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1. Introduction

The aim of the present paper is to describe the structure of the lattices of congruences on finite lattices. It will be shown that the congruences on any finite lattice L correspond to some elements of special extension of the lattice L. This extension can be effectively constructed.

We shall use the following denotations:

$\mathrm{Con}(L)$...the lattice of all congruences on the lattice L

$\mathrm{con}(u, v)$...the least congruence on the lattice L which has the elements u, v in the same class

$\mathrm{Ir}(L)$...the set of all join-irreducible elements of the lattice L

$\mathrm{At}(L)$...the set of all atoms of the lattice L

$\mathrm{At}(p)$, where $p \in L$...the set of all atoms a L satisfying the condition $a \leqslant p$

$x^{\uparrow}$...the principal filter of the element x

$x^{\downarrow}$...the principal ideal of the element x

$\langle u, v \rangle$, where $u \leqslant v$...the interval in a lattice

$u -\!\!< v$...the element u is covered by v.

A finite lattice will be called join-atomic iff each its element can be written as a join of atoms.

2. Congruences on finite join-atomic lattices

Recall the well-known notions of distributive and standard elements of a lattice. If L is a lattice, $a \in L$, the element a is called *distributive* [*standard*] if

$$a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y)[x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y)]$$

for each x, $y \in L$. It is known that each standard element of a lattice is distributive and that distributive (standard) elements correspond to congruences if the given lattice is principally complemented (relatively complemented). Each finite relatively complemented lattice is principally complemented and each finite principally complemented lattice is join-atomic. It can be easily shown that in a finite join-atomic lattice each distributive element is standard. We prefer the notion of a distributive element since it is more convenient for the proofs. Further details on distributive and standard elements can be found in [1] or [2].

The set of all distributive elements in L will be denoted by $\mathrm{Dis}(L)$.

Theorem 2.1. *Let L be a lattice with the least element 0 and $p \in L$. Let us define the relation*

$$\Theta_p \quad \{(x, y) | x \in L,\ y \in L,\ x \vee y \leq p \vee (x \wedge y)\}.$$

(i) $p \in \mathrm{Dis}(L) \Leftrightarrow \Theta_p \in \mathrm{Con}(L)$

(ii) $p \in \mathrm{Dis}(L) \Rightarrow [(x, y) \in \Theta_p \Leftrightarrow p \vee x = p \vee y$, *for each* $x, y \in L]$

(iii) $p \in \mathrm{Dis}(L) \Rightarrow \mathrm{con}(0, p) - \Theta_p$

Proof. First assume $\Theta_p \in \mathrm{Con}(L)$. Trivially, $(0, p) \in \Theta_p$, therefore $(x, p \vee x) \in \Theta_p$, $(y, p \vee y) \in \Theta_p$, $(x \wedge y, (p \vee x) \wedge (p \vee y)) \in \Theta_p$ for each $x, y \in L$. From the last inclusion and the definition of Θ_p we get $(p \vee x) \wedge (p \vee y) \leq p \vee (x \wedge y)$. Since the inverse inequality is a lattice identity, $p \in \mathrm{Dis}(L)$.

Further we assume $p \in \mathrm{Dis}(L)$. First we prove (ii). Assume $(x, y) \in \Theta_p$, then $x \vee y \leq p \vee (x \wedge y)$. This inequality is equivalent to $p \vee (x \wedge y) = (p \vee (x \wedge y)) \vee (x \vee y) = p \vee x \vee y$. Since the elements $p \vee x$, $p \vee y$ are in the interval $\langle p \vee (x \wedge y), p \vee x \vee y \rangle$, we have $p \vee x \quad p \vee y$. Conversely, if $p \vee x = p \vee y$, there is

$$x \vee y \leq p \vee x \vee y = p \vee x = p \vee y - (p \vee x) \wedge (p \vee y) = p \vee (x \wedge y)$$

and $(x, y) \in \Theta_p$. From (ii) it follows that Θ_p is an equivalence relation and it is $\vee$-stable. Now we prove that Θ_p is also $\wedge$-stable.

Choose $(x, y) \in \Theta$, $z \in L$; then $(x \wedge z) \vee (y \wedge z) \leq (x \vee y) \wedge z \leq (p \vee (x \wedge y)) \wedge (p \vee z) = p \vee [(x \wedge y) \wedge z] \quad p \vee [(x \wedge z) \wedge (y \wedge z)]$, therefore $(x \wedge z, y \wedge z) \in \Theta_p$.

There remains to be proved (iii). We denote $\lambda = \mathrm{con}(0, p)$. As $(0, p) \in \Theta_p$, there is $\lambda \subseteq \Theta_p$. Choose $(x, y) \in \Theta_p$; then $x \vee y$ is in the interval $\langle x \wedge y, p \vee (x \wedge y) \rangle$ and since $(x \wedge y, p \vee (x \wedge y)) \in \lambda$, there is $(x \wedge y, x \vee y) \in \lambda$, i.e. $(x, y) \in \lambda$.

Definition 2.1. *Let L be a finite join atomic lattice. We call a subset $B \subseteq \mathrm{At}(L)$ closed iff it has the following two properties:*

(i) $a \in \mathrm{At}(L),\ a \leq \vee B \Rightarrow a \in B$

(ii) $a \in \mathrm{At}(L),\ b \in B,\ c \in L,\ a \leq b \vee c,\ a \nleq c \Rightarrow a \in B$

Remark. Trivially, (i) and (ii) are properties of the "closure type", therefore they define two closure operators on $\mathrm{At}(L)$, which will be denoted by σ_1 and σ_2. The conjunction of (i) and (ii) is of the "closure type" again, the corresponding closure operator will be denoted by σ. The operator σ_1 can be described very simply: $\sigma(M) = \mathrm{At}(\vee M)$ for each $M \subset \mathrm{At}(L)$. Later we shall describe the operator σ_2, too.

Lemma 2.1. *Let L be a finite join-atomic lattice, $p \in L$ and let* $\mathrm{At}(p)$ *be closed. Then* $\mathrm{At}(p \vee t) = \mathrm{At}(p) \cup \mathrm{At}(t)$ *for each* $t \in L$.

Proof. The inclusion $\supseteq$ is trivial, we shall prove $\subseteq$. Choose $a \in \mathrm{At}(p \vee t) - \mathrm{At}(t)$; then $a \in \mathrm{At}(L)$, $a \not\leq t$, $a \leq p \vee t$. The last inequality can be written in another form:

$$a \leq t \vee (\vee \mathrm{At}(p)).$$

As L is finite, we can find a minimal subset $Q \subseteq \mathrm{At}(p)$ such that

$$a \leq t \vee (\vee Q).$$

There is $Q \neq \emptyset$, since $a \not\leq t$. Choose $b \in Q$ and denote

$$Q' = Q - \{b\}, \quad c = t \vee (\vee Q').$$

Then $a \in \mathrm{At}(L)$, $b \in \mathrm{At}(p)$, $c \in L$, $a \leq b \vee c$, $a \not\leq c$ (the last fact is a consequence of the minimality of Q) and as $\mathrm{At}(p)$ is closed, we have $a \in \mathrm{At}(p)$, q.e.d.

Theorem 2.2. *Let L be a finite join-atomic lattice, $p \in L$. Then $p \in \mathrm{Dis}(L) \Leftrightarrow$ $\mathrm{At}(p)$ is closed.*

Proof. Assume $p \in \mathrm{Dis}(L)$. As L is join-atomic, $p = \vee \mathrm{At}(p)$ and $\mathrm{At}(p)$ has the property (i) from Definition 2.1. Now we prove that it has also the property (ii). Assume that

$$a \in \mathrm{At}(L), \quad b \in \mathrm{At}(p), \quad c \in L, \quad a \leq b \vee c, \quad a \not\leq c.$$

We have $a \leq p \vee c$, since $b \leq p$ and $a \leq b \vee c$, and $a \wedge c = 0$ (the least element of L), since $a \in \mathrm{At}(L)$ and $a \not\leq c$. Therefore

$$p = p \vee (a \wedge c) = (p \vee a) \wedge (p \vee c) = p \vee a, \quad \text{i.e.} \quad a \leq p, \quad a \in \mathrm{At}(p).$$

We proved the implication $\Rightarrow$. The implication $\Leftarrow$ is a consequence of Lemma 2.1 and the trivial fact $\mathrm{At}(u \wedge v) = \mathrm{At}(u) \cap \mathrm{At}(v)$ for each $u, v \in L$.

Lemma 2.2. *Let x, y be elements of a lattice L with the least element 0. Then*

$$\mathrm{con}(0, x) \vee \mathrm{con}(0, y) = \mathrm{con}(0, x \vee y)$$

holds in $\mathrm{Con}(L)$.

Lemma 2.3. *Let L be a finite join-atomic lattice with the least element 0 and $\Theta \in \mathrm{Con}(L)$. Then $\Theta = \mathrm{con}(0, t)$ for some $t \in L$.*

Proof. Trivially, $\Theta = \vee \{\mathrm{con}(u, v) | u \in L, v \in L, u -\!\!< v, u\Theta v\}$. For such u, v there exists $a_{u,v} \in \mathrm{At}(L) \cap (v^{\downarrow} - u^{\downarrow})$, since L is join-atomic. The intervals $\langle u, v\rangle$ and $\langle 0, a_{u,v}\rangle$ are transposes, therefore $\mathrm{con}(u, v) = \mathrm{con}(0, a_{u,v})$. By Lemma 2.2 it suffices to take $t = \vee \{a_{u,v} | u \in L, v \in L, u -\!\!< v, u\Theta v\}$.

Lemma 2.4. *Let L be a finite join-atomic lattice with the least element* 0 *and* $t \in L$. *Then* $\operatorname{con}(0, t) = \operatorname{con}(0, p)$ *for some* $p \in \operatorname{Dis}(L)$.

Proof. We construct a sequence of sets

$$B_0 = \operatorname{At}(t), \quad B_1 = \sigma_1(B_0), \quad B_2 = \sigma_2(B_1),$$
$$B_3 = \sigma_1(B_2), \ldots, B_{2n} = \sigma_2(B_{2n-1}), \quad B_{2n+1} = \sigma_1(B_{2n}), \ldots.$$

Trivially, $B_0 \subseteq B_1 \subseteq B_2 \subseteq B_3 \subseteq \ldots$. As L is finite, this sequence becomes stable on the value $\sigma(B_0)$. If we denote $t_i = \vee B_i$, there is $t = t_0 \leqslant t_1 \leqslant t_2 \leqslant \ldots$ and $\operatorname{con}(0, t) = \operatorname{con}(0, t_0) \subseteq \operatorname{con}(0, t_1) \subseteq \ldots$. As $x = \vee \operatorname{At}(x)$ for each $x \in L$, $t_{2n+1} = \vee B_{2n+1} = \vee \sigma_1(B_{2n}) = \vee \operatorname{At}(\vee B_{2n}) = \vee \operatorname{At}(t_{2n}) = t_{2n}$ and $\operatorname{con}(0, t_{2n}) = \operatorname{con}(0, t_{2n+1})$. If we prove that $\operatorname{con}(0, t_{2n-1}) = \operatorname{con}(0, t_{2n})$ for each n, it will be trivial that the lemma holds for $p = \vee \sigma(B_0)$.

Let us define an operator τ (in general, it is not a closure operator) in the following way:

If S has the property (ii) from Definition 2.1, $\tau(S) = S$, if S has not the property (ii), there exist elements $a \in \operatorname{At}(L)$, $b \in S$, $c \in L$ such that $a \leqslant b \vee c$, $a \nleqslant c$, $a \notin S$. We choose one such triple (a, b, c) and take $\tau(S) = S \cup \{a\}$.

We construct a sequence of sets

$$S_0 = B_{2n-1}, \quad S_1 = \tau(S_0), \quad S_2 = \tau(S_1), \quad S_3 = \tau(S_2), \ldots$$

and a sequence of elements of L

$$s_0 = \vee S_0, \quad s_1 = \vee S_1, \quad s_2 = \vee S_2, \quad s_3 = \vee S_3, \ldots.$$

As $S_0 \subseteq S_1 \subseteq S_2 \subseteq S_3 \subseteq \ldots$ and L is finite, this sequence becomes stable on the value $\sigma_2(S_0) = \sigma_2(B_{2n-1})$. As $t_{2n-1} = s_0 \leqslant s_1 \leqslant s_2 \leqslant s_3 \leqslant \ldots$, $\operatorname{con}(0, t_{2n-1}) = \operatorname{con}(0, s_0) \subseteq \operatorname{con}(0, s_1) \subseteq \operatorname{con}(0, s_2) \subseteq \operatorname{con}(0, s_3) \subseteq \ldots$ and this sequence becomes stable on the value $\operatorname{con}(0, t_{2n})$. It suffices to prove that $\operatorname{con}(0, s_i) \supseteq \operatorname{con}(0, s_{i+1})$ for each i. Let us denote $\lambda = \operatorname{con}(0, s_i)$. If $S_{i+1} = S_i$, there is $s_{i+1} = s_i$ and trivially $(0, s_{i+1}) \in \lambda$. If $S_{i+1} \neq S_i$, there exist elements $a \in \operatorname{At}(L)$, $b \in S_i$, $c \in L$ such that $a \leqslant b \vee c$, $a \nleqslant c$, $a \notin S_i$, $S_{i+1} = S_i \cup \{a\}$, therefore $s_{i+1} = s_i \vee a$. As $B \leqslant \vee S_i = s_i$, there is $(0, b) \in \lambda$ and $(c, b \vee c) \in \lambda$. As $c \leqslant a \vee c \leqslant b \vee c$, there is $(c, a \vee c) \in \lambda$. As $a \wedge c = 0$, there is $(0, a) \in \lambda$. Therefore $(0, s_{i+1}) = (0, s_i \vee a) \in \lambda$, q.e.d.

Theorem 2.3. *Let L be a finite join-atomic lattice. Then* $\operatorname{Dis}(L)$ *is a sublattice of* L *and the assignment* $p \mapsto \Theta_p$ *defines an isomorphism of lattices* $\operatorname{Dis}(L) \to \operatorname{Con}(L)$.

Proof. It is known that $\operatorname{Dis}(L)$ is a $\vee$-subsemilattice of L. By Theorem 2.2, $p, q \in \operatorname{Dis}(L) \Rightarrow \operatorname{At}(p), \operatorname{At}(q)$ are closed $\Rightarrow \operatorname{At}(p \wedge q) = \operatorname{At}(p) \cap \operatorname{At}(q)$ is closed $\Rightarrow p \wedge q \in \operatorname{Dis}(L)$ and therefore $\operatorname{Dis}(L)$ is a sublattice of L. The assignment $p \mapsto \Theta_p$ is injective since $p^{\downarrow} = [0]\Theta_p$. By Theorem 2.1 and Lemma 2.2, $\Theta_{p \vee q} = \operatorname{con}(0, p \vee q) = \operatorname{con}(0, p) \vee \operatorname{con}(0, q) = \Theta_p \vee \Theta_q$ and it suffices to prove the surjectivity. Choose $\lambda \in \operatorname{Con}(L)$. By Lemma 2.3, $\lambda = \operatorname{con}(0, t)$ for some $t \in L$. By Lemma 2.4, $\lambda = \operatorname{con}(0, p) = \Theta_p$ for some $p \in \operatorname{Dis}(L)$.

Lemma 2.5. *Let L be a finite join-atomic lattice. If A_1, A_2 are closed subsets of* $\mathrm{At}(L)$, *the set $A_1 \cup A_2$ is closed.*

Proof. $A_1 = \mathrm{At}(p)$, $A_2 = \mathrm{At}(q)$, where $p = \vee A_1$, $q = \vee A_2$. By Theorem 2.2 p and q are distributive, and by Lemma 2.1 and the distributivity of $p \vee q$, $A_1 \cup A_2 = At(p) \cup \mathrm{At}(q) = \mathrm{At}(p \vee q)$ is closed.

Corollary. *The closure operator σ is topological.*

3. Congruence-preserving extensions of lattices

We shall construct a special extension of lattices which will make it possible to apply the results of the preceding paragraph to any finite lattice.

Theorem 3.1. *Let L be a finite lattice with the least element* 0, $u \in \mathrm{Ir}(L) - \mathrm{At}(L)$, $u_0 -\!\!< u$ *and let L' be a set disjoint to L such that there exists a bijection*

$$L - u_0^{\uparrow} \to L', \quad x \mapsto x''.$$

Define the relation $\leqslant$ on the set $L'' = L \cup L'$ in the following way:

$x \leqslant y$ *for* $x, y \in L$ *is defined as in* L,
$x \leqslant y'' \Leftrightarrow x \leqslant y$ *for all* $x \in L$, $y \in L - u_0^{\uparrow}$,
$x'' \leqslant y \leftrightarrow x \vee u \leqslant y$ *for all* $x \in L - u_0^{\uparrow}$, $y \in L$,
$x'' \leqslant y'' \Leftrightarrow x \leqslant y$ *for all* $x, y \in L - u_0^{\uparrow}$.

Then $\leqslant$ is an order (reflexive, antisymmetric and transitive binary relation) on L'' and $(L'', \leqslant)$ is a lattice in which L is a sublattice. Moreover,

$$\mathrm{At}(L'') = \mathrm{At}(L) \cup \{0''\}, \quad \mathrm{Ir}(L'') = (\mathrm{Ir}(L) - \{u\}) \cup \{0''\}.$$

The proof is not interesting, it is necessary to distinguish a lot of cases for the elements of L''. It can be easily proved that the lattice operations on L'' are the following ones:

$$\begin{aligned}
&x \vee y,\ x \wedge y \quad \text{are defined as in } L \text{ if} \quad x, y \in L,\\
&x \vee y'' = y'' \vee x = (x \vee y)'' \quad \text{if} \quad x \vee y \in L - u_{00}^{\uparrow},\\
&\qquad = x \vee y \vee u \quad \text{if} \quad x \vee y \in u_0^{\uparrow},\\
&x'' \vee y'' = x \vee y'',\\
&x \wedge y'' = y'' \wedge x = (x \wedge y)'' \quad \text{if} \quad x \in u^{\uparrow},\\
&\qquad x \wedge y \quad \text{if} \quad x \in L - u^{\uparrow},\\
&x'' \wedge y'' = (x \wedge y)''.
\end{aligned}$$

Definition 3.1. *The lattice L'' constructed in Theorem 3.1 will be called a simple extension of the lattice L.*

Theorem 3.2. *Let L be a finite lattice with the least element* 0, u, $v \in \mathrm{Ir}(L) - (\mathrm{At}(L) \cup \{0\})$. *Then the lattices* $(L^u)^v$ *and* $(L^v)^u$ *are isomorphic.*

Proof. The assignment

$x \mapsto x$ for $x \in L$,

$x^u \mapsto x^u$ for $x \in L - u_0\uparrow$, where $u_0 -\!< u$,

$x^v \mapsto x^v$ for $x \in L - v_0^\uparrow$, where $v_0 -\!< v$,

$(x^u)^v \mapsto (x^v)^u$ for $x \in L - (u_0^\uparrow \cup v_0^\uparrow)$

defines an isomorphism of lattices $(L^u)^v \to (L^v)^u$.

Definition 3.2. *Let L_1 be a sublattice of a lattice L_2. The extension $L_1 \subseteq L_2$ will be called congruence-preserving iff the assignment $\Theta \mapsto \Theta \cap (L_1 \times L_1)$ defines an isomorphism of lattices* $\mathrm{Con}(L_2) \to \mathrm{Con}(L_1)$.

Remark. If $L_1 \subseteq L_2$ and $L_2 \subseteq L_3$ are congruence-preserving extensions, $L_1 \subseteq L_3$ is congruence-preserving. Trivially, $L \subseteq L$ is always congruence-preserving.

Theorem 3.3. *Each simple extension of lattices is congruence-preserving.*

Proof. Let $L \subseteq L^u$ be a simple extension of lattices, $0 \neq u_0 -\!< u \in \mathrm{Ir}(L)$. The element u_0 is uniquely determined by u. Define a mapping φ: $\mathrm{Con}(L^u) \to \mathrm{Con}(L)$, $\varphi(\Theta) = \Theta \cap (L \times L)$. Trivially, this definition is correct and φ preserves the intersections. There suffices to prove that φ is injective and surjective.

Choose two different congruences Θ_1, Θ_2 on L^u; then e.g. $\Theta_2 \nsubseteq \Theta_1$. There exist elements a, $b \in L^u$ such that

$$a -\!< b, \quad (a, b) \in \Theta_2, \quad (a, b) \notin \Theta_1.$$

There are four possibilities:

(a) $a \in L$, $b \in L$; then $(a, b) \in \varphi(\Theta_2) - \varphi(\Theta_1)$, $\varphi(\Theta_1) \neq \varphi(\Theta_2)$.

(b) $a \in L'$, $b \in L'$; then $a = x^u$, $b = y^u$ for some x, $y \in L - u_0^\uparrow$. As the intervals $\langle a, b \rangle$ and $\langle x, y \rangle$ are transposes, they are collapsed by the same congruences and $(x, y) \in \varphi(\Theta_2) - \varphi(\Theta_1)$.

(c) $a \in L$, $b \in L'$; then $b = y^u$ for some $y \in L - u_0^\uparrow$. As $a \leq y -\!< y^u = b$ and $a -\!< b$, there is $a = y$, $(y, y^u) \in \Theta_2 - \Theta_1$. The intervals $\langle y, y^u \rangle$ and $\langle u_0, u \rangle$ are projective (consider the interval $\langle 0, 0^u \rangle$!), therefore $(u_0, u) \in \varphi(\Theta_2) - \varphi(\Theta_1)$.

(d) $a \in L'$, $b \in L$; then $a = x^u$ for some $x \in L - u_0^\uparrow$ and $a = x^u < x \vee u \leq b$, therefore $b = x \vee u$ and we have $(x^u, x \vee u) \in \Theta_2 - \Theta_1$. The intervals $\langle x^u, x \vee u \rangle$ and $\langle x \wedge u_0, u_0 \rangle$ are transposes, therefore $(x \wedge u_0, u_0) \in \varphi(\Theta_2) - \varphi(\Theta_1)$.

There remains to be proved the surjectivity. Take $\lambda \in \mathrm{Con}(L)$. We shall construct $\Theta \in \mathrm{Con}(L^u)$ such that $\lambda = \varphi(\Theta) = \Theta \cap (L \times L)$. There are two possibilities:

(a) If $(u_0, u) \in \lambda$, we define

$$(x, y) \in \Theta \Leftrightarrow (x, y) \in \lambda \quad \text{for} \quad x, y \in L,$$
$$(x, y^u) \in \Theta \Leftrightarrow (y^u, x) \in \Theta \Leftrightarrow (x, y) \in \quad \text{for} \quad x \in L, y \in L - u_0^\uparrow,$$
$$(x^u, y^u) \in \Theta \Leftrightarrow (x, y) \in \lambda \quad \text{for} \quad x, y \in L - u_{00}^\uparrow.$$

It is easy to prove that $\Theta\in\mathrm{Con}(L^u)$ and $\varphi(\Theta)=\lambda$.

(b) If $(u_0, u)\notin\lambda$, we define

$$(x, y)\in\Theta\Leftrightarrow(x, y)\in\lambda \quad \text{for} \quad x, y\in L,$$
$$(x, y^u)\in\Theta\Leftrightarrow(y^u, x)\in\Theta\Leftrightarrow[x=u \text{ and } (u_0, y)\in\lambda]$$
$$\text{for} \quad x\in L,\ y\in L-u_0^{\uparrow},$$
$$(x^u, y^u)\in\Theta\Leftrightarrow(x, y)\in\lambda \quad \text{for} \quad x, y\in L-u_0^{\uparrow}.$$

Now the stability of Θ is not trivial. We shall consider one of the interesting cases: $(x, y^u)\in\Theta$, x, $z\in L$, $y\in L-u_0^{\uparrow}$; we want to prove that $(x\wedge z,\ y^u\wedge z)\in\Theta$.

By the definition of Θ, $x=u$ and $(u_0, y)\in\lambda$. There are two possibilities:

(1) $z\in u^{\uparrow}$; then

$$x\wedge z=u\wedge z=u,\ y^u\wedge z=(y\wedge z)^u.$$

By the $\wedge$-stability of λ, $(u_0\wedge z,\ y\wedge z)\in\lambda$, but $u_0\wedge z=u_0\wedge u\wedge z=u_0$, therefore $(x\wedge z, (y\wedge z)^u) = (x\wedge z,\ y^u\wedge z)\in\Theta$.

(2) $z\in L-u^{\uparrow}$; then $u\wedge z<u$ and from the join-irreducibility of u we get $u\wedge z\leqq u_0$, $u_0\wedge z = u_0\wedge u\wedge z = u\wedge z = x\wedge z$, $y^u\wedge z = y\wedge z$, Therefore $(x\wedge z,\ y^u\wedge z) = (u_0\wedge z,\ y\wedge z) \in \lambda\subseteq\Theta$.

Remark. Applying the construction of a simple extension to a finite lattice L as often as possible we get an embedding of L into some join-atomic lattice $\bar{L}$. The extension $L\subseteq\bar{L}$ is congruence-preserving and by Theorem 2.3 we get an isomorphism of lattices $\mathrm{Dis}(\bar{L})\rightarrow\mathrm{Con}(L)$.

By Theorem 3.2 up to the isomorphism the lattice $\bar{L}$ does not depend on the order in which the elements of $\mathrm{Ir}(L)-(\mathrm{At}(L)\cup\{0\})$ are used in its construction.

The distributive elements of the lattice $\bar{L}$ can be found by Theorem 2.2: first we find $\sigma(\{a\})$ for all $a\in\mathrm{At}(\bar{L})$, the elements $\vee\sigma(\{a\})$ and joins of such elements are all distributive elements of $\bar{L}$.

The construction of a simple extension has some interesting properties: it preserves the distributivity of a lattice (if L is distributive, $\bar{L}$ is a Boolean algebra) but does not preserve, e.g., the meet-semidistributivity the property defined by the quasiidentity $a=b\wedge c=b\wedge d \Rightarrow a=b\wedge(c\vee d)$ — it suffices to extend the pentagon.

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КОНГРУЭНЦИИ НА КОНЕЧНЫХ СТРУКТУРАХ

Богуслав Сивак

Резюме

В статье доказаны три основных результата. Два относятся к конечным структvрам, все элементы которых представимы в виде суммы атомов: 1) дистрибутивные элементы таких структур находятся во взаимно однозначном соответствим со замкнутыми множествами некоторого оператора замыкания, 2) конгруэнции на таких структурах находятся во взаимно однозначном соответствии с дистрибутивными элементами.

Наконец показывается, как для любой конечной структуры построить такое конечное расширение, которое имеет изоморфную структуру конгруэнций и все элементы которого представимы в виде суммы атомов.