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ON A GAMBLER'S RUIN PROBLEM

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ABSTRACT. The object of this paper is to specify an explicit expression for the absorption probabilities for a random walk on the integers $-b, -b+1, \dots, -1, 0, 1, \dots, a$ which arises in a gambler's ruin problem proposed by the authors. There are two barriers, one absorbing at a and the other at $-b$, such that the random walk particle is returned to the position j, $-b \le j < a$, whenever reaches it.

1. Introduction

The classical problem of the gambler's ruin affords a classical illustration of the simple random walk with absorbing barriers. For example, Barnett (1964), Cox and Miller (1965), Feller (1968), Srinivasan and Mehata (1976), and Kannan (1979) have treated this problem. We consider here a more general problem where there are two barriers, one of which is absorbing and the other is such that the random walk particle is immediately returned to a certain position upon reaching it. We consider two players A and B with initial fortunes a and b dollars respectively. The game consists of a series of independent turns. Let X_u, $u \ge 1$, denote the B's winnings in the uth turn with

$$X_u = \begin{cases} 1 & \text{with probability } p, \\ -1 & \text{with probability } q, \\ 0 & \text{with probability } r, \end{cases}$$

where $p+q+r=1$ and $0<p,q<1$, $0 \le r < 1$. Thus, we have

$$S_n = \sum_{u=1}^{n} X_u$$

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which represents B's net gain at the end of n turns. If at any stage $S_n = -b$, B is ruined, player A immediately donates j dollars of his fortune to player B, $0 \le j < a$. The game ends when B has gained all A's fortune and A is ruined, i.e., $S_n = a$. Thus S_n is a random walk with two barriers, one of which, at a, is absorbing, and the other, at $-b$, such that the random walk particle is returned to the position j, $-b \le j < a$, whenever reaching it. In this paper, an explicit expression for the absorption probabilities is derived. Cox and Miller [3; p. 33] (1965) solved the classical random walk problem with absorbing barriers at the points $-b$ and a; however, the probability of absorption at time n is found to be erroneous. The correct formula can be found in Gulati and Hill [6] (1981). The special case, when the barriers are symmetrically placed on either side of the origin has been given by Munford [6] (1981). If we replace the absorbing barrier at $-b$ by an impenetrable (reflecting) barrier at $j = 0$, -1's are never accumulated if S_n reaches zero. The absorption probabilities have been deduced by El-Shehawy [4] (1992).

2. Generating function for the absorption probabilities

Let $f(n, a \mid i_0)$ denote the probability that the particle is absorbed at a at time n given that its initial position was i_0, $-b \le i_0 \le a$. The absorption probabilities $f(n, a \mid i_0)$ must satisfy

$$f(n,a \mid i_0) = pf(n-1, a \mid i_0+1) + rf(n-1, a \mid i_0) + qf(n-1, a \mid i_0-1) \qquad (n = 1, 2, \dots; \quad i_0 = -b+1, \dots, -1, 0, 1, \dots, a-1), \tag{1}$$

where

$$f(0, a \mid i_0) = 0 \quad (i_0 = -b, \dots, -1, 0, 1, \dots, a-1),$$
$$f(0, a \mid a) = 1, \qquad f(n, -b \mid i_0) = f(n, j \mid i_0) \quad (n = 1, 2, \dots).$$

Introducing the generating function

$$G_{i_0,a}(t) = \sum_{n=0}^{\infty} f(n, a \mid i_0)\, t^n. \tag{2}$$

Following Cox and Miller [3] (1965) we deduce that

$$G_{i_0,a}(t) = H_{i_0}(t)/H_a(t), \qquad i_0 = -b, \dots, -1, 0, 1, \dots, a, \tag{3}$$

where

$$H_{i_0}(t) = \big(h_1(t)\big)^{b+i_0} - \big(h_2(t)\big)^{b+i_0} + [q/p]^{b+i_0}\Big[\big(h_1(t)\big)^{j-i_0} - \big(h_2(t)\big)^{j-i_0}\Big],$$

and $h_1(t)$, $h_2(t)$ are the roots of the quadratic equation

$$pth^2 - (1-rt)h + qt = 0\,, \tag{4}$$

p, q and r are the probabilities of taking one step to the right, to the left and remaining in position, respectively. Writing $X(t) = 1 - rt$ and $Y(t) = \sqrt{4pqt^2 - (1-rt)^2}$, we have

$$h_{1,2}(t) = (2pt)^{-1}\big[X(t) \pm \mathrm{i}\ Y(t)\big]\,, \qquad \mathrm{i} = \sqrt{-1}\,,$$

and

$$\begin{aligned} H_{i_0}(t) &= (2pt)^{-(b+i_0)}\Big[\big(X(t)+\mathrm{i}\,Y(t)\big)^{b+i_0} - \big(X(t)-\mathrm{i}\,Y(t)\big)^{b+i_0}\Big] \\ &\quad + (2pt)^{i_0-j}[q/p]^{b+i_0}\Big[\big(X(t)+\mathrm{i}\,Y(t)\big)^{j-i_0} - \big(X(t)-\mathrm{i}\,Y(t)\big)^{j-i_0}\Big]\,. \end{aligned} \tag{5}$$

Accordingly, by expanding both the numerator and denominator of (3) in powers of $Y(t)$ and noting that only odd powers of $Y(t)$ occur, thus $Y(t)$ is cancelled, leaving a ratio of two polynomials in t, each of degree at most $a+b-1$, and consequently a partial fraction expansion of it is available. Using complex variables notation, we have

$$X(t) = |z|\cos\phi\,, \qquad Y(t) = |z|\sin\phi\,,$$

where

$$|z| = \sqrt{X^2(t) + Y^2(t)} = 2\sqrt{pq}t\,, \quad \text{and} \quad \phi = tan^{-1}\big(Y(t)/X(t)\big)\,.$$

It is useful to observe that

$$1 - rt = 2\sqrt{pq}t\cos\phi\,, \quad \text{or} \quad t = \big(r + 2\sqrt{pq}\cos\phi\big)^{-1}\,, \tag{6}$$

and $h_{1,2}(t) = \sqrt{q/p}e^{\pm\mathrm{i}\phi}$. Formula (3) becomes

$$G_{i_0,a}(t) = T_{i_0}(\phi)/T_a(\phi)\,, \tag{7}$$

where

$$T_{i_0}(\phi) = \big[\sqrt{p/q}\big]^{a-i_0}\Big[\big(\sqrt{p}\big)^{b+j}\sin(b+i_0)\phi + \big(\sqrt{q}\big)^{b+j}\sin(j-i_0)\phi\Big]\,.$$

The denominator of (7) is found to have $a+b-1$ distinct roots. A study of the function $\gamma(\phi)$, where

$$\gamma(\phi) = \frac{\sin(a+b)\phi}{\sin(a-j)\phi} - \big[\sqrt{q/p}\big]^{b+j} \tag{8}$$

shows that if $a-j = \big[\sqrt{p/q}\big]^{b+j}(b+a)$, the roots of the equation $\gamma(\phi) = 0$ give distinct roots of $T_a(\phi) \in [0,\pi)$. If $a-j < \big[\sqrt{p/q}\big]^{b+j}(b+a)$, there are $a+b-1$

distinct real roots ϕ_ν $(\nu = 1, 2, \dots, a+b-1)$ of (8). The corresponding roots of $H_a(t)$ are then

$$t_\nu = \frac{1}{r + 2\sqrt{pq}\cos\phi_\nu}, \qquad \nu = 1, 2, \dots, a+b-1.$$

If $a - j > \left[\sqrt{p/q}\right]^{b+j}(b+a)$, there are only $a+b-2$ distinct roots ϕ_ν $(\nu = 2, 3, \dots, a+b-1)$ that give distinct roots t_ν of $H_a(t)$. The remaining root of $H_a(t)$ is given by

$$t_1 = \frac{1}{r + 2\sqrt{pq}\cosh\phi_1},$$

where ϕ_1 is the unique root of the equation,

$$\sinh(a-j)\phi = \left[\sqrt{p/q}\right]^{b+j}\sinh(a+b)\phi.$$

3. Explicit expression for the absorption probabilities

THEOREM. *We have*

$$f(n, a \mid i_0) = -2\sqrt{pq}\Bigg[M^0(\phi_1) + \sum_{\nu=2}^{a+b-1} T_{i_0}(\phi_\nu)\left(\tfrac{\partial T_a(\phi)}{\partial\phi}\big|_{\phi=\phi_\nu}\right)^{-1} \cdot \\ \cdot \left(r + 2\sqrt{pq}\cos\phi_\nu\right)^{n-1}\sin\phi_\nu\Bigg], \tag{9}$$

where

$$M^0(\phi_1) = M(\phi_1)\cdot\begin{cases} \left[r + 2\sqrt{pq}\cos\phi_1\right]^{n+1}, & a-j < \left(\sqrt{p/q}\right)^{b+j}(a+b), \\ \left[1 - \left(\sqrt{p} - \sqrt{q}\right)^2\right]^{n+1}, & a-j = \left(\sqrt{p/q}\right)^{b+j}(a+b), \\ \left[r + 2\sqrt{pq}\cosh\phi_1\right]^{n+1}, & a-j > \left(\sqrt{p/q}\right)^{b+j}(a+b), \end{cases}$$

and

$$M(\phi_1) = \begin{cases} T_{i_0}(\phi_1)\left[\tfrac{\partial T_a(\phi)}{\partial\phi}\big|_{\phi=\phi_1}\right]^{-1}\left[r + 2\sqrt{pq}\cos\phi_1\right]^{-2}\sin\phi_1, \\ \qquad a-j < \left(\sqrt{p/q}\right)^{b+j}(a+b), \\ -2(a-i_0)(a+b)^{-1}(a-j)^{-1}(2a+b-j)^{-1}\left[r + 2\sqrt{pq}\right]^{-2}, \\ \qquad a-j = \left(\sqrt{p/q}\right)^{b+j}(a+b), \\ -T_{i_0}(\mathrm{i}\,\phi_1)\left[\tfrac{\partial T_a(\phi)}{\partial\phi}\big|_{\phi=\mathrm{i}\,\phi_1}\right]^{-1}\left[r + 2\sqrt{pq}\cosh\phi_1\right]^{-2}\sinh\phi_1, \\ \qquad a-j > \left(\sqrt{p/q}\right)^{b+j}(a+b). \end{cases}$$

P r o o f . Employing the partial fraction method, formula (3) can be written in the form

$$G_{i_0,a}(t) = \sum_{n=0}^{\infty}\left(\sum_{\nu=1}^{a+b-1} \frac{\eta_\nu}{t_{\nu+1}^{n+1}}\right) t^n ,$$

where the constants η_ν are given by,

$$\eta_\nu = -H_{i_0}(t)\left[\tfrac{\partial H_a(t)}{\partial t}\right]^{-1}\Big|_{t=t_\nu} . \tag{10}$$

Differentiating the quadratic equation (4) with respect to t we get

$$\frac{\partial h_{1,2}(t)}{\partial t} = t^{-2} h_{1,2}^2(t)\left[q - p h_{1,2}^2(t)\right]^{-1} . \tag{11}$$

The study of (8) with (3), (10) and (11) finally yield

$$\eta_\nu = \frac{-2\sqrt{pq}\left[\sin(b+i_0)\phi_\nu + \left(\sqrt{q/p}\right)^{b+j}\sin(j-i_0)\phi_\nu\right] t_\nu^2 \sin\phi_\nu}{\left(\sqrt{q/p}\right)^{a-i_0}\left[(a+b)\cos(a+b)\phi_\nu + \left(\sqrt{q/p}\right)^{b+j}(j-a)\cos(j-a)\phi_\nu\right]} ,$$
$$\nu = 2, 3, \dots, a+b-1 \tag{12}$$

and

$$\eta_1 = -2\sqrt{pq}M(\phi_1) ,$$

where

$$\phi_1 = \begin{cases} \cos^{-1}\dfrac{1-rt_1}{2\sqrt{pq}\,t_1} & \text{if } a-j \le \left(\sqrt{p/q}\right)^{b+j}(a+b) , \\ \cosh^{-1}\dfrac{1-rt_1}{2\sqrt{pq}\,t_1} & \text{if } a-j > \left(\sqrt{p/q}\right)^{b+j}(a+b) , \end{cases}$$

t_1 is the smallest root in absolute value of $H_a(t)$. Formula (9) follows. □

We see that with the appropriate change of notation, expression (9) agrees with that of M u n f o r d [8] (1981) in the case $j = 0$, $p > q$; and with that of W e e s a k u l's [10] (1961) and B l a s i [2] (1976) for the particular case $j = 0$, $b = 1$, $r = 0$, replacing i_0, a, ϕ_ν, η_ν with $b-u$, b, α_ν, ρ_ν and interchanging p and q respectively.

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