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Ján Borsík; Jozef Doboš

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ON METRIZATION OF THE UNIFORMITY OF A PRODUCT OF METRIC SPACES

JÁN BORSÍK and JOZEF DOBOŠ

Let T be a nonempty set. Denote by $\mathcal{M}(T)$ the set of all mappings $f\colon \{x\in R^T;\ \forall t\in T\colon x(t)\geqq 0\}\to R$ such that $d(x, y) = f(\{d_t(x(t), y(t))\}_{t\in T})$ is a metric on the set $\prod_{t\in T} M_t$ for every collection of metric spaces $\{(M_t, d_t)\}_{t\in T}$.

In [3] we have established a necessary and sufficient condition for the product topology on $\prod_{t\in T} M_t$ to be metrized by d. A natural question arises whether we can investigate metrizability by the metric d of the product uniformity on $\prod_{t\in T} M_t$. The special case when the index set T has exactly one element was solved in [2]. The present paper gives a complete answer regarding any index set T. The necessary and sufficient condition is formulated in Theorem.

For elements of the uniform spaces theory we refer to [1].

Definition 1. *Let $D=\{(M_t, d_t)\}_{t\in T}$ be a collection of metric spaces. Define a mapping $\varrho_D\colon \left(\prod_{t\in T} M_t\right)^2\to R^T$ by*

$$(\varrho_D(x, y))(t)=d_t(x(t), y(t)) \tag{1}$$

for each $x, y\in\prod_{t\in T} M_t$, $t\in T$.

Definition 2. *Let T be a nonempty set. Suppose R^T to be ordered coordinate-wise, i.e.*

$$x\leqq y \text{ if and only if } x(t)\leqq y(t) \text{ for each } t\in T.$$

Define a function $\Theta\colon T\to R$ by $\Theta(t)=0$ for each $t\in T$. Put $T^+=\{x\in R^T\colon x\geqq\Theta\}$. Denote by $\mathcal{M}(T)$ the set of all functions $f\colon T^+\to R$ such that $f\circ\varrho_D$ is a metric for every collection of metric spaces $D=\{(M_t, d_t)\}_{t\in T}$.

Lemma 1. *Let $f\in\mathcal{M}(T)$. Then*

$$\forall x, y\in T^+\colon x\leqq 2y \Rightarrow f(x)\leqq 2\cdot f(y), \tag{2}$$

(3) $$\forall x \in T^{+}: f(x)=0 \Leftrightarrow x=\Theta .$$

Proof. See [3].

Definition 3. *Let (M, d) be a metric space. Define a uniformity $\mathscr{U}$ on the set M as follows:*

(4) $$\mathscr{U}=\{A \subset M^{2} ; \exists \varepsilon>0: d^{-1}(\langle 0, \varepsilon)) \subset A\} .$$

Definition 4. *Let $\{(M_{t}, d_{t})\}_{t \in T}$ be a collection of metric spaces. Let $\{(M_{t}, \mathscr{U}_{t})\}_{t \in T}$ be a collection of uniform spaces defined according to (4). Denote by $\mathscr{U}_{D}$ the product uniformity of the collection $\{(M_{t}, \mathscr{U}_{t})\}_{t \in T}$, i.e.*

(5) $$\mathscr{U}_{D}=\{A \subset\Big(\prod_{t \in T} M_{t}\Big)^{2}: \exists F \subset T, F \neq \emptyset \text{ finite } \forall t \in F \ \exists U_{t} \in \mathscr{U}_{t}:$$
$$\bigcap_{t \in F}(\pi_{t} \times \pi_{t})^{-1}(U_{t}) \subset A\},$$

where π_{t} is the projection.

Denote by $\mathscr{U}_{f}$ the uniformity on the set $\prod_{t \in T} M_{t}$ derived by (4) from the metric $f \circ \varrho_{D}$.

Lemma 2. *Let $D=\{(M_{t}, d_{t})\}_{t \in T}$ be a collection of metric spaces. Let $f \in \mathcal{M}(\Gamma)$. Then $\mathscr{U}_{D} \subset \mathscr{U}_{f}$.*

Proof. Let $U \in \mathscr{U}_{D}$. Then by (5) we have $\exists F \subset T, F \neq \emptyset$ *finite* $\forall t \in F\ \exists U_{t} \in \mathscr{U}_{t}$: $\bigcap_{t \in F}(\pi_{t} \times \pi_{t})^{-1}(U_{t}) \subset U$. Let $t \in F$. Since $U_{t} \in \mathscr{U}_{t}$, there is, according to (4), a positive ε_{t} such that

$$d_{t}^{-1}(\langle 0, \varepsilon_{t})) \subset U_{t} .$$

Denote $V_{t}=(\pi_{t} \times \pi_{t})^{-1}(d_{t}^{-1}(\langle 0, \varepsilon_{t})))$. Then we have

(6) $$\bigcap_{t \in F} V_{t} \subset \bigcap_{t \in F}(\pi_{t} \times \pi_{t})^{-1}(U_{t}) \subset U .$$

Define a mapping $A: F \rightarrow R^{T}$ as follows:

$$(A(t))(i)= \begin{cases}2 \varepsilon, & \text{if } i=t \\ 0, & \text{if } i \in T-\{t\}\end{cases}$$

for each $t \in F, i \in T$.

Let $t \in F$. Denote $\delta_{t}=f(A(t)) / 2$ (according to (3) we have $\delta_{t}>0$) and $W_{t} = (f \circ \varrho_{D})^{-1}(\langle 0, \delta_{t}))$.

Then $W_{t} \in \mathscr{U}_{f}$ for all $t \in F$, therefore

(7) $$\bigcap_{t \in F} W_{t} \in \mathscr{U}_{f} .$$

We show that $W_t \subset V_t$ for all $t \in F$. Let $t \in F$, $(x, y) \in W_t$. Then $f(\varrho_D(x, y)) < \delta = f(A(t))/2$, therefore from (2) it follows that $\neg(\varrho_D(x, y) \geqq A(t)/2)$. i.e.

$$d_t(x(t), y(t)) = (\varrho_D(x, y))\,(t) < (A(t))\,(t)/2 = \varepsilon .$$

Therefore $(x, y) \in V_t$. Then by (6) we have $\bigcap_{t \in F} W_t \subset \bigcap_{t \in F} V_t \subset U$ and therefore by (7) we get $U \in \mathcal{U}_f$.

Proposition 1. *Let $D = \{(M_t, d_t)\}_{t \in T}$ be a collection of metric spaces. Let $f \in \mathcal{M}(T)$ be a mapping continuous at the point Θ. Then $\mathcal{U}_D = \mathcal{U}_f$.*

Proof. By lemma 2 it suffices to prove that $\mathcal{U}_f \subset \mathcal{U}_D$. Let $U \in \mathcal{U}_f$. Then according to (4) there is a positive ε such that

$$(f \circ \varrho_D)^{-1}(\langle 0, \varepsilon)) \subset U . \tag{8}$$

Since f is continuous at the point Θ, we have

$$\exists F \subset T, F \neq \emptyset \text{ finite } \exists \gamma > 0\ \forall y \in T^+ : \quad (\forall t \in F : y(t) < \gamma) \Rightarrow f(y) < \varepsilon . \tag{9}$$

Denote $V = \bigcap_{t \in F} (\pi_t \times \pi_t)^{-1}(d_t^{-1}(\langle 0, \gamma)))$. Then $V \in \mathcal{U}_D$.

We show that $V \subset (f \circ \varrho_D)^{-1}(\langle 0, \varepsilon))$.

Let $(x, y) \in V$. Then $d_t(x(t), y(t)) < \gamma$ for all $t \in T$, therefore from (4) we get

$$(f \circ \varrho_D)\,(x, y) = f(\varrho_D(x, y)) < \varepsilon, \quad \text{i.e.} \quad (x, y) \in (f \circ \varrho_D)^{-1}(\langle 0, \varepsilon)).$$

Therefore $V \subset (f \circ \varrho_D)^{-1}(\langle 0, \varepsilon))$. Then from (8) we have $V \subset U$, therefore $U \in \mathcal{U}_D$.

Definition 5. *Let $D = \{(M_t, d_t)\}_{t \in T}$ be a collection of metric spaces. Define*

$$I_D = \{t \in T : \sup \mathit{Im}\ d_t < \infty\} , \tag{10}$$

$$S_D = \{t \in T : \forall \varepsilon > 0 : d_t^{-1}((0, \varepsilon)) \neq \emptyset\} . \tag{11}$$

Theorem. *Let $D = \{(M_t, d_t)\}_{t \in T}$ be a collection of metric spaces. Let M_t be a nonempty set for each $t \in T$. Let $f \in \mathcal{M}(T)$. Then $\mathcal{U}_D = \mathcal{U}_f$ if and only if*

$$\forall \varepsilon > 0\ \exists F \subset T, F \neq \emptyset \text{ finite } \exists \delta > 0\ \forall \alpha \in N^{(T-(I_D \cup F))}\ \exists a \in T^+ :$$

$$\forall t \in T - (I_D \cup F) : a(t) \geqq \alpha(t), \tag{A}$$
$$\forall t \in I_D - F : a(t) \geqq \sup \mathit{Im}\ d_t, \tag{B}$$
$$\forall t \in F \cap S_D : a(t) \geqq \delta, \tag{C}$$
$$f(a) < \varepsilon . \tag{D}$$

Proof. *Necessity.*

Let $\varepsilon > 0$. Since $\mathcal{U}_f \subset \mathcal{U}_D$, we have $(f \circ \varrho_D)^{-1}(\langle 0, \varepsilon/2)) \in \mathcal{U}_D$. Therefore according to (5) we have

$$\exists F \subset T, F \neq \emptyset \text{ finite } \forall t \in F\ \exists U_t \in \mathcal{U}_t : \bigcap_{t \in F} (\pi_t \times \pi_t)^{-1}(U_t) \subset (f \circ \varrho_D)^{-1}(\langle 0, \varepsilon/2)).$$

Let $t\in F$. Since $U_t\in\mathscr{U}_t$, according to (4) there exists $\gamma_t>0$ such that $d_t^{-1}((0,\gamma_t))\subset U_t$. Denote $V=\bigcap_{t\in F}(\pi_t\times\pi_t)^{-1}(d_t^{-1}((0,\gamma_t)))$. Then obviously

$$V\subset(f\circ\varrho_D)^{-1}((0,\varepsilon/2)). \tag{12}$$

Let $t\in F\cap S_D$. Then there are $u_t,\ v_t\in M_t$ such that

$$0<d_t(u_t,v_t)<\gamma_t. \tag{13}$$

Put

$$\delta=\min\{d_t(u_t,v_t):t\in F\cap S_D\}>0 \tag{14}$$

(in case of $F\cap S_D=\emptyset$ let $\delta>0$ be arbitrary).

Let $\alpha\in N^{(T-(I_D\cup F))}$. Let $t\in T-I_D$. Then there are $p_t,\ q_t\in M_t$ such that

$$d_t(p_t,q_t)\geqq\alpha(t). \tag{15}$$

Denote $J=\{t\in I_D:\sup Im\ d_t>0\}$.

Let $t\in J$. Then there are $r_t,\ s_t\in M_t$ such that

$$d_t(r_t,s_t)>(1/2)\cdot\sup Im\ d_t. \tag{16}$$

Let $t\in T$. Since M_t is a nonempty set, choose an arbitrary element $w_t\in M_t$.

Define the mappings $x,\ y\colon T\to\bigcup_{t\in T}M_t$ as follows:

$$x(t)=\begin{array}{c}u_t\\p_t\\r_t\\w_t\end{array}\qquad y(t)=\begin{array}{cl}v_t & \text{for } t\in F\cap S_D\\ q_t & \text{for } t\in T-(I_D\cup F)\\ s_t & \text{for } t\in J-F\\ w_t & \text{for } t\in[I_D-(J\cup F)]\cup(F-S_D).\end{array}$$

Denote $a=2\cdot\varrho_D(x,y)$.

Now we show that a satisfies the conditions (A), (B), (C), (D). "A": Let $t\in T-(I_D\cup F)$. Then according to (15) we have

$$a(t)=(2\cdot\varrho_D(x,y))\,(t)=2d_t(x(t),y(t))=2d_t(p_t,q_t)\geqq\alpha(t).$$

"B": Let $t\in I_D-(J\cup F)$. Then we obtain

$$a(t)=2d_t(x(t),y(t))=2d_t(w_t,w_t)=0=\sup Im\ d_t.$$

Let $t\in J-F$. Then from (16) we have

$$a(t)=2d_t(x(t),y(t))=2d_t(r_t,s_t)>2\cdot(1/2)\cdot\sup Im\ d_t=\sup Im\ d_t.$$

Therefore $a(t)\geqq\sup Im\ d_t$ for all $t\in I_D-F$.

"C": Let $t\in F\cap S_D$. Then from (14) we get

$$a(t)=2d_t(x(t),\,y(t))=2d_t(u_t,\,v_t)\geqq\delta\,.$$

"D": Let $t\in F\cap S_D$. Then according to (13) we have

$$d_t(x(t),\,y(t))=d_t(u_t,\,v_t)<\gamma_t\,.$$

Let $t\in F-S_D$. Then

$$d_t(x(t),\,y(t))=d_t(w_t,\,w_t)=0<\gamma_t\,.$$

Therefore $d_t(x(t),\,y(t))<\gamma_t$ for each $t\in F$, i.e. $(x,\,y)\in V$. Then from (2) and (13) we obtain

$$f(a)\leqq 2f(\varrho_D(x,\,y))=2(f\circ\varrho_D)\,(x,\,y)<2\cdot\varepsilon/2=\varepsilon\,.$$

Sufficiency. By lemma 2 it suffices to prove that $\mathscr{U}_f\subset\mathscr{U}_D$. Let $U\in\mathscr{U}_f$. Then according to (4) there is a positive ε such that

$$(f\circ\varrho_D)^{-1}(\langle 0{,}2\varepsilon))\subset U\,. \tag{17}$$

Then by the hypotheses we have

$$\exists F\subset T,\;F\neq\emptyset\;\textit{finite}\;\exists\delta>0\;\forall\alpha\in N^{(T-(I_D\cup F))}\;\exists a\in T^+:(A)-(D).$$

Let $t\in F-S_D$. Then there is $\gamma_t>0$ such that $d_t^{-1}((0,\,\gamma_t)=\emptyset$. Denote $\gamma=$ $=\min\,\{\gamma_t\colon t\in F-S_D\}>0$, in case of $F-S_D=\emptyset$ let $\gamma>0$ be arbitrary. Then

$$d_t^{-1}((0,\,\gamma))=\emptyset\;\;\text{for each}\;\;t\in F-S_D\,. \tag{18}$$

Denote $A=\bigcap\limits_{t\in F}(\pi_t\times\pi_t)^{-1}(d_t^{-1}(\langle 0,\,\min\,\{\gamma,\,\delta\})))$. Then $A\in\mathscr{U}_D$.

Let $(x,\,y)\in A$. Then

$$d_t(x(t),\,y(t))<\min\,\{\gamma,\,\delta\}\;\;\text{for each}\;\;t\in F. \tag{19}$$

Let $t\in T$. Then there is a positive integer n_t such that

$$d_t(x(t),\,y(t))\leqq n_t\,.$$

Define a mapping $\alpha\colon(T-(I_D\cup F))\to N$ by

$$\alpha(t)=n_t\,.$$

Then by the hypothesis there is $a\in T^+$ satisfying $(A)-(D)$. We show that $\varrho_D(x,\,y)\leqq a$.

Let $t\in I_D-F$. Then from (8) we have

$$d_t(x(t),\,y(t))\leqq\sup\,Im\;d_t\leqq a(t)\,.$$

Let $t \in F \cap S_D$. Then from (19) and (C) we obtain

$$d_t(x(t), y(t)) \leqq \delta \leqq a(t).$$

Let $t \in T - (I_D \cup F)$. Then from (A) we get

$$d_t(x(t), y(t)) \leqq \alpha(t) \leqq a(t).$$

Let $t \in F - S_D$. Then from (19) and (18) we have

$$d_t(x(t), y(t)) = 0 \leqq a(t).$$

Therefore $(\varrho_D(x, y))(t) = d_t(x(t), y(t)) \leqq a(t)$ for each $t \in T$, i.e. $\varrho_D(x, y) \leqq a$. Then according to (2) and (D) we obtain

$$(f \circ \varrho_D)(x, y) = f(\varrho_D(x, y)) \leqq 2 \cdot f(a) < 2\varepsilon,$$

therefore $(x, y) \in (f \circ \varrho_D)^{-1}(\langle 0, 2\varepsilon))$, i.e. $A \subset (f \circ \varrho_D)^{-1}(\langle 0, 2\varepsilon))$. Then according to (17) we obtain $A \subset U$, therefore $U \in \mathscr{U}_D$.

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Matematický ústav SAV
Komenského 14
041 54 Košice

966 54 Tekovské Nemce 261
okr. Žiar nad Hronom

О МЕТРИЗАЦИИ РАВНОМЕРНОЙ СТРУКТУРЫ ПРОИЗВЕДЕНИЯ МЕТРИЧЕСКИХ ПРОСТРАНСТВ

Ян Борсик—Йозеф Добош

Резюме

Пусть T — некоторое непустое множество. Обозначим T^+ множество всех неотрицательных вещественных функций, определенных на множестве T. Пусть $f: T^+ \to R$ — функция, для которой

$$d(x, y) = f(d_t(x(t), y(t)))$$

является метрикой на множестве $\prod_{t \in T} M_t$ для каждого семейства метрических пространств (M_t, d_t) $(t \in T)$. В настоящей работе мы предлагаем необходимое и достаточное условие метризации равномерной структуры произведения при помощи метрики d.