $(\mathscr{L}, \mathscr{T})$. The topology $\mathscr{T}$ on $\mathscr{L}$ should be connected, in a sense, with the state of the corresponding physical system. In mathematical formalism, we assume that there is defined a finite measure μ on $\mathscr{L}$ and the constructed topology is associated with this measure. It should be noted that the logic as a topological space has been investigated in [7]. However, the topology defined there has not been linked with measure.

1. Preliminaries

Let $(\mathscr{L}, \leqq, \perp)$ be a triple, where $\mathscr{L}$ is a set partially ordered by $\leqq$, having a largest element, 1, and a smallest element, 0, and being endowed with an orthocomplementation relation $\perp: \mathscr{L} \to \mathscr{L}$. Let the following properties be satisfied ($a, b \in \mathscr{L}$):

i) $(a^{\perp})^{\perp} = a$,

ii) $a \leqq b$ implies $b^{\perp} \leqq a^{\perp}$,

iii) $a \vee a^{\perp} = 1$,

iv) $a \leqq b$ implies $b = a \vee (a \vee b^{\perp})^{\perp}$,

v) $\bigvee_{i=1}^{\infty} a_i$ exists in $\mathscr{L}$ for every sequence a_i of pairwise orthogonal elements of $\mathscr{L}$ (a_i is orthogonal to a_j — abbr. $a_i \perp a_j$ — if $a_i \leqq a_j^{\perp}$).

If $\mathscr{L}$ satisfies i)—v), then it is called a *quantum logic* (briefly a *logic*). Throughout the paper, let $\mathscr{L}$ denote always a logic.

The elements $a, b \in \mathscr{L}$ are said to be compatible (abbr. $a \leftrightarrow b$) if there exist mutually orthogonal elements $a_1, b_1, c \in \mathscr{L}$ such that $a = a_1 \vee c$, $b = b_1 \vee c$. These elements are determined uniquely ([9]). Thus, we can denote a_1 as $a - b$ and b_1 as $b - a$. The element $a_1 \vee b_1$ will then be denoted by $a \triangle b$.

A function $\nu: \mathscr{L} \to \langle -\infty, \infty \rangle$ is called a *signed measure* if the following two conditions are satisfied:

i) $\nu(0) = 0$,

ii) $\nu\left(\bigvee_{i=1}^{\infty} a_i\right) = \sum_{i=1}^{\infty} \nu(a_i)$ for every sequence a_i of mutually orthogonal elements of $\mathscr{L}$.

If, moreover, ν is nonnegative, then ν is called a *measure*. If $\nu(1) = 1$ for a measure ν, then ν is called a *state*. A signed measure ν is bounded if $|\nu(a)| \leqq K$, $a \in \mathscr{L}$, for some real K. We shall denote $M(\mathscr{L})$ the set of all bounded signed measures and $S(\mathscr{L})$ the set of all states on $\mathscr{L}$.

Let μ be a measure and ν a signed measure on $\mathscr{L}$. Then ν is said to be *absolutely continuous with respect to* μ (denoted $\nu \ll_{\varepsilon} \mu$), if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu(a) < \delta$ implies $|\nu(a)| < \varepsilon$, $a \in \mathscr{L}$. Denote by $M_{\mu}(\mathscr{L})$ the set of all bounded signed measures ν on $\mathscr{L}$ such that $\nu \ll_{\varepsilon} \mu$.

Typical examples of logics are the Boolean σ-algebras and the lattice $\mathscr{L}(H)$ of all closed subspaces of a separable (real or complex) Hilbert space H. Let us adopt the following notation. Let $[H]$ denote the Banach space of all bounded operators on H with the usual operator norm. If $A \in [H]$, then A^* is the adjoint operator. Put $|A| = (A^*A)^{1/2}$. If $A \in [H]$, then A is of the *trace class* if $\Sigma\,(|A|\,\varphi_i, \varphi_i) < \infty$ for some orthogonal base $\{\varphi_i\}$. In this case the sum $\operatorname{tr} A = \Sigma\,(A\varphi_i, \varphi_i)$ is called a trace of A and, as known, it is independent of the choice of a base. The trace class operators form a separable Banach space with respect to the norm τ: $\tau(A) = \operatorname{tr}|A|$. Moreover, if $A, B \in [H]$ and A is of the trace class, then AB and BA are also of the trace class and we have the inequalities

$$\tau(AB) \leqq \tau(A)\,\|B\|$$

$$\tau(BA) \leqq \tau(A)\,\|B\|$$
$$\|A\| \leqq \tau(A),$$

where $\|\ \|$ denotes the usual operator norm (see [8]).

According to the famous Gleason theorem and its generalization (see [1], [2]), there exists a one-to-one correspondence between bounded signed measures on $\mathscr{L}(H)$, where $\dim H \geqq 3$, and hermitean operators of the trace class. Indeed, $v \in M(\mathscr{L}(H))$ iff v is of the form $v(M) = \operatorname{tr} TP^M$, $M \in \mathscr{L}(H)$, where P^M is the orthogonal projector corresponding to M and T is a hermitean trace class operator. A signed measure v is nonnegative iff T is a positive operator. The operator corresponding to v will be denoted by T_v.

2. Logic as a topological space

Naturally, a reasonable topology on $\mathscr{L}$ should be connected with algebraic operations of $\mathscr{L}$. Let $(\mathscr{L}, \mathscr{T})$ be a topological space. Put $\mathscr{G} = \{(a, b) \in \mathscr{L} \times \mathscr{L}: a \perp b\}$. Let $\mathscr{T} \times \mathscr{T}$ be the usual product topology and $\mathscr{T}_0$ the relative topology on $\mathscr{G}$ induced by $\mathscr{T} \times \mathscr{T}$.

Definition 2.1. *We say that $(\mathscr{L}, \mathscr{T})$ is a topological logic if the following conditions are true:*

i) *The orthocomplementation $\perp: a \mapsto a^\perp$ is a homeomorphism of $(\mathscr{L}, \mathscr{T})$ into itself.*

ii) *The mapping $\vee: (a, b) \mapsto a \vee b$ is a continuous mapping from $(\mathscr{G}, \mathscr{T}_0)$ into $(\mathscr{L}, \mathscr{T})$.*

The proofs of the following two lemmas are obvious.

Lemma 2.1. *$(\mathscr{L}, \mathscr{T})$ is a topological logic iff the following conditions are satisfied:*

i) *if a_α is a net in $\mathscr{L}$, $a \in \mathscr{L}$ and $a_\alpha \to a$ in $(\mathscr{L}, \mathscr{T})$, then $a_\alpha^\perp \to a^\perp$ in $(\mathscr{L}, \mathscr{T})$,*

ii) *if a_α, b_α are nets in $\mathscr{L}$, $a, b \in \mathscr{L}$, $a_\alpha \perp b_\alpha$, $a \perp b$, $a_\alpha \to a$ and $b_\alpha \to b$ in $(\mathscr{L}, \mathscr{T})$, then $a_\alpha \vee b_\alpha \to a \vee b$ in $(\mathscr{L}, \mathscr{T})$.*

Of course, if $(\mathscr{L}, \mathscr{T})$ satisfies the first countability axiom and we change in the above lemma all nets for sequences, the validity of Lemma 2.1 will be preserved.

Lemma 2.2. *Let $(\mathscr{L}, \mathscr{T})$ be a topological logic. Let a_α, b_α be nets in $\mathscr{L}$, $a, b \in \mathscr{L}$, $a_\alpha \leqq b_\alpha$, $a \leqq b$. If $a_\alpha \to a$ and $b_\alpha \to b$ in $(\mathscr{L}, \mathscr{T})$, then $b_\alpha - a_\alpha \to b - a$ in $(\mathscr{L}, \mathscr{T})$.*

Assuming that $S(\mathscr{L})$ is nonempty, we can exhibit a simple example of a topology which converts $\mathscr{L}$ into a topological logic. Define a pseudometric d on $\mathscr{L}$ as follows:

$$d(a, b) = \sup \{|s(a) - s(b)|,\ s \in S(\mathscr{L})\}.$$

Denote by $\mathcal{T}_d$ the topology induced by d. Then $(\mathcal{L}, \mathcal{T}_d)$ is a topological logic. Throughout this paper, let d and $\mathcal{T}_d$ have always this meaning. It can be easily seen that in the case of a set σ-algebra $\mathcal{S}$ the topology $\mathcal{T}_d$ is discrete. In the case of $\mathcal{L}(H)$, where $\dim H \geqq 3$, $d(M, N) = \|P^M - P^N\|$, $M, N \in \mathcal{L}(H)$ (see [3], Theorem 6.1).

Now, let μ be an arbitrary fixed finite measure on $\mathcal{L}$. We intend to obtain a topology associated with μ. In the classical measure theory, there is known an example of such a topology. If $\mathcal{S}$ is a set σ-algebra and μ a finite measure on $\mathcal{S}$, then the function $\varrho_\mu : \varrho_\mu(A, B) = \mu(A \triangle B)\,(A, B \in \mathcal{S})$ is a pseudometric. Denote by $\mathcal{T}_{\varrho_\mu}$ the topology induced by ϱ_μ. Naturally, $(\mathcal{S}, \mathcal{T}_{\varrho_\mu})$ is a topological logic because the operations of forming the union and complementation are continuous ([4]). In the case of a general logic, there is impossible to define a topology in such a simple way because there exist noncompatible pairs of elements of $\mathcal{L}$. However, by means of a measure we can define a distance of comparable pairs.

Definition 2.2. *Let μ be a finite measure on $\mathcal{L}$. A topology $\mathcal{T}$ on $\mathcal{L}$ is called a μ-topology if for every $a \in \mathcal{L}$ and every net $a_\alpha \in \mathcal{L}$ such that $a_\alpha \leqq a$ $(a \leqq a_\alpha)$ the following statement is true*:

$$a_\alpha \to a \quad in \quad (\mathcal{L}, \mathcal{T}) \quad iff \quad \mu(a - a_\alpha) \to 0 \ (\mu(a_\alpha - a) \to 0).$$

3. Topology on $\mathcal{L}$ (H)

Throughout this section, let μ be an arbitrary fixed finite measure on $\mathcal{L}$ (H), $\dim H \geqq 3$. We shall construct for μ a μ-topology $\mathcal{T}$ such that $(\mathcal{L}(H), \mathcal{T})$ is a pseudometrizable topological logic. Denote by $N(T_\mu)$ the null space of T_μ and by $S(T_\mu)$ its orthogonal complement. Then μ is of the form $\mu(M) = \Sigma\,\lambda_i(P^M \varphi_i, \varphi_i)$, $M \in \mathcal{L}(H)$, where $\{\varphi_i\}$ is the orthonormal system of eigenvectors of T_μ and $\{\lambda_i\}$ the system of the corresponding eigenvalues.

For every $u \in S(T_\mu)$ and $\varepsilon > 0$, define the relation $U_{u,\varepsilon}$ as follows:

$$U_{u,\varepsilon} = \{(M, N) \in \mathcal{L}(H) \times \mathcal{L}(H),\ \|P^M u - P^N u\| < \varepsilon\}.$$

The family of all such relations is a prebase of uniformity ([6], Theorem 6.3). The uniform topology induced by this uniformity will be called *the topology of strong convergence with respect to* μ and denoted by $\mathcal{T}_{sc}(\mu)$. A sequence M_n converges to M in $(\mathcal{L}(H), \mathcal{T}_{sc}(\mu))$ iff for every $u \in S(T_\mu)$

$$\lim_{n \to \infty} \|P^{M_n} u - P^M u\| = 0,$$

i.e. iff for every $u \in S(T_\mu)$ the sequence $P^{M_n} u$ converges strongly to $P^M u$.

Theorem 3.1. $(\mathcal{L}(H), \mathcal{T}_{sc}(\mu))$ *is a pseudometrizable topological logic.*

Proof. Assume the more complicated case of $\dim S(T_\mu) = \infty$. A uniform space is pseudometrizable iff its uniformity possesses a countable base. Denote by $\mathscr{V}$ the countable family of all sets of the form $\bigcap_{i=1}^{k} U_{\varphi_i, \varepsilon}$, where ε is a positive rational and k a positive integer. We shall prove that $\mathscr{V}$ is a base of the uniformity inducing $\mathscr{T}_{sc}(\mu)$. It will suffice to show that for every $u \in S(T_\mu)$ and $\varepsilon > 0$ there exist a positive rational ε_0 and an integer k such that $\bigcap_{i=1}^{k} U_{\varphi_i, \varepsilon_0} \subset \subset U_{u, \varepsilon}$. Let $\varepsilon > 0$ and $u \in S(T_\mu)$ be given. Then $u = \sum_{i=1}^{\infty} (u, \varphi_i)\varphi_i$. Choose the integer k and the rational ε_0 such that $\left\| \sum_{i=k+1}^{\infty} (u, \varphi_i)\varphi_i \right\| < \frac{\varepsilon}{4}$ and $0 < \varepsilon_0 < < \frac{\varepsilon}{2k\|u\|}$. Let $(M, N) \in \bigcap_{i=1}^{k} U_{\varphi_i, \varepsilon_0}$. Hence,

$$\|P^M u - P^N u\| \leqq \left\| P^M \sum_{i=k+1}^{\infty} (u, \varphi_i)\varphi_i \right\| + \left\| P^N \sum_{i=k+1}^{\infty} (u, \varphi_i)\varphi_i \right\| +$$

$$+ \sum_{i=1}^{k} \|u\| \, \|P^M \varphi_i - P^N \varphi_i\| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + k\|u\|\varepsilon_0 < \varepsilon.$$

Thus, $(M, N) \in U_{u, \varepsilon}$. The pseudometrizability is therefore proved. Now, let $M_n \to M$ and $N_n \to N$ in $(\mathscr{L}(H), \mathscr{T}_{sc}(\mu))$, where $M_n \perp N_n$ and $M \perp N$. Then, for every $u \in S(T_\mu)$,

$$\|P^{M_n^\perp} u - P^{M^\perp} u\| = \|P^{M_n} u - P^M u\|$$

and

$$\|P^{M_n \vee N_n} u - P^{M \vee N} u\| = \|P^{M_n} u + P^{N_n} u - P^M u - P^N u\| \leqq$$

$$\leqq \|P^{M_n} u - P^M u\| + \|P^{N_n} u - P^N u\|.$$

Thus, $M_n^\perp \to M^\perp$ and $M_n \vee N_n \to M \vee N$ in $(\mathscr{L}(H), \mathscr{T}_{sc}(\mu))$. The theorem is proved.

Lemma 3.1. *Let $\{u_i\}_{i=1}^{\infty}$ be an orthonormal system in H, where $u = \sum_{i=1}^{\infty} (u, u_i) u_i$. If A_n is a sequence of operators of $[H]$ such that $\lim_{n \to \infty} \|A_n u_i\| = 0$, $(i = 1, 2, \ldots)$ and $\|A_n\| \leqq K$ for some real K, then, also, $\lim_{n \to \infty} \|A_n u\| = 0$.*

Proof. Given $\varepsilon > 0$, choose integers j and n_0 such that

$$\left\| \sum_{i=k+1}^{\infty} (u, u_i) u_i \right\| < \frac{\varepsilon}{2K} \quad \text{and} \quad \|A_n u_i\| < \frac{\varepsilon}{2\|u\| j}$$

for every $n \geqq n_0$ and $i = 1, \ldots, j$. Hence, for every $n \geqq n_0$,

$$\|A_n u\| \leqq \sum_{i=1}^{j} |(u, u_i)| \quad \|A_n u_i\| + K\left\| \sum_{i=j+1}^{\infty} (u, u_i) u_i \right\| < \varepsilon.$$

The lemma is proved.

Lemma 3.2. *A sequence M_n converges to 0 in $(\mathscr{L}(H), \mathscr{T}_{sc}(\mu))$ iff $\lim_{n\to\infty} \mu(M_n) = 0$.*

Proof. Assume again that $\dim S(T_\mu) = \infty$. Let $M_n \to 0$ in $(\mathscr{L}(H), \mathscr{T}_{sc}(\mu))$. Hence, $\lim_{n\to\infty} \|P^{M_n}\varphi_i\| = 0$, $(i = 1, 2, \ldots)$. Given $\varepsilon > 0$, there exist integers k and n_0 such that $\sum_{i=k+1}^{\infty} \lambda_i < \frac{\varepsilon}{2}$ and $\lambda_i(P^{M_n}\varphi_i, \varphi_i) < \frac{\varepsilon}{2k}$ for every $n \geqq n_0$ and $i = 1, \ldots, \ldots, k$. Hence, for every $n \geqq n_0$,

$$\mu(M_n) = \sum_{i=k+1}^{\infty} \lambda_i(P^{M_n}\varphi_i, \varphi_i) + \sum_{i=1}^{k} \lambda_i(P^{M_n}\varphi_i, \varphi_i) < \frac{\varepsilon}{2} + k\frac{\varepsilon}{2k} = \varepsilon.$$

Thus, $\lim_{n\to\infty} \mu(M_n) = 0$. On the other hand, let $\lim_{n\to\infty} \mu(M_n) = 0$. This implies that $\lim_{n\to\infty} \|P^{M_n}\varphi_i\| = 0$. Hence, using Lemma 3.1, we have $M_n \to 0$ in $(\mathscr{L}(H), \mathscr{T}_{sc}(\mu))$. Lemma is proved.

Theorem 3.2. *$\mathscr{T}_{sc}(\mu)$ is a μ-topology. Moreover, if $M_n \leftrightarrow M$, then $M_n \to M$ in $(\mathscr{L}(H), \mathscr{T}_{sc}(\mu))$ iff $\lim_{n\to\infty} \mu(M_n \triangle M) = 0$.*

Proof. It suffices to prove the second assertion. Let $\lim_{n\to\infty} \mu(M_n \triangle M) = \lim_{n\to\infty} \mu(M_n - M) + \mu(M - M_n) = 0$. According to Lemma 3.2 the sequences $M_n - M$ and $M - M_n$ converge to 0 in $(\mathscr{L}(H), \mathscr{T}_{sc}(\mu))$, i.e. $\lim_{n\to\infty} \|P^{M_n - M}u\| = \lim_{n\to\infty} \|P^{M - M_n}u\| = 0$ for every $u \in S(T_\mu)$. Hence, according to equality

$$\|P^{M_n}u - P^M u\|^2 = \|P^{M_n - M}u\|^2 + \|P^{M - M_n}u\|^2,$$

one obtains $M_n \to M$ in $(\mathscr{L}(H), \mathscr{T}_{sc}(\mu))$. The opposite implication can be proved dually.

The following theorem shows that $\mathscr{T}_{sc}(\mu)$ is induced by a pseudometric which can be explicitly expressed.

Theorem 3.3. *$\mathscr{T}_{sc}(\mu)$ is induced by the following pseudometric p:*

$$p(M, N) = \operatorname{tr}|(P^M - P^N)T_\mu|, \qquad M, N \in \mathscr{L}(H).$$

Proof. The function p is a pseudometric because the function $\tau(A) = \operatorname{tr}|A|$ is a norm on the space of trace class operators. Further, it is necessary and sufficient to prove that $p(M_n, M) \to 0$ iff $\lim_{n\to\infty} \|P^{M_n}u - P^M u\| = 0$, for every $u \in S(T_\mu)$ and every sequence M_n and $M \in \mathscr{L}(H)$. We assume the case of $\dim S(T_\mu) = \infty$. Let $\lim_{n\to\infty} \|P^{M_n}u - P^M u\| = 0$ for every $u \in S(T_\mu)$. Denote by P_i the projector corresponding to the one-dimensional subspace generated by φ_i. We shall prove first that $\lim_{n\to\infty} \operatorname{tr}|(P^{M_n} - P^M)P_i| = 0$, $i = 1, 2, \ldots$

$$\operatorname{tr}|(P^{M_n} - P^M)P_i| = \tau((P^{M_n} - P^M)P_i) = \tau((P^{M_n} - P^M)P_iP_i) \leqq$$
$$\leqq \|(P^{M_n} - P^M)P_i\| \, \tau(P_i) = \|(P^{M_n} - P^M)P_i\| = \|(P^{M_n} - P^M)\varphi_i\|.$$

Of course, the last expression converges to 0. Now, let $\varepsilon > 0$ be given. There exists integer k such that $\sum_{i=k+1}^{\infty} \lambda_i = \operatorname{tr}\left|\sum_{i=k+1}^{\infty} \lambda_i P_i\right| < \frac{\varepsilon}{4}$. Further, there exists n_0 such that $\lambda_i \operatorname{tr}|(P^{M_n} - P^M)P_i| < \frac{\varepsilon}{2k}$ for $n \geqq n_0$, $i = 1, \ldots, k$. Hence, for $n \geqq n_0$,

$$p(M_n, M) = \operatorname{tr}\left|(P^{M_n} - P^M)\left(\sum_{i=1}^{k} \lambda_i P_i + \sum_{i=k+1}^{\infty} \lambda_i P_i\right)\right| \leqq$$
$$\sum_{i=1}^{k} \lambda_i \operatorname{tr}|(P^{M_n} - P^M)P_i| + \|P^{M_n} - P^M\| \operatorname{tr}\left|\sum_{i=k+1}^{\infty} \lambda_i P_i\right| < \varepsilon.$$

Thus, $\lim_{n\to\infty} p(M_n, M) = 0$. On the other hand, if $\lim_{n\to\infty} p(M_n, M) = 0$, then the inequality $\|A\| \leqq \tau(A)$ implies $\lim_{n\to\infty} \|(P^{M_n} - P^M)T_\mu u\| = 0$ for every $u \in H$. If we put $u = \varphi_i$, we obtain $\lim_{n\to\infty} \lambda_i \|(P^{M_n} - P^M)\varphi_i\| = 0$, i.e. $\lim_{n\to\infty} P^{M_n}\varphi_i = P^M\varphi_i$. Put $A_n = P^{M_n} - P^M$. According to Lemma 3.1, $P^{M_n}u$ converges strongly to $P^M u$ for every $u \in S(T_\mu)$. Thus, $M_n \to M$ in $(\mathscr{L}(H), \mathscr{T}_{sc}(\mu))$. The theorem is proved.

Let us return to the space $(\mathscr{S}, \varrho_\mu)$ and observe some of its properties. $(\mathscr{S}, \varrho_\mu)$ is complete and sometimes separable (e.g. if $\mathscr{S}$ is countably generated (see [4])). As we shall see, the situation for $(\mathscr{L}(H), \mathscr{T}_{sc}(\mu))$ is similar.

Theorem 3.4 *$(\mathscr{L}(H), \mathscr{T}_{sc}(\mu))$ is a complete separable uniform space.*

Proof. For showing the completeness, let M_n be a cauchy sequence in $(\mathscr{L}(H), \mathscr{T}_{sc}(\mu))$. This implies that all sequences $P^{M_n}u$, where $u \in S(T_\mu)$, are cauchy in H. Hence, by completeness of H, we have the vector $Bu = \lim_{n\to\infty} P^{M_n}u$. B is an operator defined on $S(T_\mu)$ with values in H. Denote by $R(B)$ the range of B and by M the closure of $R(B)$. We shall prove that $M_n \to M$ in $(\mathscr{L}(H), \mathscr{T}_{sc}(\mu))$. It will suffice to prove $P^M u = Bu$ for every $u \in S(T_\mu)$. $Bu = P^M u$ iff $Bu \in M$ and $u - Bu \in M^\perp$. We have to show that Bu satisfies both conditions. Of course, $Bu \in M$. Further, it will suffice to show that $u - Bu \perp R(B)$. Let $y \in R(B)$, i.e. let $y = Bz$, $z \in S(T_\mu)$. Hence, $y = \lim_{n\to\infty} P^{M_n}z$. Naturally, $u - P^{M_n}u \perp P^{M_n}z$. Thus, $u - Bu \perp y$. The completeness is proved.

The separability is an immediate consequence of the separability of the space of trace class operators with respect to the norm τ ([8]).

4. The preliminary uniqueness theorem for $\mathscr{L}(H)$

Let us summarize the properties of the space $(\mathscr{S}, \mathscr{T}_{\varrho_\mu})$:

i) $\mathscr{T}_{\varrho_\mu}$ is a μ-topology,

ii) $(\mathscr{S}, \mathscr{T}_{\varrho_\mu})$ is a pseudometrizable topological logic,

iii) $\mathscr{T}_{\varrho_\mu}$ is weaker then $\mathscr{T}_d$,

iv) Every $\nu \in M_\mu(\mathscr{S})$ is a continuous function from $(\mathscr{S}, \mathscr{T}_{\varrho_\mu})$ into R.

Properties i) and iii) are evident. For ii) and iv), see [4]. Moreover:

Theorem 4.1. *$\mathscr{T}_{\varrho_\mu}$ is the unique topology satisfying the above properties* i)—iv).

Proof. Let $\mathscr{T}$ be a topology satisfying i)—iv). Assume $A_n \to A$ in $(\mathscr{S}, \mathscr{T})$. Define measure $\mu_A \in M_\mu(\mathscr{S})$ via $\mu_A(E) = \mu(A \cap E)$, $E \in \mathscr{S}$. Measures μ and μ_A are continuous mappings, hence, $\lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \mu_A(A_n) = \mu(A)$. This implies $\lim_{n\to\infty} \mu(A\backslash A_n) = \lim_{n\to\infty} \mu(A_n\backslash A) = 0$. Thus, $A_n \to A$ in $(\mathscr{S}, \mathscr{T}_{\varrho_\mu})$. On the other hand, let $A_n \to A$ in $(\mathscr{S}, \mathscr{T}_{\varrho_\mu})$. Hence, $A_n\backslash A$ and $A\backslash A_n$ converge to $\emptyset$ because $\mathscr{T}$ is a μ-topology. According to ii), $A_n \cup A = (A_n\backslash A) \cup A \to \emptyset \cup A = A$ in $(\mathscr{S}, \mathscr{T})$. Finally, by Lemma 2.2, $A_n = (A_n \cup A)\backslash(A\backslash A_n) \to A$ in $(\mathscr{S}, \mathscr{T})$. The theorem is proved. (Of course, the property iii) was not necessary for the proof.)

The validity of the above theorem is perhaps not so surprising as the fact that the same theorem holds for a measure μ on $\mathscr{L}(H)$, $\dim H \geqq 3$. We shall prove it later. In this section, we prove only a preliminary uniqueness theorem. In the following μ has the same meaning as in the previous section.

Definition 4.1. *A topology $\mathscr{T}$ on $\mathscr{L}(H)$ is said to be μ-regular if the following assertion is true: if $M_n \to M$ in $(\mathscr{L}(H), \mathscr{T})$, then $M_n \to M$ in $(\mathscr{L}(H), \mathscr{T}_{sc}(\mu))$, too.*

Of course, $\mathscr{T}_{sc}(\mu)$ is μ-regular. Further, we use the following notation. If $u \in H$, then $[u]$ denotes the subspace generated by u.

Lemma 4.1. *Let $\mathscr{T}$ be a topology on $\mathscr{L}(H)$, let $\mathscr{T} \subset \mathscr{T}_d$. If $v_n \in H$, $n = 1, 2, \ldots$, $v \in H$, $v \neq 0$ and $\lim_{n\to\infty} \|v_n - v\| = 0$, then $[v_n] \to [v]$ in $(\mathscr{L}(H), \mathscr{T})$.*

Proof. It suffices to prove the above lemma for the case $\|v_n\| = \|v\| = 1$. For every $u \in H$ we have

$$\begin{aligned}&\|P^{[v_n]}u - P^{[v]}u\| = \|(u, v_n)v_n - (u, v)v\| \leqq \\ &\|(u, v_n)v_n - (u, v)v_n\| + \|(u, v)v_n - (u, v)v\| \leqq \\ &|(u, v_n - v)| + |(u, v)|\,\|v_n - v\| \leqq 2\|v_n - v\|\,\|u\|.\end{aligned}$$

Hence, $\|P^{[v_n]} - P^{[v]}\| \leqq 2\|v_n - v\|$. Therefore, $[v_n] \to [v]$ in $(\mathscr{L}(H), \mathscr{T}_d)$. Thus, by the assumption, $[v_n] \to [v]$ also in $(\mathscr{L}(H), \mathscr{T})$.

Lemma 4.2. *If $M, N \in \mathscr{L}(H)$, $N \subset M$, $u \in H$ and $P^M u \in N$, then $P^N u = P^M u$.*

Proof. Clearly, $P^M u - P^N u = P^{M-N} u$. By the assumption the left-hand side of this equality is in N, while the right-hand side lies in $M - N$. Thus, both sides have to be zero.

We can now prove the preliminary uniqueness theorem.

Theorem 4.2. *Let μ be a finite measure on $\mathscr{L}(H)$, where* $\dim H \geqq 3$. *Then there exists a unique topology $\mathscr{T}$ satisfying the following conditions*:

i) *$\mathscr{T}$ is a μ-topology,*

ii) *$(\mathscr{L}(H), \mathscr{T})$ is a pseudometrizable topological logic,*

iii) *$\mathscr{T}$ is weaker than $\mathscr{T}_d$,*

iv) *$\mathscr{T}$ is μ-regular.*

Proof. Assume that $\mathscr{T}$ is a topology with properties i)—iv). We have to prove that $\mathscr{T} = \mathscr{T}_{sc}(\mu)$. Let ϱ be pseudometric inducing $\mathscr{T}$. It will suffice to show that, for every M_n, $M \in \mathscr{L}(H)$, $M_n \to M$ in $(\mathscr{L}(H), \mathscr{T})$ iff $M_n \to M$ in $(\mathscr{L}(H), \mathscr{T}_{sc}(\mu))$. Of course, one of these implications is obvious. The μ-regularity of $\mathscr{T}$ implies that from $M_n \to M$ in $(\mathscr{L}(H), \mathscr{T})$ there follows $M_n \to M$ in $(\mathscr{L}(H), \mathscr{T}_{sc}(\mu))$. For the second implication it will be sufficient to show that every sequence M_n converging to M in $(\mathscr{L}(H), \mathscr{T}_{sc}(\mu))$ contains a subsequence M_{n_k} such that $\lim_{k \to \infty} \varrho(M_{n_k}, M) = 0$. Let M_n and M be given such that $M_n \to M$ in $(\mathscr{L}(H), \mathscr{T}_{sc}(\mu))$, i.e. $\lim_{n \to \infty} \|P^{M_n}u - P^M u\| = 0$ for every $u \in S(T_\mu)$. We now divide the proof into two steps. We assume first the special case $M = P^M(S(T_\mu))$, while in the second step this limitation is omitted. Throughout the proof, we consider the more complicated case of $\dim M = \infty$.

I. Let $P^M(S(T_\mu)) = M$. Denote $s_1 = \sup\{\mu([v]), v \in M\}$. We can choose a vector $u_1 \in S(T_\mu)$ and a non-zero vector $v_1 \in M$ such that $v_1 = P^M u_1$ and $\mu([v_1]) > s_1 - 1$. Denote $M^{(2)} = M - [v_1]$ and $s_2 = \sup\{\mu([v]), v \in M^{(2)}\}$. Then we can choose $u_2 \in S(T_\mu)$ and non-zero $v_2 \in M^{(2)}$ such that $v_2 = P^M u_2$ and $\mu([v_2]) > s_2 - \frac{1}{2}$, and so on. Thus, we can define by the induction sequences u_k of the elements of $S(T_\mu)$, v_k of mutually orthogonal non-zero elements of M and $M^{(k)}$ of elements of $\mathscr{L}(H)$ such that the following conditions 1.—3. are true:

1. $P^M u_k = v_k$, $(k = 1, 2, \ldots)$,
2. $M^{(1)} = M$, $M^{(k)} = M - \bigvee_{i=1}^{k-1} [v_i]$, $(k = 2, 3, \ldots)$, $v_k \in M^{(k)}$, $(k = 1, 2, \ldots)$,
3. $\mu([v_k]) > s_k - \frac{1}{k}$, where $s_k = \sup\{\mu([v]), v \in M^{(k)}\}$, $(k = 1, 2, \ldots)$.

Then Lemma 4.2 implies

$$v_k = P^M u_k = P^{M^{(k)}} u_k, \qquad (k = 1, 2, \ldots). \tag{1}$$

Further, put

$$M_n^{(1)} = M_n, \quad M_n^{(k)} = M_n^{(k-1)} - [P^{M_n^{(k-1)}} u_{k-1}], \qquad (n = 1, 2, \ldots, k = 2, 3, \ldots).$$

By the assumption, $\lim_{n \to \infty} P^{M_n} u_1 = P^M u_1$. Hence, by Lemma 4.1, $[P^{M_n} u_1] \to [P^M u_1]$ in $(\mathscr{L}(H), \mathscr{T}_{sc}(\mu))$. Then Lemma 2.2 implies that $M_n^{(2)} \to M^{(2)}$ in $(\mathscr{L}(H), \mathscr{T}_{sc}(\mu))$. From this it follows that $\lim_{n \to \infty} P^{M_n^{(2)}} u_2 = P^{M^{(2)}} u_2 = v_2$. Again by Lemma 4.1 and 2.2, $M_n^{(3)} \to M^{(3)}$ in $(\mathscr{L}(H), \mathscr{T}_{sc}(\mu))$, and so on. Thus, we can prove by induction that

$$M_n^{(k)} \to M^{(k)} \quad \text{for} \quad n \to \infty \quad \text{in} \quad (\mathscr{L}(H), \mathscr{T}_{sc}(\mu)), \qquad k = 1, 2, \ldots \tag{2}$$

In what follows, we shall prove the equality $M = \bigvee_{k=1}^{\infty} [v_k]$. In the opposite case, there would exist a non-zero vector $w \in M$, $w \perp v_k$, $k = 1, 2, \ldots$ Then the assumption $M = P^M(S(T_\mu))$ implies that $\mu([w]) > 0$. The finiteness of μ yields that $\lim_{k \to \infty} \mu([v_k]) = 0$. Hence, there exists integer p with the property $\mu([w]) >$ $> \mu([v_p]) + \frac{1}{p} > s_p = \sup\{\mu([v]), v \in M^{(p)}\}$. However, $w \in M^{(k)}$ $(k = 1, 2, \ldots)$, — a contradiction. The equality $M = \bigvee_{k=1}^{\infty} [v_k]$ is proved.

From the previous equality, from the lower continuity of μ and from the fact that $\mathscr{T}$ is a μ-topology, one obtains that $\lim_{k \to \infty} \varrho\left(\bigvee_{i=1}^{k} [v_i], M\right) = 0$. Let m be an arbitrary integer. Then there exists integer $k(m)$ such that $\varrho\left(\bigvee_{i=1}^{k(m)} [v_i], M\right) < \frac{1}{2m}$. Put $v_{n,k} = P^{M_n^{(k)}} u_k$, $(n, k = 1, 2, \ldots)$. By (1) and (2), $\lim_{n \to \infty} \|v_{n,k} - v_k\| = 0$, $k =$ $= 1, 2, \ldots$ Hence, according to Lemma 4.1, $[v_{n,k}] \to [v_k]$ for $n \to \infty$ in $(\mathscr{L}(H), \mathscr{T})$, $k = 1, 2, \ldots$ For $k_1 \neq k_2$, we have $[v_{n,k_1}] \perp [v_{n,k_2}]$ and $[v_{k_1}] \perp [v_{k_2}]$. Since $(\mathscr{L}(H), \mathscr{T})$ is a topological logic, we have $\bigvee_{i=1}^{k(m)} [v_{n,i}] \to \bigvee_{i=1}^{k(m)} [v_i]$ for $n \to \infty$ in $(\mathscr{L}(H), \mathscr{T})$. Then for m there exists $n(m)$ such that $\varrho\left(\bigvee_{i=1}^{k(m)} [v_{n(m),i}], \bigvee_{i=1}^{k(m)} [v_i]\right) < \frac{1}{2m}$. We see that $\varrho\left(\bigvee_{i=1}^{k(m)} [v_{n(m),i}], M\right) < \frac{1}{m}$. The latter consideration may be reproduced for every integer m. Let us choose indices $n(m)$ in such a way that $n(m)$ is an increasing sequence. Put $N_{n(m)} = \bigvee_{i=1}^{k(m)} [v_{n(m),i}]$. Then $N_{n(m)} \to M$ for $m \to \infty$ in $(\mathscr{L}(H), \mathscr{T})$. The μ-regularity of $\mathscr{T}$ implies that $N_{n(m)} \to M$ also in $(\mathscr{L}(H), \mathscr{T}_{sc}(\mu))$. Hence, according to Lemma 2.2, $M_{n(m)} - N_{n(m)} \to 0$ in $(\mathscr{L}(H), \mathscr{T}_{sc}(\mu))$. This implies $\lim_{m \to \infty} \mu \cdot$ $\cdot (M_{n(m)} - N_{n(m)}) = 0$. $\mathscr{T}$ is also a μ-topology. Thus, $M_{n(m)} - N_{n(m)} \to 0$ also in $(\mathscr{L}(H), \mathscr{T})$. Finally,

$$M_{n(m)} = (M_{n(m)} - N_{n(m)}) \vee N_{n(m)} \to 0 \vee M = M \quad \text{for} \quad m \to \infty \quad \text{in} \quad (\mathscr{L}(H), \mathscr{T}).$$

Summarizing, M_n contains a subsequence converging to M in $(\mathscr{L}(H), \mathscr{T})$.
II. Let us omit the assumption of $M = P^M(S(T_\mu))$. $P^M(S(T_\mu))$ is closed, i.e. it is an element of $\mathscr{L}(H)$. Let us prove that $M - P^M(S(T_\mu)) \perp S(T_\mu)$. In fact, if $u \in M - P^M(S(T_\mu))$ and $v \in S(T_\mu)$, then

$$(u, v) = (u, P^M v) + (u, P^{M^\perp} v)$$

and both scalar products on the right-hand side are evidently zero. It follows from the proved orthogonality that $\mu(M - P^M(S(T_\mu))) = 0$. Hence, according to the fact that $\mathscr{T}$ is a μ-topology, we have $\varrho(M, P^M(S(T_\mu))) = 0$. Thus, $M_n \to$ $\to M$ in $(\mathscr{L}(H), \mathscr{T})$ implies $M_n \to P^M(S(T_\mu))$. For the sake of simplicity, write $N = P^M(S(T_\mu))$. By Lemma 4.2, $N = P^N(S(T_\mu))$. Hence, from the step I of our proof we obtain the existence of a subsequence $M_{n(m)}$ converging to $N =$ $= P^M(S(T_\mu))$ in $(\mathscr{L}(H), \mathscr{T})$. Then the equality $\varrho(M, P^M(S(T_\mu))) = 0$ implies that $M_{n(m)} \to M$ in $(\mathscr{L}(H), \mathscr{T})$, too. The theorem is proved.

5. Topologies induced by measures defined on a general logic

As we have seen, the spaces $(\mathscr{S}, \mathscr{T}_{\varrho_\mu})$ and $(\mathscr{L}(H), \mathscr{T}_{sc}(\mu))$ have many similar or even identical properties. The following question naturally arises: Are $\mathscr{T}_{\varrho_\mu}$ and $\mathscr{T}_{sc}(\mu)$ special cases of a topology with similar properties, which is induced, in a way, by a measure defined on a general logic? As we shall see, the answer is yes.

Let μ be a finite measure on a logic $\mathscr{L}$. For every $\nu \in M_\mu(\mathscr{L})$ and every $\varepsilon > 0$, define a relation $U_{\nu, \varepsilon} = \{(a, b) \in \mathscr{L} \times \mathscr{L}, |\nu(a) - \nu(b)| < \varepsilon\}$. The family of all such relations is a prebase of a uniformity on $\mathscr{L} \times \mathscr{L}$ ([6], Theorem 6.3). The topology induced by this uniformity will be called *the topology of absolute continuity with respect to* μ and denoted by $\mathscr{T}_{ac}(\mu)$. Evidently, a net a_α converges to $a \in \mathscr{L}$ in $(\mathscr{L}, \mathscr{T}_{ac}(\mu))$ iff $\nu(a_\alpha) \to \nu(a)$ for every $\nu \in M_\mu(\mathscr{L})$.

We say that a signed measure on $\mathscr{L}$ possesses a Jordan decomposition if it can be written as a difference of two measures.

Theorem 5.1. *Let μ be a finite measure on $\mathscr{L}$. Then the following assertions are true:*

i) *$(\mathscr{L}, \mathscr{T}_{ac}(\mu))$ is a topological logic,*
ii) *$\mathscr{T}_{ac}(\mu)$ is a μ-topology,*
iii) *If every $\nu \in M_\mu(\mathscr{L})$ possesses a Jordan decomposition, then $\mathscr{T}_{ac}(\mu) \subset \mathscr{T}_d$,*
iv) *If $M_\mu(\mathscr{L})$ is a separable norm space with respect to the usual suprem norm, then $(\mathscr{L}, \mathscr{T}_{ac}(\mu))$ is pseudometrizable,*
v) *Every $\nu \in M_\mu(\mathscr{L})$ is a continuous map from $(\mathscr{L}, \mathscr{T}_{ac}(\mu))$ into R.*

Proof. i) If $\nu(a_\alpha) \to \nu(a)$ for every $\nu \in M_\mu(\mathscr{L})$, then also $\nu(a_\alpha^\perp) \to \nu(a^\perp)$. Similarly, if $\nu(a_\alpha) \to \nu(a)$ and $\nu(b_\alpha) \to \nu(b)$, where $a_\alpha \perp b_\alpha$ and $a \perp b$, then

$$\nu(a_\alpha \vee b_\alpha) = \nu(a_\alpha) + \nu(b_\alpha) \to \nu(a) + \nu(b) = \nu(a \vee b).$$

ii) Let $a_\alpha \leqq a$. If $a_\alpha \to a$ in $(\mathscr{L}, \mathscr{T}_{ac}(\mu))$, then $\mu(a_\alpha) \to \mu(a)$. Thus, $\mu(a - a_\alpha) \to 0$. On the other hand, if $\mu(a - a_\alpha) \to 0$, then $\nu(a - a_\alpha) \to 0$ for every $\nu \in M_\mu(\mathscr{L})$, i.e. $\nu(a_\alpha) \to \nu(a)$. Thus, $a_\alpha \to a$ in $(\mathscr{L}, \mathscr{T}_{ac}(\mu))$. Analogously in the case $a \leqq a_\alpha$.
iii) The proof is evident.
iv) Let $M_\mu(\mathscr{L})$ be separable with respect to the norm

$$\|\nu\| = \sup\{|\nu(a)|, a \in \mathscr{L}\}.$$

Let C be a countable dense subset of $M_\mu(\mathscr{L})$. The necessary and sufficient condition for pseudometrizability of $(\mathscr{L}, \mathscr{T}_{ac}(\mu))$ is the existence of a countable base for the uniformity inducing $\mathscr{T}_{ac}(\mu)$. Denote by $\mathscr{D}$ the family of all sets of the form $\bigcap_{i=1}^{k} U_{\nu_i, \varepsilon}$, where k is a positive integer, $\nu_i \in C$, $i = 1, \ldots, k$, and ε is a positive rational. Obviously, $\mathscr{D}$ is countable. Further, it will suffice to prove that for every $\nu \in M_\mu(\mathscr{L})$ and $\varepsilon > 0$ there exist $\pi \in C$ and a rational $\varepsilon_0 > 0$ such that $U_{\pi, \varepsilon_0} \subset U_{\nu, \varepsilon}$. Given $\nu \in M_\mu(\mathscr{L})$ and $\varepsilon > 0$, there exists positive rational $\varepsilon_0 < \frac{\varepsilon}{3}$ and $\pi \in C$ such that $\|\nu - \pi\| < \varepsilon_0$. Now, let $(a, b) \in U_{\pi, \varepsilon_0}$. Then

$$\begin{aligned} |\nu(a) - \nu(b)| &\leqq |\nu(a) - \pi(a)| + |\pi(a) - \pi(b)| + |\pi(b) - \nu(b)| \\ &\leqq 2\|\pi - \nu\| + |\pi(a) - \pi(b)| < 3\varepsilon_0 < \varepsilon. \end{aligned}$$

Thus, $(a, b) \in U_{\nu, \varepsilon}$. Summarizing, $\mathscr{D}$ is the base of the uniformity. The assertion is proved.
v) Proof is evident.
Theorem 5.1 is proved.

In what follows, we shall prove that as $\mathscr{T}_{\varrho_\mu}$ as $\mathscr{T}_{sc}(\mu)$ are special cases of $\mathscr{T}_{ac}(\mu)$.

Theorem 5.2. *If μ is a finite measure on the set σ-algebra $\mathscr{S}$, then $\mathscr{T}_{\varrho_\mu} = \mathscr{T}_{ac}(\mu)$.*

Proof. If $A_\alpha \to A$ in $(\mathscr{S}, \mathscr{T}_{\varrho_\mu})$, then it follows from the continuity of any $\nu \in M_\mu(\mathscr{S})$ that $\nu(A_\alpha) \to \nu(A)$, i.e. $A_\alpha \to A$ in $(\mathscr{S}, \mathscr{T}_{ac}(\mu))$. On the other hand, if $A_\alpha \to A$ in $(\mathscr{S}, \mathscr{T}_{ac}(\mu))$, then both $\mu(A_\alpha)$ and $\mu_A(A_\alpha)$ converge to $\mu(A)$. This implies that both $\mu(A \setminus A_\alpha)$ and $\mu(A_\alpha \setminus A)$ converge to 0. Thus, $\varrho_\mu(A_\alpha, A) \to 0$. The theorem is proved.

Further, μ denotes a finite measure defined on $\mathscr{L}(H)$, where $\dim H \geqq 3$.

Lemma 5.1. *$M(\mathscr{L}(H))$ is separable with respect to the suprem norm. Thus, $M_\mu(\mathscr{L}(H))$ is also separable.*

Proof. Let D be a countable dense subset of the unit sphere in H. Let C

be the set of all $\nu \in M(\mathscr{L}(H))$ of the form $\nu(M) = \sum_{i=1}^{k} \lambda_i (P^M \varphi_i, \varphi_i)$, $M \in \mathscr{L}(H)$, where λ_i are rationals, $\varphi_i \in D$, $i = 1, \ldots, k$, k is a positive integer. Then C is countable. Let us show the density of C in $M(\mathscr{L}(H))$. Let a measure $\pi \in M(\mathscr{L}(H))$ and $\varepsilon > 0$ be given. Then π is of the form $\pi(M) = \sum_{i=1}^{\infty} \xi_i (P^M \psi_i, \psi_i)$, where $\|\psi_i\| = 1$, $\xi_i \geqq 0$ and $\sum_{i=1}^{\infty} \xi_i < \infty$. There exists integer k such that $\sum_{i=k+1}^{\infty} \cdot \xi_i < \frac{\varepsilon}{2}$. Choose positive rationals λ_i and vectors $\varphi_i \in D$ such that $|\lambda_i - \xi_i| < \frac{\varepsilon}{4k}$ and $\|\varphi_i - \psi_i\| < \frac{\varepsilon}{8k\pi(H)}$, $i = 1, \ldots, k$. Define measure $\nu_1 \in C$ as follows: $\nu_1(M) = \sum_{i=1}^{k} \lambda_i (P^M \varphi_i, \varphi_i)$, $M \mu \mathscr{L}(H)$. Then, for every $M \in \mathscr{L}(H)$, we have

$$|\pi(M) - \nu_1(M)| \leqq \sum_{i=k+1}^{\infty} \xi_i + \left| \sum_{i=1}^{k} \xi_i (P^M \psi_i, \psi_i) - \lambda_i (P^M \varphi_i, \varphi_i) \right|$$

$$\leqq \frac{\varepsilon}{2} + \sum_{i=1}^{k} |\xi_i (P^M \psi_i, \psi_i) - \xi_i (P^M \varphi_i, \varphi_i)| + \sum_{i=1}^{k} |\xi_i (P^M \varphi_i, \varphi_i) - \lambda_i (P^M \varphi_i, \varphi_i)|$$

$$\leqq \frac{\varepsilon}{2} + \sum_{i=1}^{k} \xi_i |(P^M \psi_i, \psi_i) - (P^M \varphi_i, \psi_i)| + \sum_{i=1}^{k} \xi_i |(P^M \varphi_i, \psi_i) - (P^M \varphi_i, \varphi_i)| + \frac{\varepsilon}{4}$$

$$\leqq \frac{\varepsilon}{2} + \sum_{i=1}^{k} \xi_i \|P^M \psi_i - P^M \varphi_i\| + \sum_{i=1}^{k} \xi_i \|P^M \varphi_i\| \, \|\psi_i - \varphi_i\| + \frac{\varepsilon}{4}$$

$$\leqq \frac{\varepsilon}{2} + \sum_{i=1}^{k} \pi(H) \frac{\varepsilon}{8k\pi(H)} + \sum_{i=1}^{k} \pi(H) \frac{\varepsilon}{8k\pi(H)} + \frac{\varepsilon}{4} = \varepsilon.$$

Hence, $\|\pi - \nu_1\| \leqq \varepsilon$. Further, if π is a signed measure of $M(\mathscr{L}(H))$, then there exist measures π_1 and π_2 such that $\pi = \pi_1 - \pi_2$. As we have seen, there exist measures π_3, π_4 of C such that $\|\pi_1 - \pi_3\| < \frac{\varepsilon}{2}$ and $\|\pi_2 - \pi_4\| < \frac{\varepsilon}{2}$. Then $\pi_3 - \pi_4 \in C$ and $\|\pi - (\pi_3 - \pi_4)\| < \varepsilon$. Thus, C is dense in $M(\mathscr{L}(H))$. The lemma is proved.

Lemma 5.2. *$\mathscr{T}_{ac}(\mu)$ is μ-regular.*

Proof. We shall prove it only for the complex case. Let $\varphi \in S(T_\mu)$ and $\psi \in H$ be given. Define on $\mathscr{L}(H)$ a complex function $w_{\varphi, \psi}$ as follows:

$$w_{\varphi, \psi}(M) = (P^M \varphi, \psi), \qquad M \in \mathscr{L}(H).$$

If M_n is a sequence of orthogonal subspaces of $\mathscr{L}(H)$, then

$$w_{\varphi,\psi}\left(\bigvee_{n=1}^{\infty} M_n\right) = (P^{\vee M_n}\varphi, \psi) = \left(\sum_{n=1}^{\infty} P^{M_n}\varphi, \psi\right) =$$

$$= \sum_{n=1}^{\infty} (P^{M_n}\varphi, \psi) = \sum_{n=1}^{\infty} w_{\varphi,\psi}(M_n).$$

Thus, $w_{\varphi,\psi}$ is σ-additive. Obviously, the real and imaginary parts $\operatorname{Re} w_{\varphi,\psi}$, $\operatorname{Im} w_{\varphi,\psi}$ are bounded signed measures on $\mathscr{L}(H)$. Moreover, both of them belong to $M_\mu(\mathscr{L}(H))$. In fact, if $\lim \mu(M_n) = 0$ for a sequence $M_n \in \mathscr{L}(H)$, then, according to Lemma 3.2, $\lim_{n\to\infty} \|P^{M_n}\varphi\| = 0$. Hence, $\lim_{n\to\infty} w_{\varphi,\psi}(M_n) = 0$. Now, suppose that $M_n \to M$ in $(\mathscr{L}(H), \mathscr{T}_{ac}(\mu))$. Hence, we necessarily obtain

$$\lim_{n\to\infty} \operatorname{Re} w_{\varphi,\psi}(M_n) = \operatorname{Re} w_{\varphi,\psi}(M) \quad \text{and} \quad \lim_{n\to\infty} \operatorname{Im} w_{\varphi,\psi}(M_n) = \operatorname{Im} w_{\varphi,\psi}(M).$$

This implies that

$$\lim_{n\to\infty} (P^{M_n}\varphi, \psi) = (P^M\varphi, \psi). \tag{1}$$

Since ψ is an arbitrary vector of H, one obtains that the sequence $P^{M_n}\varphi$ converges weakly to $P^M\varphi$. Putting $\psi = \varphi$ in (1), we have $\lim_{n\to\infty} \|P^{M_n}\varphi\| = \|P^M\varphi\|$. This fact and the weak convergence imply the strong convergence $\lim_{n\to\infty} \cdot \|P^{M_n}\varphi - P^M\varphi\| = 0$ (see Problem 14 in [5]). The lemma is proved.

Theorem 5.3. *If μ is a finite measure on $\mathscr{L}(H)$, where* $\dim H \geqq 3$, *then* $\mathscr{T}_{sc}(\mu) = \mathscr{T}_{ac}(\mu)$.

P r o o f. It follows from Theorem 5.1, lemmas 5.1, 5.2 and from the uniqueness theorem 4.2.

6. The final version of the uniqueness theorem on $\mathscr{L}(H)$

In the preliminary uniqueness theorem 4.2, the condition iv) has a meaning only for $\mathscr{L}(H)$. In the following uniqueness theorem, this defect will be removed. We obtain the same uniqueness theorem as in the case of a measure defined on a set σ-algebra.

Theorem 6.1. *Let μ be a finite measure on $\mathscr{L}(H)$, where* $\dim H \geqq 3$. *Then there exists a unique topology $\mathscr{T}$ satisfying the following conditions* i)—iv):

i) *$\mathscr{T}$ is a μ-topology,*
ii) *$(\mathscr{L}(H), \mathscr{T})$ is a pseudometrizable topological logic,*
iii) *$\mathscr{T}$ is weaker than $\mathscr{T}_d$,*
iv) *Every $\nu \in M_\mu(\mathscr{L}(H))$ is a continuous map from $(\mathscr{L}(H), \mathscr{T})$ into R.*

This topology is $\mathscr{T}_{ac}(\mu) = \mathscr{T}_{sc}(\mu)$.

Proof. We know already that $\mathcal{T}_{ac}(\mu)$ $(=\mathcal{T}_{sc}(\mu))$ fulfils conditions i)—iv). Let $\mathcal{T}$ be an arbitrary topology satisfying i)—iv) and let $M_n \to M$ in $(\mathcal{L}(H), \mathcal{T})$. According to iv), $M_n \to M$ in $(\mathcal{L}(H), \mathcal{T}_{ac}(\mu))$, too. By Lemma 5.2, $\mathcal{T}_{ac}(\mu)$ is μ-regular, hence $M_n \to M$ in $(\mathcal{L}(H), \mathcal{T}_{sc}(\mu))$. Thus, $\mathcal{T}$ is also μ-regular.

Then from Theorem 4.2 it immediately follows that $\mathcal{T} = \mathcal{T}_{sc}(\mu)$. The theorem is proved.

REFERENCES

[1] DVUREČENSKIJ, A.: On convergences of signed states, Math. Slovaca 28, 1978, 289—295.

[2] GLEASON, A. M.: Measures on the closed subspaces of a Hilbert space, J. Math. Mech. 6, 1957, 885—893.

[3] GUDDER, S. P.: Spectral methods for a generalized probability theory, Trans. Amer. Math. Soc., 119, 1965, 428—442.

[4] HALMOS, P. R.: Measure Theory, Van Nostrand, Princeton, 1968.

[5] HALMOS, P. R.: A Hilbert Space Problem Book, Van Nostrand, Princeton, 1967.

[6] KELLEY, J. L.: General Topology, Van Nostrand, New York, 1955.

[7] SARYMSAKOV, T. A.—AJUPOV, Š. A.—CHADŽIJEV, DŽ.—ČILIN, V. I.: Uporjadočennyje algebry, FAN, Taškent, 1983.

[8] SCHATTEN, R.: Norm Ideals of Completely Continuous Operators, Springer, Berlin, 1970.

[9] VARADARAJAN, V. S.: Probability in physics and a theorem on simultaneous observability, Com. Pure Appl. Math., 15, 1962, 189—217.

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ТОПОЛОГИИ НА КВАНТОВОЙ ЛОГИКЕ ИНДУЦИРОВАННЫЕ МЕРАМИ

Vladimír Palko

Резюме

В классической теории меры известно, что если μ-конечная мера на σ-поле $\mathcal{S}$, то функция $\varrho_\mu : \varrho_\mu(A, B) = \mu(A \triangle B)$, $A, B \in \mathcal{S}$, является псевдометрикой. В работе занимаемся проблематикой топологии, порождённой конечной мерой на логике $\mathcal{L}$. Для меры μ на $\mathcal{L}$ определяется топологическое пространство $(\mathcal{L}, \mathcal{T})$. Пространство $(\mathcal{S}, \varrho_\mu)$ является специальным случаем этого пространства.