Surjit Singh Khurana

Dieudonné property

Mathematica Slovaca, Vol. 52 (2002), No. 5, 549--553

Persistent URL: http://dml.cz/dmlcz/130862

Math. Slovaca, 52 (2002), No. 5, 549–553

DIEUDONNÉ PROPERTY

SURJIT SINGH KHURANA

(Communicated by Miloslav Duchoň)

ABSTRACT. Let X_0 be a locally compact Hausdorff space, $C_0(X_0)$ the space of all scalar-valued bounded continuous functions on X_0 vanishing at infinity, and X a one-point compactification of X_0. We derive the Dieudonné property of $C_0(X_0)$ from the Dieudonné property of $C(X)$. The result is extended to $C_0(X_0, E)$, E a Banach space.

1. Introduction and notations

In [7], for a locally compact space X_0, the Dieudonné property of $C_0(X_0)$, endowed with the sup-norm topology, is proved. In this paper, this property is established as a simple consequence of the known Dieudonné property of $C(X)$ when X is compact. The result is then extended to $C_0(X_0, E)$, E being a Banach space.

In this paper, X_0 is a locally compact Hausdorff space, K the field of real or complex numbers (called scalars), $C_0(X_0)$ the space of all scalar-valued bounded continuous functions on X_0 vanishing at infinity, and X a one-point compactification of X_0; this point is called the point at infinity and we will denote it by p. $C(X)$ will denote the space of all K-valued continuous functions on X. We have $C_0(X_0) = \{f \in C(X) : f(p) = 0\}$. $C_0(X_0)$ and $C(X)$ are taken with the sup-norm topology. The duals of $C_0(X_0)$ and $C(X)$ are denoted by $M_0(X_0)$ and $M(X)$. Also $M_0(X_0) = \{\mu \in M(X) : |\mu|(\{p\}) = 0\}$.

We fix an increasing net of functions $\{g_\alpha\}$ in $C_0(X_0)$, $0 \le g_\alpha \uparrow 1$. For a $\mu \in M_0(X_0)$, we have

$$(\forall f \in C(X))\big(\mu(f) = \lim \mu(f g_\alpha)\big).$$

For locally convex spaces, the notations and results of [8], [6] will be used. For topological spaces, we refer to [2]. For topological measure theory, notations

2000 Mathematics Subject Classification: Primary 46E10, 28C05; Secondary 46G10, 28B05.
Keywords: Mackey topology, quasi-complete.

and results of [10], [9], [5] will be used. All locally convex spaces are assumed to be Hausdorff and over K.

For a locally convex space E, E', E'' will denote its dual and bidual; let

$$H = \left\{ f \in E'' : \ (\exists \{x_k\} \subset E)\left(x_k \xrightarrow{\sigma(E'',E')} f\right)\right\}.$$

H is called the *Baire subspace of* E'' *of first class* ([1; p. 646]).

E is said to have *Dieudonné* property if every equicontinuous, absolutely convex subset of E', compact in $\sigma(E',H)$ topology, is also compact in $\sigma(E',E'')$ topology. It is well-known ([3]) that $C(X)$ has Dieudonné property.

2. Main results

LEMMA 1. *Suppose* H *is the Baire subspace of first class of* $(C(X))''$ *and let* $F = \big(M(X), \tau(M(X),H)\big)$. *Then* F *is quasi-complete.*

Proof. Take a Cauchy, bounded net $\{\nu_\alpha\}$ in F. It is norm bounded. By taking subnets, if necessary, we assume that $\nu_\alpha \to \mu \in M(X)$ in $\sigma\big(M(X), C(X)\big)$-topology. Assume also that $\nu_\alpha \to \nu \in \tilde{F}$, the completion of F. We want to prove that $\nu = \mu$. They are evidently equal on $C(X)$. Take an $f \in H$ and a sequence $\{f_n\} \subset C(X)$ such that

$$f_n \to f \qquad \text{in} \quad \big((C(X))'', \sigma\big((C(X))'', M(X)\big)\big).$$

Since $\big((C(X))'', \tau\big((C(X))'', M(X)\big)\big)$ is complete,

$$P = \left\{ \sum_{n=1}^{\infty} c_n (f_n - f) : \ \sum_{n=1}^{\infty} |c_n| \le 1 \right\}$$

is absolutely convex and compact in $\big((C(X))'', \sigma\big((C(X))'', M(X)\big)\big)$ ([6; p. 249, 20.9.(6)]). Since $P \subset H$, $\nu_\alpha \to \nu$ uniformly on P. Thus $\mu(f) = \nu(f)$, and so $\mu = \nu$ on H. This proves the result. □

THEOREM 2. $C_0(X_0)$ *has Dieudonné property.*

Proof. Let H_0 be the Baire subspace of $(C_0(X_0))''$ of first class. This means elements of H_0 are limits of bounded, pointwise convergent sequences in $C_0(X_0)$. Thus $H_0 \subset (C(X))''$. Let H be the subspace of $(C(X))''$, generated by 1 and H_0. H is the Baire subspace of $(C(X))''$ of first class. By Lemma 1, $\big(M(X), \tau\big(M(X), H\big)\big)$ is quasi-complete.

Take a bounded, absolutely convex, compact $P \subset M_0(X_0)$ with the topology $\sigma\big(M_0(X_0), H_0\big)$. Now we will prove that P is compact in $\big(M(X), \sigma\big(M(X), H\big)\big)$.

Take a sequence $\{\mu_n\}$ in P and let μ_∞ be a cluster point of this sequence in $\sigma(M_0(X_0), H_0)$. Take an increasing sequence $\{g_n\} \subset \{g_\alpha\}$ such that $\lim_{k\to\infty} \mu_n(g_k) = \mu_n(1)$, $1 \le n < \infty$. Now $\lim g_n = g_0 \in H_0$. This proves that μ_∞ is the cluster point of $\{\mu_n\}$ in $(M(X), \sigma(M(X), H))$. Since $C(X)$ has Dieudonné property, the proof is complete. □

Now we come to the vector case. Let E be a Banach space such that for every compact Hausdorff space Y, $C(Y,E)$, with the sup-norm, has Dieudonné property. We will prove that $C_0(X_0, E)$ has Dieudonné property.

We denote by $C_0(X_0,E)$ the space of all E-valued continuous functions on X_0 vanishing at infinity, and by $C(X,E)$, the space of all E-valued continuous functions on X, both with the supremum norm topology. We have $C_0(X_0,E) = \{f \in C(X,E) : f(p) = 0\}$. The dual of $C(X,E)$ is denoted by $M(X,E')$ ([5]). It is a routine to prove that $(C_0(X_0,E))' = \{\mu \in M(X,E') : |\mu|(\{p\}) = 0\}$. Since elements of E can be considered as constant functions in $C(X,E)$, we have $C(X,E) = C_0(X_0,E) \oplus E$.

As in the scalar case, for a $\mu \in (C_0(X_0,E))'$, we have

$$(\forall f \in C(X,E))\,(\mu(f) = \lim \mu(f g_\alpha)) .$$

We begin with a simple lemma:

Lemma 3. *Suppose $\{f_n\}$ is a bounded sequence in $C(X,E)$ such that $f_n(x)$ is weakly Cauchy for each x in X. Then $\lim_{n\to\infty} \mu(f_n)$ exists for every $\mu \in M(X,E')$.*

Proof. We consider $C(X,E)$ as a subspace of $C(X \times S)$, S being the closed unit ball of E' with weak*-topology. Take a $\mu \in M(X,E')$ and extend μ to $C(X \times S)$ without increasing its norm. Since $\{f_n\}$ is pointwise Cauchy on $X \times S$, the result follows from the dominated convergence theorem. □

Now we come to Dieudonné property of $C_0(X_0,E)$.

Theorem 4. *If E is a Banach space such that, for every compact Hausdorff space Y, $C(Y,E)$ has Dieudonné property, then $C_0(X_0,E)$ has Dieudonné property.*

Proof. Let H_0 be the Baire subspace of $(C_0(X_0,E))''$ of first class. This means elements of H_0 are the limits of bounded, pointwise, weakly Cauchy sequences in $C_0(X_0,E)$. Thus $H_0 \subset (C(X,E))''$ (Lemma 3). Let E_1 be the limits, in E'', of weak Cauchy sequences in E, and H be the subspace of $(C(X,E))''$, generated by E_1 and H_0. As in the scalar case, H is the Baire subspace of $(C(X,E))''$ of first class. Proceeding exactly as in Lemma 1, we can prove that $(M(X,E'), \tau(M(X,E'),H))$ is quasi-complete.

Now for a bounded, absolutely convex, compact

$$P \subset \Big((C_0(X_0,E))', \sigma\big((C_0(X_0,E))', H_0\big) \Big),$$

to prove that P is compact in $\big(M(X,E'), \sigma(M(X,E'),H)\big)$, we take a sequence $\{\mu_n\}$ in P. Let μ_∞ be a cluster point of this sequence in

$$\Big((C_0(X_0,E))', \sigma\big((C_0(X_0,E))', H_0\big) \Big).$$

Take an increasing sequence $\{g_n\} \subset \{g_\alpha\}$ such that $\lim_{k\to\infty} |\mu_n|(g_k) = |\mu_n|(1)$, $1 \le n < \infty$. Fix an $x_0 \in E_1$ and take a sequence $\{x_k\} \subset E$ such that $x_k \to x_0$ in $\big(E'', \sigma(E'',E')\big)$. Put $\phi = \lim_{k\to\infty} x_k g_k$ (note $\{x_k g_k\}$ is point-wise weakly Cauchy in $C(X,E)$). From $\big|\mu\big((1-g_k)x_k\big)\big| \le |\mu|\big((1-g_k)\|x_k\|\big) \to 0$ for every $\mu \in M(X,E')$, it follows that $\mu_n(\phi) = \mu_n(x_0)$, $1 \le n < \infty$. This proves that μ is a cluster point of $\{\mu_n\}$ in $\big(M(X,E'), \sigma(M(X,E'),H)\big)$. This proves the result. □

Remark 5. It is proved in [4] that if a Banach space does not contain ℓ_1, then, for every compact Hausdorff space Y, $C(Y,E)$ has Dieudonné property.

Acknowledgement

The author is very grateful to the referee for some very useful suggestions which have improved the paper.

REFERENCES

[1] EDWARDS, R. E.: *Functional Analysis. Theory and Applications*, Holt Reinhart and Winston, New York-Chicago-San Francisco-Toronto-London, 1965.

[2] GILLMAN, L.—JERRISON, M.: *Rings of Continuous Functions*, D. Van Nostrand Comp., Inc., Princeton-Toronto-London-New York, 1960.

[3] GROTHENDIECK, A.: *Sur les applicationes linéaires faiblement compactes d'espace du type $C(K)$*, Canad. J. Math. **5** (1953), 129–173.

[4] KALTON, N. J.—SAAB, E.—SAAB, P.: *On Dieudonné property of $C(\Omega, E)$*, Proc. Amer. Math. Soc. **96** (1986), 50–52.

[5] KHURANA, S. S.: *Topologies on spaces of continuous vector-valued functions*, Trans. Amer. Math. Soc. **241** (1978), 195–211.

[6] KÖTHE, G.: *Topological Vector Spaces I*, Springer-Verlag, New York, 1969.

[7] PANCHAPAGESAN, T. V.: *A simple proof of Grothendieck theorem on Dieudonné property on $C_0(T)$*, Proc. Amer. Math. Soc. **129** (2001), 823–832.

[8] SCHAEFFER, H. H.: *Topological Vector spaces*, Springer Verlag, New York, 1986.

[9] WHEELER, R. F.: *Survey of Baire measures and strict topologies*, Exposition. Math. **2** (1983), 97–190.

[10] VARADARAJAN, V. S.: *Measures on topological spaces*, Amer. Math. Soc. Transl. Ser. 2 **48** (1965), 161–220.

Received June 21, 2001
Revised June 11, 2002

Department of Mathematics
University of Iowa
Iowa City
Iowa 52242
U. S. A.

E-mail: skhurana@blue.weeg.uiowa.edu