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ON MULTISEQUENCES AND THEIR APPLICATION TO PRODUCTS OF SEQUENTIAL SPACES

SALIOU SITOU

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ABSTRACT. Michael's theorem on sequentiality of products, Arhangel'skii's theorem on the Fréchetness and Nogura-Shibakov estimate of the sequential order of products admit a common generalization, obtained with the aid of multisequences.

1. Introduction

Let X be a topological space and let $A \subset X$. We say that A is *sequentially closed* if no sequence in A converges to a point outside A. Let the *sequential adherence* of A be the set of limits of sequences in A. From now on, cl and $\mathrm{cl}_{\mathrm{seq}}$ denote respectively the closure and the *sequential adherence* operation in X. So let $\mathrm{cl}^0_{\mathrm{seq}} A = A$ and for each ordinal $\alpha > 0$, $\mathrm{cl}^{\alpha}_{\mathrm{seq}} A = \mathrm{cl}_{\mathrm{seq}}\Big(\bigcup\limits_{\beta<\alpha} \mathrm{cl}^{\beta}_{\mathrm{seq}} A\Big)$. It is well known that $\mathrm{cl}^{\omega_1}_{\mathrm{seq}} A = \mathrm{cl}^{\omega_1+1}_{\mathrm{seq}} A$.[1] The *sequential closure* of A is by definition $\mathrm{cl}_{\mathrm{Tseq}} A = \mathrm{cl}^{\omega_1}_{\mathrm{seq}} A$. A subset A is sequentially closed if and only if $A = \mathrm{cl}_{\mathrm{Tseq}} A$. Sequentially closed sets yield a topology. A topology is *sequential* if it coincides with the topology generated by its sequentially closed subsets. More precisely, a topology is sequential if and only if for all $A \subset X$, $\mathrm{cl}\, A \subset \mathrm{cl}_{\mathrm{Tseq}} A$. The *sequential order* $\sigma(x; A)$ of a point $x \in X$ relative to a subset A is by definition the least ordinal α such that $x \in \mathrm{cl}^{\alpha}_{\mathrm{seq}} A$. The sequential order $\sigma(x)$ of x is: $\sigma(x) = \sup\{\sigma(x; A),\ A \subset X \text{ such that } x \in \mathrm{cl}^{\omega_1}_{\mathrm{seq}} A\}$. The sequential order $\sigma(X)$ of the whole space is $\sigma(X) = \sup\limits_{x\in X} \sigma(x)$. Recall that a Fréchet (Fréchet-Urysohn) space is a space in which $\mathrm{cl}\, A = \mathrm{cl}_{\mathrm{seq}} A$ for each subset A.

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[1] ω_1 is the first uncountable ordinal.

E. Michael proved in [10] that the product of a sequential space with a regular locally countably compact sequential space is sequential. A. V. Arhangel'skii proved in [2] that the product of an *infinitely subtransversal* (the definition will be given in Section 3) Fréchet space with a regular locally countably compact Fréchet space is Fréchet. The aim of this article is to show that the use of multisequences enables us to provide simple proofs of some classical theorems such as the above quoted theorems of E. Michael and A. V. Arhangel'skii. In fact, this new method gives precise bounds for the sequential order of the product of a sequential space with a sequential and regular locally countably compact space and enables us to generalize the above theorem of Arhangel'skii. The results of this article form part of the third chapter of my Ph.D thesis [12] written under the supervision Professor S. Dolecki.

2. Multisequences

Let $\mathrm{Seq} = \Big(\bigcup\limits_{n\in\mathbb{N}} \mathbb{N}^n, \sqsubseteq\Big)$ be the set of finite sequences of natural numbers ordered by concatenation. For s and t in Seq, we denote (s,t) the concatenation of s and t (hence, if $s = (n_0, n_1, n_2, \dots, n_k)$ and $t = (m_0, m_1, \dots, m_p)$ then $(s,t) = (n_0, n_1, \dots, n_k, m_0, m_1, m_2, \dots, m_p))$ and $s \sqsubseteq t$ (resp. $s \sqsubset t$), if there exists $q \in \mathrm{Seq}$ (resp. q in Seq, q different from the empty sequence) such that $t = (s,q)$. The order $\sqsubseteq$ is well-founded and its length function is: $l(\emptyset) = 0$ and for $s = (n_0, n_1, \dots, n_k)$, $l(s) = k+1$.

DEFINITION 2.1. A subtree $T \subset \mathrm{Seq}$ is called *well-capped* ([5]) if each of its non empty subsets has a maximal element.

DEFINITION 2.2. An *index tree* is a well-capped subtree T of Seq with the following properties:

$$\begin{cases} \forall\limits_{s,t\in\mathrm{Seq}} & t \in T \text{ and } s \sqsubseteq t \implies s \in T\,; \\ \forall\limits_{t\in T}\ \exists\limits_{n\in\mathbb{N}} & (t,n) \in T \implies \forall\limits_{n\in\mathbb{N}} (t,n) \in T\,. \end{cases}$$

Remark 2.3. If T is an index tree then $(T, \sqsupseteq)$ is well-founded. So it possesses a rank function that fulfils,

$$r(t) = r(t,T) = \min\Big\{\alpha \in \mathrm{Seq} : \forall\limits_{s\sqsupset t} r(s) < \alpha\Big\}\,.$$

If T is an index tree, then for all $t \in T$, the length of t is finite.

DEFINITION 2.4. The *rank* of an index tree T is by definition

$$r(T) = r(\emptyset, T).$$

The following lemma is well known.

LEMMA 2.5. ([5; Lemma 6]) *For each countable ordinal α, there exists an index tree T of rank α. Moreover, if $(t,n) \in T$ then $r(t) = \lim_{n\to+\infty} r(t,n)$.*

DEFINITION 2.6. Let T be an index tree, $\max T$ the set of all maximal elements of T and X a set. A multisequence[2] in X is a mapping defined on $\max T$ into X. If X is a topological space, then we say that the multisequence f converges to a point x if it admits an extension $\widetilde{f}\colon T \to X$ such that

$$\forall_{t\in T\setminus \max T} \quad \widetilde{f}(t) = \lim_{n\to+\infty} \widetilde{f}(t,n) \quad \text{and} \quad x = \widetilde{f}(\emptyset).$$

The extended multisequence $\widetilde{f}$ is *irreducible* if it is one to one and if for each sequence $(t_k) \subset T$ and for each index $t \in T$, $\widetilde{f}(t) = \lim_k \widetilde{f}(t_k)$, then $t_k = t$ for all but finitely many k or $t_k = (t, n_k)$ with $\lim_{k\to+\infty} n_k = +\infty$.

The *sequential order* of a convergent multisequence f is defined by

$$\sigma(f) = \sigma\big(f(\emptyset); f(\max T)\big). \tag{1}$$

THEOREM 2.7. ([3], [12]) *Let X be a topological space, $A \subset X$ and $x \in X$. Then, $x \in \mathrm{cl}_{\mathrm{Tseq}} A$ if and only if there exists in A a multisequence that converges to x.*

THEOREM 2.8. ([3], [4]) *Let X be a topological space, $A \subset X$ and $x \in X$. If $\sigma(x; A) = \alpha$, then there exists a multisequence $f\colon \max T \to A$ of order α that converges to x.*

3. Sequentiality and upper bound of the sequential order of the product of two sequential spaces

It is well known that the product of two sequential spaces need not be sequential even when the topologies are Fréchet. There are several examples illustrating this ([1], [6], [11], [13]). The following example seems to be new and particularly simple. It shows that the product of two Fréchet spaces need not be sequential nor of sequential order equal 1.

[2]Let us note that in [7], [8], P. Kratochvíl introduced a notion of multisequence which is similar to ours.

EXAMPLE 3.1. Let $X = \{x_{(n,k,p)},\ (n,k,p) \in \mathbb{N}^3\} \cup \{x_n,\ n \in \mathbb{N}\} \cup \{x\}$ be a set such that for each $n \neq m$, $x_n \neq x_m$, for each $(n,k,p) \neq (n',k',p')$, $x_{(n,k,p)} \neq x_{(n',k',p')}$ and for each n, $x_{(n,k)} = x_n$ for all $k \in \mathbb{N}$. Equip X with the following Hausdorff topology: the points $x_{(n,k,p)}$ are isolated, a neighborhood base of the point x is the countable set $\mathcal{N}(x) = \{A_p,\ p \in \mathbb{N}\}$ where

$$A_p = \{x\} \cup \{x_n,\ n \geq p\} \cup \{x_{(n,k,p)};\ (n,k,p) \in \mathbb{N}^3,\ n \geq p\}$$

and a neighborhood base of a point x_n is the uncountable set $\mathcal{N}(x_n) = \{B_f,\ f \in \mathbb{N}^{\mathbb{N}}\}$ where $B_f = \{x_n\} \cup \{x_{(n,k,p)},\ k \in \mathbb{N},\ p \geq f(k)\}$. It is obvious that $x_{(n,k,p)} \underset{p}{\longrightarrow} x_{(n,k)} \underset{k}{=} x_n \underset{n}{\longrightarrow} x$. Let $Y = \{y_{(m,q)},\ (m,q) \in \mathbb{N}^2\} \cup \{y\}$ be topological spaces equipped with the following topology: the points $y_{(m,q)}$ are isolated and a neighborhood base of the point y is the uncountable set $\mathcal{N}(y) = \{B_g,\ g \in \mathbb{N}^{\mathbb{N}}\}$ where $B_g = \{y\} \cup \{y_{(m,q)};\ (m,q) \in \mathbb{N}^2,\ q \geq g(m)\}$. It is obvious that for each m, $y_{(m,q)} \underset{q}{\longrightarrow} y$, so $(y_{(m,q)})$ is a fan. Note that X and Y are respectively an untraversable arrow and an untraversable fan ([3], [4], [12]).

I claim that the product $X \times Y$ is not sequential.

In fact, let $A = \{(x_{(n,k,k)}, y_{(k,n)}),\ (n,k) \in \mathbb{N}^2\}$. Because the only convergent sequences in A are stationary sequences, A is sequentially closed. But $(x,y) \in \operatorname{cl} A$ so that if $W \in \mathcal{N}(x,y)$ is a neighborhood of (x,y), then it contains a product $U \times V$ where $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$. Accordingly, (because x is first countable) W contains a subset of form: $\{x_{(n,k,p)};\ n \geq n_0$ and $(k,p) \in \mathbb{N}^2\}$ $\times \{y_{(m,q)};\ m \in \mathbb{N}$ and $q \geq q_m\}$. Let $s > n_0$ and $t > q_s$. Then $(x_{s,t,t}, y_{t,s}) \in W \cap A$. Thus $X \times Y$ is not sequential.

Each subset A of $X \times Y$ defines a multifunction $(A\colon X \rightrightarrows Y)$ to the effect that $Ax = \{y \in Y:\ (x,y) \in A\}$. By A^- we understand the inverse multifunction $A^- y = \{x \in X:\ (x,y) \in A\}$. If $V \subset X$, then $AV = \bigcup_{x \in V} Ax$; if $W \subset Y$, then $A^- W = \bigcup_{y \in W} A^- y = \{x:\ Ax \cap W \neq \emptyset\}$.

LEMMA 3.2. *Let X be a sequential space, Y a sequential regular locally countably compact space and $A \subset X \times Y$. If $(x,y) \in \operatorname{cl} A$, then there exists a tree T of rank $r(T) \leq \sigma(X) + \sigma(Y)$, a subtree R of T of rank $r(R) \leq \sigma(Y)$ and multisequences $f\colon \max T \to X$ and $g\colon \max T \to Y$ such that $f \times g$ is valued in A and converges to (x,y) and the restriction $f|_R$ of f to R has the constant value x.*

P r o o f. Let $\mathcal{N}(y)$ be the set of closed countably compact neighborhoods of y and let $W \in \mathcal{N}(y)$. It is obvious that $x \in \operatorname{cl} A^- W = \operatorname{cl}_{\mathrm{Tseq}} A^- W$. Let

$\alpha = \sigma(x; A^{-}W)$. Then there exists a tree of rank α and, by virtue of Theorem 2.8, a multisequence $f_0^W\colon \max S^W \to A^{-}W$ of order α that converges to x. Let $g_0^W\colon \max S^W \to W$ be a multisequence such that $(f_0^W \times g_0^W)(\max S^W) \subset A$. Because W is countably compact and closed, we may assume that the multisequence g_0^W converges to a point $y_W \in W$. Thus the multisequence $f_0^W \times g_0^W\colon \max S^W \to A$ converges to a (x, y_W). Let $F = \{y^W : W \in \mathcal{N}(y)\}$, then $y \in \operatorname{cl} F$. By virtue of Theorem 2.8, there exists a multisequence $g_0\colon \max R \to F$ that converges to y such that $\sigma(g_0) = \sigma(y; F) = r(R)$, thus $r(R) \le \sigma(Y)$. Define $T = \{(t, S^W) : t \in \max R\} \cup \{R\}$ (where $(t, S^W) := \{(t, s) : s \in W\}$), we get that $r(T) = r(S) + r(R) \le \sigma(X) + \sigma(Y)$. Let $f\colon T \to X$ defined by

$$\begin{cases} \forall_{t\in\max T,\, s\in S^W} \quad f(t,s) = f_0^W(s)\,, \\ \forall_{t\in R} \quad f(t) = x\,, \end{cases}$$

and $g\colon T \to Y$ defined by

$$\begin{cases} \forall_{t\in\max T,\, s\in S^W} \quad g(t,s) = g_0^W(s)\,, \\ \forall_{t\in R} \quad g(t) = g_0(t)\,. \end{cases}$$

The multisequences $f\colon \max T \to X$ and $g\colon \max T \to Y$ converge respectively to x and y and the multisequence $f \times g$ valued in A converges to (x, y). The proof is complete. □

The following theorem implies [9; Theorem 4.2.b] of E. Michael and [11; Theorem 2.3] of T. Nogura and A. Shibakov.

THEOREM 3.3. *Let X be a sequential space and let Y be a sequential regular locally countably compact space. Then $X \times Y$ is sequential and $\sigma(X \times Y) \le \sigma(X) + \sigma(Y)$.*

Proof. By virtue of Lemma 3.2, if $A \subset X \times Y$ and $(x, y) \in A$, then there exists a tree T of rank less than or equal to $\sigma(X) + \sigma(Y)$ and a multisequence defined on T that converges to (x, y). So $(x, y) \in \operatorname{cl}_{\mathrm{Tseq}} A$, thus $X \times Y$ is sequential. On the other hand, because the order of a multisequence is less than or equal to the rank of the tree on which it is defined, we get that for each $A \subset X \times Y$, and for each $(x, y) \in \operatorname{cl} A$, $\sigma\big((x, y); A\big) \le \sigma(X) + \sigma(Y)$. □

Let us recall, following A. V. Arhangel'skii, that a fan $\big(x_{(n,k)}\big)$ converging to x is (α_3) (*infinitely subtransversal*) if it has a subfan $\big(x_{(n_p,k_q)}\big)$ such that each sequence $\big(x_{(n_{p_m},k_{q_m})}\big)_m$ converges to x. A topology is *infinitely subtranversal* if each fan converging to a point x is infinitely subtransversal.

COROLLARY 3.4. ([1; Theorem 5.16]) The product of an infinitely subtransversal Fréchet space X with a Fréchet regular and locally countably compact space Y is a Fréchet space.

Proof. According to Theorem 3.3, it is enough to prove that the sequential order of $X \times Y$ is equal to 1. So, let $A \subset X \times Y$, $(x, y) \in \operatorname{cl} A$. By virtue of Lemma 3.2, there exists a bisequence $(x_{(n,k)}, y_{(n,k)})$ in A that converges to (x, y) such that for each k, $x_{(n,k)} \underset{n}{=} x$, so $(x_{(n,k)})$ is a fan that converges to x. As X in infinitely subtransversal we may assume that the fan $(x_{(n,k)})$ is epitransversal ([4], [12]). On the other hand, the bisequence $y_{(n,k)} \underset{k}{\longrightarrow} y_n \underset{n}{\longrightarrow} y$ converges to y in the Fréchet space Y. So there exists a sequence $(y_{(n_m,k_m)})_m$ that converges to y. Thus $(x_{(n_m,k_m)}, y_{(n_m,k_m)})$ is a sequence in A that converges to (x, y). □

THEOREM 3.5. *If X is a sequential infinitely subtransversal space of order α and if Y is a Fréchet regular locally countably compact space, then $X \times Y$ is sequential and its sequential order equals α.*

Proof. It is obvious that the sequential order of this product is greater than or equal to α. On the other hand, according to the preceding the product is sequential.

Let $A \subset X \times Y$ and $(x, y) \in \operatorname{cl} A$. By virtue of Lemma 3.2, there are multisequences $f \colon T \to X$ and $g \colon T \to Y$ such that $(f \times g)(\max T)$ is valued in A and converges to (x, y) and a subtree R of T of rank less than or equal 1 such that the restriction $f|_R$ of f to R has the constant value x. If $r(R) = 0$ then $r(T) = \alpha$ and then $\sigma\big((x, y); A\big) \le \alpha$. If $r(R) = 1$ then for each n, $f(n) = x$ so $f(n, k)$ is a fan converging to x. So because X is infinitely subtransversal we may assume that it is epitransversal [3]. Hence $g(n, k) \underset{k}{\longrightarrow} g(n) \underset{n}{\longrightarrow} y$ is a bisequence converging to y in the Fréchet space Y. So as in the proof of Theorem 3.3, we may reduce the rank of T to α so $\sigma\big((x, y); A\big) \le \alpha$. The proof is complete. □

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Département de mathématiques
Université de Bourgogne
BP 400
F-21011 Dijon cedex
FRANCE

E-mail: sitou@satie.u-bourgogne.fr