Mathematica Slovaca

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Mathematica Slovaca, Vol. 46 (1996), No. 2-3, 285--289

Persistent URL: http://dml.cz/dmlcz/131457

Math. Slovaca, **46** (1996), No. 2–3, 285–289

Mathematica Slovaca
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Slovak Academy of Sciences

Dedicated to Professor Tibor Šalát
on the occasion of his 70th birthday

NOTE ON A LOCAL INVERTIBILITY[1]

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(*Communicated by Milan Medveď*)

ABSTRACT. It is shown, if any affinely small perturbation of a C^2-smooth map has at most one zero point near a zero point of the map, then the linearization of the map at the zero point is invertible. Hence, that result is an inverse of the well-known fact that the linear invertibility implies local nonlinear invertibility.

It is well known that the linear invertibility implies local nonlinear invertibility. More precisely, let us consider a map $F\colon X \to Y$, $F(0) = 0$, where F is C^1-smooth, and X, Y are Banach spaces. If $DF(0)$ is invertible, then any C^1-small perturbation of F has a unique zero point near 0. Now we shall study a reverse problem.

THEOREM. *Consider a C^2-smooth map $F\colon X \to Y$ satisfying $F(0) = 0$ and assume $DF(0)$ is Fredholm with index 0.*

If there exist a neighbourhood $U \subset X$ of 0 and numbers $K > 0$, $\delta > 0$ such that for any linear bounded mapping $B\colon X \to Y$, $\|B\| \le K$, the perturbation $\varepsilon B + F$, $0 \le \varepsilon \le \delta$, has the only zero point 0 in U, then $DF(0)$ is invertible.

Note, if there is a number K satisfying the assumption of the above theorem, then this assumption holds with any $K > 0$ and the same neighbourhood U. Of course, we must take another $\delta > 0$. If we are interested in the invertibility of $DF(x_0)$ for a general fixed x_0 satisfying $F(x_0) = 0$, then Theorem is applied with perturbations of the form $\varepsilon(B - Bx_0) + F$, where B has the properties of Theorem. Indeed, we apply Theorem for the map $x \mapsto F(x + x_0)$. The perturbation terms $\varepsilon(B - Bx_0)$ are affinely small.

AMS Subject Classification (1991): Primary 58C15, 58F14, 58F30.
Key words: local invertibility, bifurcation, perturbation.

[1]Supported by Grant GA-SAV 2/999369/93.

Proof of Theorem. Let us suppose $DF(0)$ is not invertible. Since $DF(0)$ is Fredholm with index 0, we have

$$\dim\ker DF(0) = m < \infty \quad \text{and} \quad \operatorname{codim}\operatorname{im} DF(0) = m .$$

We take the decomposition

$$\begin{aligned} X_1 \oplus X_2 &= X , & X_1 &= \ker DF(0) , \\ Y_1 \oplus Y_2 &= Y , & Y_2 &= \operatorname{im} DF(0) , \end{aligned}$$

and the projections $P = I - Q$, $Q\colon Y \to Y_1$, $P\colon Y \to Y_2$. We assume $X_1 = Y_1 = \mathbb{R}^m$. Let us take any invertible matrix $A\colon \mathbb{R}^m \to \mathbb{R}^m$ with the norm $\|A\| = 1$ and consider the perturbation $\varepsilon B + F$ for $\varepsilon \geq 0$ small, where B is a linear map defined by

$$PB = 0 \quad \text{and} \quad QB(x_1 + x_2) = Ax_1 \quad \forall x_{1,2} \in X_{1,2} .$$

Our assumptions imply that 0 is the only zero point of $\varepsilon B + F$ in the set U for any $\varepsilon \geq 0$ small. On the other hand, let us solve

$$\varepsilon Bx + F(x) = 0 , \qquad x \in U ,$$

which is equivalent to

$$\begin{aligned} \varepsilon Ax_1 + QF(x_1 + x_2) &= 0 , \qquad x_{1,2} \in X_{1,2} , \\ PF(x_1 + x_2) &= 0 . \end{aligned}$$

Since $PF(0) = 0$ and $D_{x_2}PF(0)$ is invertible, the above equation is reduced on

$$\varepsilon Ax_1 + QF\bigl(x_1 + x_2(x_1)\bigr) = 0 ,$$

where $x_2(\cdot)$ is the solution of $PF(x_1 + x_2) = 0$, $x_2(0) = 0$.

Note that

a) x_1 is considered on a fixed neighbourhood $\tilde{U} \subset \{x \in U \mid x_2 = 0\}$ of $0 \in X_1 = \mathbb{R}^m$;

b) the map $x_2(\cdot)$ is C^2-smooth.

By putting $\bar{F}(x_1) = QF\bigl(x_1 + x_2(x_1)\bigr)$, $x_1 \in \tilde{U}$, we have just proved the existence of the neighbourhood $\tilde{U}$ of $0 \in X_1 = \mathbb{R}^m$ and a number $\tilde{\delta} > 0$ such that for any invertible matrix A, $A\colon \mathbb{R}^m \to \mathbb{R}^m$, $\|A\| = 1$, the map $\varepsilon A + \bar{F}$, $0 \leq \varepsilon \leq \tilde{\delta}$, has the only zero point 0 in $\tilde{U}$.

Hence

$$\deg(\bar{F}, \tilde{U}, 0) = \deg(\varepsilon A + \bar{F}, \tilde{U}, 0) = \deg(\varepsilon A + \bar{F}, B_\varepsilon, 0) ,$$

where $B_\varepsilon = \{\varepsilon^2 z \mid |z| \leq 1\}$, and deg is the Brouwer degree. Here ε is sufficiently small.

Furthermore, by using $\bar{F} \in C^2$, $\bar{F}(0) = 0$, $D\bar{F}(0) = 0$, we have

$$\varepsilon A\varepsilon^2 z + \bar{F}(\varepsilon^2 z) = \varepsilon^3\big(Az + O(\varepsilon)\cdot O(|z|)\big).$$

We have used here the C^2-smoothness of F. Since A is invertible, we have

$$\deg(\varepsilon A+\bar{F}, B_\varepsilon, 0) = \deg\big(A+O(\varepsilon)\cdot O(|z|), B_1, 0\big) = \deg(A, B_1, 0).$$

Summarizing we see

$$\deg(\bar{F}, \tilde{U}, 0) = \deg(A, B_1, 0).$$

But A is an arbitrary invertible matrix satisfying $\|A\| = 1$. Hence $\deg(\bar{F}, \tilde{U}, 0)$ has to vary. We arrive at the contradiction. Hence $DF(0)$ is invertible. The proof is finished. □

Remark 1. We can weaken the C^2-smoothness of F in Theorem by assuming

(a) $F\colon X \to Y$ is C^1-smooth and Fredholm with index 0 satisfying $F(0)=0$;
(b) there exist a neighbourhood $U \subset X$ of 0 and a number $\delta > 0$ such that any δ-small C^1-smooth perturbation $\tilde{F}$ of F with $\tilde{F}(0) = 0$ has the only zero point 0 in U.

To prove this assertion, let us assume $DF(0)$ is not invertible. First, by the proof of Theorem, we can consider the case $X = Y = \mathbb{R}^m$, $F(0) = 0$, $DF(0) = 0$. Furthermore, by using a cut-off function method [4; 3.128 Lemma], [5; p. 314], there is a C^1-smooth map $G\colon \mathbb{R}^m \to \mathbb{R}^m$ such that

$$G = -F \qquad \text{on a neighbourhood of } 0 \in \mathbb{R}^m;$$
$$\|G(x)\| \le \delta, \quad \|DG(x)\| \le \delta \quad \forall x\in\mathbb{R}^m.$$

Hence the perturbation $F + G$ has the only zero point 0 in U. But $F + G = 0$ near the point 0. This contradiction gives that $DF(0)$ has to be invertible.

Remark 2. The statement of Remark 1 is true when the C^1-smoothness is replaced by the C^r-smoothness, $1 \le r \le \infty$, in the assumptions (a), (b) of Remark 1.

Remark 3. The main difference between Theorem and Remarks 1, 2 is as follows: If perturbing terms have to be smoother than an unperturbed map, then we can apply only Theorem. For instance, assume

1) $F\colon X \to Y$ is C^r-smooth, $r \ge 2$, and Fredholm with index 0 satisfying $F(0) = 0$;
2) there exist a neighbourhood $U \subset X$ of 0 and a number $\delta > 0$ such that for any δ-small C^{r+1}-smooth map G satisfying $G(0) = 0$ the perturbation $F + G$ has the only zero point 0 in U.

Then we are able to apply only Theorem to assert that $DF(0)$ is invertible.

This theorem is important in the bifurcation theory. Since it ensures that, if a bifurcating branch of solutions of an equation is locally unique, then the linearizations of that equation at those bifurcating solutions are invertible. Some results of that kind have been already proved. But those proofs are much more technically tedious. Moreover, the above theorem presents a general result independent of a special form of an investigated equation.

We illustrate that idea in the following example. Let us consider a smooth diffeomorphism $f_e \colon \mathbb{R}^m \to \mathbb{R}^m$, $e \in \mathbb{R}$, and f_0 has a one-parametric family

$$\Gamma = \big\{ \{x_n(c)\}_{-\infty}^{\infty} \mid c \in \mathbb{R} \big\}$$

of homoclinic orbits tending to a hyperbolic fixed point. We put

$$X = \big\{ \{a_n\}_{-\infty}^{\infty} \mid \sup_n |a_n| < \infty \,, \; a_n \in \mathbb{R}^m \big\} \,,$$

$$F_e\big(\{y_n\}_{-\infty}^{\infty}\big)_n = y_{n+1} - f_e(y_n) \,.$$

Since $F_0\big(\{x_n(c)\}_{-\infty}^{\infty}\big) = 0$, differentiating we have

$$\ker DF_0\big(\{x_n(c)\}_{-\infty}^{\infty}\big) \supset \operatorname{span}\{x_n'(c)\}_{-\infty}^{\infty} \,.$$

Let us assume $\dim \ker DF_0\big(\{x_n(c)\}\big) = 1$ for any c small. Then, by applying the Lyapunov-Schmidt reduction (see [1; p. 358]), we obtain uniformly for e small a bifurcation equation of the equation $F_e\big(\{y_n\}_{-\infty}^{\infty}\big) = 0$ of the form

$$Q(c,e) = 0 \,, \qquad Q(\cdot\,,\cdot) \in \mathbb{R} \,,$$

where c is inherited from Γ. Hence $Q(c,0) = 0$. Generally, Q is smooth, thus $Q(c,e) = e \cdot M(c,e)$, and this implies that $M(c,0)$ is the Melnikov function for this problem, i.e., if

$$\exists c_0 \quad M(c_0,0) = 0 \text{ and } \frac{\partial}{\partial c} M(c_0,0) \neq 0 \,,$$

then, by the implicit function theorem, $Q(c,e) = 0$ has a unique solution $c = c(e)$, $c(0) = c_0$, for $e \neq 0$ small. But then $F_e(z) = 0$ has a solution $z = z(e)$ satisfying $z(0) = \big\{x_n(c_0)\big\}_{-\infty}^{\infty}$, and thus f_e has a homoclinic orbit near Γ for any e small. Furthermore, the condition

$$\exists c_0 \quad M(c_0,0) = 0 \text{ and } \frac{\partial}{\partial c} M(c_0,0) \neq 0$$

implies much more. Namely, let us fix a small $e_0 \neq 0$ and consider any perturbation $\tilde{F}_e(z) = F_e(z) + e\mu B\big(z - z(e_0)\big)$ of F_e, where $B \colon X \to X$ is a linear bounded operator satisfying $\|B\| \leq 1$, and $\mu > 0$ is a small number. By applying the above procedure of the equation $F_e = 0$, we see this condition gives that

$\tilde{F}_e(z) = 0$, $e \neq 0$, has also a unique solution $\tilde{z}(e)$ near $z(e)$ uniformly for any μ sufficiently small. Note $\tilde{F}_{e_0}\big(z(e_0)\big) = 0$ and $F_{e_0}\big(z(e_0)\big) = 0$. Hence we can apply the above theorem to obtain that $DF_{e_0}\big(z(e_0)\big)$ is invertible. It is known [2], [3]: the invertibility of $DF_e\big(z(e)\big)$, $e \neq 0$, implies that the predicted homoclinic orbit of f_e is transversal. Note, an explicit computation of the invertibility of $DF_e\big(z(e)\big)$, $e \neq 0$, is tedious (see [3; Theorem 4.1]).

Acknowledgements

It is a pleasure to thank Professor Flaviano Battelli for useful discussions when the author was visiting the Istituto di Biomatematica, Università, Urbino, Italy during September 1993.

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Received January 13, 1993
Revised March 15, 1994

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